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The detectability of asymmetric distributions deviating from normality due to small skewness

Piotr Sulewski[a]

Abstract. The aim of this article is to test the ability of goodness-of-fit tests (GoFTs) to detect any deviations from normality. A very specific case is considered, namely the deviation from normality consisting in the coincidence of asymmetry and small γ_1 skewness. The first step in achieving the aforementioned aim is to compile a set of normality-oriented GoFTs commonly recommended for use, as described in the recently published literature. The second step is to create a family of asymmetric distributions with a non-constant γ_1, further referred to as alternatives. The formulas for calculating γ_1 are provided for each alternative. To compare the alternatives with the normal distribution, a relevant similarity measure is applied. The third step involves running a Monte Carlo simulation. The study investigates 21 GoFTs and 13 alternatives. The obtained results show that the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n GoFTs prove most effective in detecting asymmetric distributions that deviate from normality due to small skewness, equal to even 0.05.

Keywords: normality, goodness-of-fit test, skewness
JEL: C1, C6

1. Introduction

Numerous goodness-of-fit tests (GoFTs) are discussed in the statistics-related literature. The most common normality test procedures available in statistical software are the Kolmogorov-Smirnov (KS) test (Kolmogorov, 1933; Smirnov, 1948), the Lilliefors (LF) test (Lilliefors, 1967), the Cramer-von Mises (CVM) test (Cramér, 1928), the Anderson-Darling (AD) test (Anderson & Darling, 1952), and the Shapiro-Wilk (SW) test (Shapiro & Wilk, 1965). The Power R package (Lafaye de Micheaux & Tran, 2016) from the R software proved the most useful to the research undertaken in this paper. The package offers a large set of generators of pseudo-random numbers that follow probability distributions which are used both frequently and sporadically. Moreover, the package provides many GoFTs for normality, uniformity and laplacity (see Section 4).

In the recent years, many articles have been devoted to GoFTs for normality, e.g.: Afeez et al. (2018), Ahmad and Khan (2015), Aliaga et al. (2003), Arnastauskaitė et al. (2021), Bayoud (2021), Bonett and Seier (2002), Bontemps and Meddahi (2005), Brys et al. (2008), Coin (2008), Desgagné et al. (2023), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Gel and Gastwirth (2008), Hernandez (2021),

[a] Pomeranian University, Institute of Exact and Technical Sciences, Arciszewskiego Street 22, 76-200 Słupsk, Poland, e-mail: piotr.sulewski@apsl.edu.pl, ORCID: https://orcid.org/0000-0002-0788-6567.

Kellner and Celisse (2019), Khatun (2021), Marange and Qin (2019), Mbah and Paothong (2015), Mishra et al. (2019), Nosakhare and Bright (2017), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Wijekularathna et al. (2020), Yap and Sim (2011), and Yazici and Yolacan (2007).

In this article, we focus on GoFTs for normality recommended for use when the alternatives are asymmetric, i.e. the skewness (γ_1) is non-zero (see Table 1). The results of applying the Monte Carlo method to assess the power of GoFTs are presented in Section 4.

Asymmetric distributions can be divided into distributions with constant γ_1 and non-constant γ_1. Distributions with constant γ_1 include exponential (EXP), Gumbel (GU), half-logistic (HL), half-Normal (HN), log-Weibull (LW) or extreme-value, or Maxwell (MX) and Rayleigh (Ry) distributions. Section 3 is devoted to distributions with non-constant γ_1.

The research results presented in the article by Sulewski (2022a) inspired further research and form the core of this article. That paper discusses the Easily Changeable Kurtosis (ECK) distribution. The ECK enables testing the ability of GoFTs to detect the deviations from normality by negative excess kurtosis $\bar{\gamma}_2$. The article shows that the most popular GoFTs do not distinguish the ECK distribution of negative $\bar{\gamma}_2$ (even $\bar{\gamma}_2 = -0.3$) from normal distribution. This is the case even when sample size $n = 30, 50$ and significance level $\alpha = 0.05$. The findings presented in the author's other works entitled 'Goodness-of-fit testing for normality when alternative distributions have undefined or constants skewness and kurtosis' and 'On the detectability of symmetric distributions that deviate from normality due to small excess kurtosis' (currently reviewed) also motivated further research in the discussed area. The common feature of this article and those mentioned in this paragraph is the testing of GoFTs.

The review of the recent statistics-related literature shows that $\gamma_1 \in [-0.25, 0.25]$ does not dominate in testing for normality. It is very interesting to see how the GoFT responds to samples coming from alternatives close to normal distribution. In this article, we will focus on γ_1 values close to zero. In other words, we use the values of alternative parameters to obtain the desired γ_1 values and similarity measure values of the alternatives to normal distribution.

The aim of this article is to test the ability of GoFTs to detect deviations from normality. A very specific case is considered, namely the deviation from normality consisting in the coincidence of asymmetry and small γ_1 values. The first step toward achieving the aforementioned aim is to compile a set of normality-oriented GoFTs commonly recommended for use, mainly on the basis of the review of recently-published source literature. The second step is to create a family of asymmetric distributions with non-constant γ_1, further referred to as alternatives. Formulas for calculating the γ_1 and $\bar{\gamma}_2$ values are provided for each distribution. In order to compare the alternatives with normal distribution, an appropriate similarity measure is applied. The third step involves performing a Monte Carlo simulation. The study is based on the use of 21 GoFTs and 13 alternatives.

The article is organised as follows: Section 2 presents 21 GoFTs for normality recommended in the literature as fit for use when the alternatives are asymmetric. Section 3 is devoted to the similarity measure of the normal distribution to the alternative distribution. Moreover, this part of the study presents asymmetric distribution with non-constant γ_1. Section 4 analyses the results of the Monte Carlo simulations. The summary and conclusions, compiled in Section 5, close the paper.

2. Goodness-of-fit tests for the Monte Carlo simulation

Hypothesis H_0 states that the data come from normal distribution. Hypothesis H_1 negates H_0. Table 1 presents the studied 21 GoFTs for normality (sorted by year) recommended in the literature in the recent years ($n \leq 100$) when alternatives are asymmetric. These GoFTs are used in the Monte Carlo simulations (see Section 4).

Table 1. GoFTs for normality when alternatives are asymmetric ($n \leq 100$)

GoFT	Recommended by
Anderson-Darling test (AD) (Anderson & Darling, 1952)	Afeez et al. (2018), Khatun (2021), Yap and Sim (2011)
Shapiro-Wilk test (SW) (Shapiro & Wilk, 1965)	Afeez et al. (2018), Bayoud (2021), Coin (2008), Hernandez (2021), Khatun (2021), Mishra et al. (2019), Romao et al. (2010), Wijekularathna et al. (2020), Yap and Sim (2011)
Kurtosis test (KT) (Shapiro et al., 1968)	Mishra et al. (2019)
D'Agostino skewness test (AS) (D'Agostino, 1970)	Mishra et al. (2019)
Shapiro-Francia test (SF) (Shapiro & Francia, 1972)	Khatun (2021), Nosakhare and Bright (2017)
D'Agostino-Pearson test (AP) (D'Agostino & Pearson, 1973)	Mishra et al. (2019)
Ryan-Joiner test (RJ) (Ryan & Joiner, 1976)	Nosakhare and Bright (2017)
T_{1n} test (T_{1n}) (LaRiccia, 1986)	Torabi et al. (2016)

Table 1. GoFTs for normality when alternatives are asymmetric ($n \leq 100$) (cont.)

GoFT	Recommended by
Jarque-Bera test (JB) (Jarque & Bera, 1987)	Brys et al. (2008), Yazici and Yolacan (2007)
1st Hosking test (H_1) (Hosking, 1990)	Arnastauskaitė et al. (2021)
1st Cabana-Cabana test (CC) (Cabaña & Cabaña, 1994)	Uyanto (2022)
Chen-Shapiro test (CS) (Chen & Shapiro, 1995)	Romao et al. (2010)
Adjusted Jarque-Bera test (AJB) (Urzua, 1996)	Nosakhare and Bright (2017)
ZA Zhang-Wu test (ZA) (Zhang & Wu, 2005)	Romao et al. (2010), Sulewski (2019), Uhm and Yi (2021), Uyanto (2022)
ZC Zhang-Wu test (ZC) (Zhang & Wu, 2005)	Romao et al. (2010), Uhm and Yi (2021)
β_3^2 Coin test (β_3^2), (Coin, 2008)	Coin (2008)
H_n test (H_n) Torabi et al. (2016)	Torabi et al. (2016)
X_{APD} test (X_{APD}) (Desgagné & Lafaye de Micheaux, 2018)	Desgagné et al. (2023)
B_v test (B_v) Tavakoli et al. (2019)	Tavakoli et al. (2019)
Modified Lilliefors test ($LF_{\overline{\alpha},\overline{\beta}}$) (Sulewski, 2022b)	Sulewski (2022b)
Delta test (δ) Bayoud (2021)	Bayoud (2021)

Source: author's work.

3. The similarity measure and the alternatives

3.1. Similarity measure

Let $f(x;\boldsymbol{\theta})$ be a probability density function (PDF) of an alternative distribution with vector of parameters $\boldsymbol{\theta}$. Similarity measure M of the alternative to the null distribution is defined as (Sulewski, 2022b)

$$M(\boldsymbol{\theta};\mu,\sigma) = \int_{-\infty}^{\infty} min[f(x;\boldsymbol{\theta}), \phi(x;\mu,\sigma)]dx, \quad (1)$$

where $\phi(x;\mu,\sigma)$ is the PDF of the normal distribution. $M(\boldsymbol{\theta};\mu,\sigma)$ takes the values of $[0,1]$ and equals 1 when the PDFs are identical. More details on distance and similarity measures can be found e.g. in Sulewski (2021).

3.2. Alternative distributions

Asymmetric alternatives with non-constant γ_1 used in Monte Carlo simulations can be divided into two groups. The first and second group includes monolithic and compound distributions, respectively, used in GoFTs for normality in recent articles. These alternatives are:

- Group I: beta (B), chi-squared (χ^2), gamma (G), generalised power (GP), inverse Gaussian (IG), lognormal (LOG), power normal (PN), SB Johnson (SB), Skew-flexible-normal (SFN), skew-normal (SN), SU Johnson (SU) and Weibull (W) distributions;
- Group II: location-contaminated normal (LCN), Gumbel-normal (GN), Laplace mixture (LM), Laplace-normal (LN), normal distribution with a plasticising component (NDPC), normal mixture (NM), plasticising component mixture (PCM), skew-normal mixture (SNM) and Weibull-normal (WN) distributions.

See Table 2 for more details. The distributions used in at least two articles (marked in bold) have been selected for the Monte Carlo simulation (see Section 4).

Table 2. Asymmetric alternatives (A) with non-constant γ_1 used in GoFTs for normality in the recent literature (in alphabetical order)

A	Article
B	Afeez et al. (2018), Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
χ^2	Arnastauskaitė et al. (2021), Bayoud (2021), Bontemps and Meddahi (2005), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Nosakhare and Bright (2017), Razali and Wah (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Wijekularathna et al. (2020)
G	Arnastauskaitė et al. (2021), Bayoud (2021), Desgagné and Lafaye de Micheaux (2018), Noughabi and Arghami (2011), Razali and Wah (2011), Romao et al. (2010), Tavakoli et al. (2019), Torabi et al. (2016), Uhm and Yi (2021), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
GN	Sulewski (2022b)
GP	Desgagné et al. (2023), Desgagné and Lafaye de Micheaux (2018)
IG	Tavakoli et al. (2019)
LCN	Coin (2008), Yap and Sim (2011)
LM	Sulewski (2022b)
LN	Sulewski (2022b)
LOG	Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Gel at el. (2007), Marange and Qin (2019), Noughabi and Arghami (2011), Romao et al. (2010), Sulewski (2019), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Wijekularathna et al. (2020), Yap and Sim (2011), Yazici and Yolacan (2007)
NDPC	Sulewski (2022b)

Table 2. Asymmetric alternatives (A) with non-constant γ_1 used in GoFTs for normality in the recent literature (in alphabetical order) (cont.)

A	Article
NM	Romao et al. (2010), Sulewski (2022b)
PCM	Sulewski (2022b)
PN	Sulewski (2022b)
SB	Sulewski (2019), Sulewski (2022b), Torabi et al. (2016)
SFN	Sulewski (2022b)
SN	Bayoud (2021), Sulewski (2022b), Torabi et al. (2016), Uyanto (2022)
SU	Sulewski (2019), Torabi et al. (2016)
SNM	Sulewski (2022b)
W	Afeez et al. (2018), Ahmad and Khan (2015), Arnastauskaitė et al. (2021), Bayoud (2021), Coin (2008), Desgagné and Lafaye de Micheaux (2018), Nosakhare and Bright (2017), Noughabi and Arghami (2011), Romao et al. (2010), Sulewski (2022b), Tavakoli et al. (2019), Torabi et al. (2016), Uyanto (2022), Yap and Sim (2011), Yazici and Yolacan (2007)
WN	Sulewski (2022b)

Source: author's work.

The family of alternatives also includes two very interesting distributions ideally suited to the subject of this work, namely the Edgeworth series (ES) and the Pearson (P) distributions. Their parameters are γ_1 and $\bar{\gamma}_2$.

Let $\phi(x;0,1)$ and $\Phi(x;0,1)$ be the PDF and the cumulative density function (CDF) of the $N(0,1)$ distribution, respectively. Below, for the analysed alternatives, the PDF, the $M(\boldsymbol{\theta};0,\sigma)$ maximum value, and the $\gamma_1(\boldsymbol{\theta}),\bar{\gamma}_2(\boldsymbol{\theta}),\boldsymbol{\theta}(\gamma_1),\boldsymbol{\theta}(\bar{\gamma}_2)$ formulas are shown. The alternatives are presented in alphabetical order.

1. Beta distribution

$$f_B(x;a,b)=\frac{x^{a-1}(1-x)^{b-1}}{B(a,b)},\ x\in[0,1]\ (a>0,b>0)$$

$$M(11.372,11.372;0.5,0.105)=0.990$$

$$\gamma_1(a,b)=\frac{2(b-a)\sqrt{a+b+1}}{(a+b+2)\sqrt{ab}}(\gamma_1\in R),\ \gamma_1(a,b)=-\gamma_1(b,a)$$

$$\bar{\gamma}_2(a,b)=\frac{6[(a-b)^2(a+b+1)-ab(a+b+2)]}{ab(a+b+2)(a+b+3)}\ (\bar{\gamma}_2\geq-2)$$

2. Chi-squared distribution

$$f_{\chi^2}(x;k) = \frac{x^{0.5k-1}exp(-0.5x)}{2^{0.5k}\Gamma(0.5k)},\ x \geq 0\ (k > 0)$$

$$M(92.498; 91.47,13.506) = 0.973$$

$$\gamma_1(\mathrm{k}) = \sqrt{\frac{8}{k}}\ (\gamma_1 > 0)\ ,\ \ k(\gamma_1) = \frac{8}{\gamma_1^2},\ \ \bar{\gamma}_2(k) = \frac{12}{k}\ (\bar{\gamma}_2 > 0),\ \ k(\bar{\gamma}_2) = \frac{12}{\bar{\gamma}_2}$$

3. Gamma distribution

$$f_{\boldsymbol{G}}(x;a,b) = \frac{x^{c-1}exp(-x/a)}{a^c\Gamma(c)},\ \ x \geq 0\ (a > 0, b > 0)$$

$$M(0.06,80.166; 4.815,0.543) = 0.979$$

$$\gamma_1(\mathrm{b}) = \frac{2}{\sqrt{b}}\ (\gamma_1 > 0), b(\gamma_1) = \frac{4}{\gamma_1^2},\ \ \bar{\gamma}_2(b) = \frac{6}{b}\ (\bar{\gamma}_2 > 0),\ \ b(\bar{\gamma}_2) = \frac{6}{\bar{\gamma}_2}$$

4. Generalised power distribution (Komunjer, 2007)

Let $g(x;a,b) = 2a^b(1-a)^b[a^b + (1-a)^b]^{-1}$, $(0 < a < 1, b > 0)$, then

$$f_{\boldsymbol{GP}}(x;a,b) = \frac{g(x;a,b)^{\frac{1}{b}}}{\Gamma\left(1+\frac{1}{b}\right)} exp\left\{-\frac{g(x;a,b)}{\left[\frac{1}{2} + sgn(x)\left(\frac{1}{2} - a\right)\right]^b}|x|^b\right\},\ \ x \in R$$

$$M(0.5,2; 0,0.707) = 1.$$

Let $\alpha_k = \int_{-\infty}^{\infty} x^k f_{\boldsymbol{GP}}(x;a,b)$, then

$$\gamma_1(\mathrm{a,b}) = \frac{\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3}{(\alpha_2 - \alpha_1^2)^{1.5}}\ (-10 < \gamma_1 < 10),\ \ \gamma_1(\mathrm{a,b}) = -\gamma_1(1-\mathrm{a,b})$$

$$\bar{\gamma}_2(a,b) = \frac{\alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4}{(\alpha_2 - \alpha_1^2)^2} - 3\ (\bar{\gamma}_2 > -1.2).$$

5. Location contaminated normal distribution

$$f_{LCN}(x;a,w) = w\phi(x;a,1) + (1-w)\phi(x;0{,}1),\ x \in R\ (0 \leq w \leq 1,\ a > 0)$$

$$M(0{,}1;\ 0{,}1) = M(a,0;\ 0{,}1) = 1$$

$$\gamma_1(a,w) = \frac{a^3 w(2w^2 - 3w + 1)}{(a^2 w - a^2 w^2 + 1)^{1.5}}\ (\gamma_1 \in R)$$

$$\bar{\gamma}_2(a,w) = \frac{a^4 w(-6w^3 + 12w^2 - 7w + 1)}{(a^2 w - a^2 w^2 + 1)^2}\ (\bar{\gamma}_2 \geq -2)$$

$$a(\bar{\gamma}_2,w) = \sqrt{\frac{\left(\frac{\bar{\gamma}_2}{6w^2 - 6w + 1} + \sqrt{\frac{\bar{\gamma}_2}{12w^3 - 6w^4 - 7w^2 + w}}\right)(6w^2 - 6w + 1)}{\bar{\gamma}_2 w^2 - 6w + 6w^2 - \bar{\gamma}_2 w + 1}}$$

$$w(\bar{\gamma}_2,a) = \frac{a + \sqrt{\frac{4\bar{\gamma}_2 + \bar{\gamma}_2 a^2 + 4a^2 + 2\sqrt{a^4 - 4a^2\bar{\gamma}_2 - 24\bar{\gamma}_2}}{\bar{\gamma}_2 + 6}}}{2a}$$

6. Lognormal distribution

$$f_{LOG}(x;a,b) = \frac{1}{xb\sqrt{2\pi}} exp\left[-\frac{1}{2}\left(\frac{ln(x) - a}{b}\right)^2\right],\ x > 0\ (a \in R,\ b > 0)$$

$$M(0.103{,}0.096;\ 1.106{,}0.106) = 0.974$$

$$\gamma_1(b) = [exp(b^2) + 2]\sqrt{exp(b^2) - 1}\ (\gamma_1 \geq 0)$$

$$\bar{\gamma}_2(b) = exp(4b^2) + 2exp(3b^2) + 3exp(2b^2) - 6\ (\bar{\gamma}_2 \geq 0)$$

7. Normal mixture distribution

$$f_{NM}(x;\ a,b,w) = w\phi(x;\ 0{,}1) + (1-w)\phi(x;\ a,b),$$
$$x \in R\ (a \in R,\ 0 \leq w \leq 1,\ b > 0)$$

$$M(a,b,1;\ 0{,}1) = M(0{,}1{,}0;\ 0{,}1) = M(0{,}1,\omega;\ 0{,}1) = 1$$

$$\gamma_1(a,b,w) = \frac{-w(2w^2 - 3w + 1)a^3 + aw(3w - 3b^2 w + 3b^2 - 3)}{[w(a^2 - b^2 - a^2 w + 1) + b^2]^{1.5}}\ (\gamma_1 \in R)$$

$$\gamma_1(a,b,w) = -\gamma_1(-a,b,w)$$

$$\bar{\gamma}_2(a,b,w) \frac{(6w^2 - 6w + 1)a^4 + a^2(12b^2 w - 12w - 6b^2 + 6) + 3b^4 - 6b^2 + 3}{(w - w^2)^{-1}[w(a^2 - b^2 - a^2 w + 1) + b^2]^2} (\bar{\gamma}_2 \geq -2)$$

8. SB distribution (Johnson, 1949)

$$f_{SB}(x;a,b) = \frac{b}{x(1-x)}\phi\left[a + bln\left(\frac{x}{1-x}\right);0,1\right],\ x \in [0,1]\ (a \in R,\ b > 0)$$

$$M(0,2.669;\ 0.5,0.093) = 0.999$$

Let $\alpha_k = \int_0^1 x^k f_{\boldsymbol{SB}}(x;a,b)$, then

$$\gamma_1(\mathrm{a,b}) = \frac{\alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3}{(\alpha_2 - \alpha_1^2)^{1.5}}(\gamma_1 \in R),\ \ \gamma_1(\mathrm{a,b}) = -\gamma_1(-\mathrm{a,b})$$

$$\bar{\gamma}_2(a,b) = \frac{\alpha_4 - 4\alpha_1\alpha_3 + 6\alpha_1^2\alpha_2 - 3\alpha_1^4}{(\alpha_2 - \alpha_1^2)^2} - 3\ (\bar{\gamma}_2 \geq -2)$$

9. Skew-normal distribution (Azzalini, 1985)

$$f_{SN}(x;a) = 2\phi(x;0,1)\ \Phi(ax;0,1),\ \ x \in R\ (a \in R)$$

$$M(0;\ 0,1) = 1$$

$$\gamma_1(\mathrm{a}) = \frac{a^3\sqrt{2}(4-\pi)}{(\pi - 2a^2 + \pi a^2)^{1.5}}\ (-1 < \gamma_1 < 1),\ \gamma_1(\mathrm{a}) = -\gamma_1(-\mathrm{a})$$

$$\bar{\gamma}_2(a) = \frac{4a^4(2\pi - 6)}{(\pi - 2a^2 + \pi a^2)^2}\ (0 \leq \bar{\gamma}_2 \leq 0.869)$$

$$a(\bar{\gamma}_2) = \pm\sqrt{\frac{\pi\left(2\bar{\gamma}_2\sqrt{\pi-3} + 6\sqrt{2\bar{\gamma}_2} - \pi\bar{\gamma}_2\sqrt{\pi-3} - 2\pi\sqrt{2\bar{\gamma}_2}\right)}{\sqrt{\pi-3}(4\bar{\gamma}_2 - 8\pi - 4\pi\bar{\gamma}_2 + \pi^2\bar{\gamma}_2 + 24)}}$$

10. SU distribution (Johnson, 1949)

$$f_{SU}(x;b,c,d) = \frac{d}{\sqrt{x^2+b^2}}\phi\left[c + dsinh^{-1}\left(\frac{x}{b}\right);0,1\right],\ \ x \in R\ (b > 0, c \in R, d > 0)$$

$$M(1.375,0,11.129;\ 0,0.124) = 0.998$$

Let

$$W = exp(d^{-2}),\ \ K_1 = W^2(W^4 + 2W^3 + 3W^2 - 3)cosh\left(\frac{4c}{d}\right),$$

$$K_2 = 4W^2(W+2)cosh\left(\frac{2c}{d}\right),\ V = \frac{b^2}{2}(W-1)\left[Wcosh\left(\frac{2c}{d}\right) + 1\right],$$

then

$$\gamma_1(\mathrm{c,d}) = \frac{-b^3\sqrt{W}(W-1)^2\left[W(W+2)sinh\left(\frac{3c}{d}\right)+3sinh\left(\frac{c}{d}\right)\right]}{4V^{1.5}} \ (\gamma_1 \in R)$$

$$\gamma_1(\mathrm{c,d}) = -\gamma_1(-\mathrm{c,d})$$

$$\bar{\gamma}_2(c,d) = \frac{b^4(W-1)^2[K_1+K_2+6W+3]}{8V^2} - 3 \ (\bar{\gamma}_2 \geq 2).$$

11. Weibull distribution (Weibull, 1951)

$$f_W(x;a,b) = \frac{b}{a^b}x^{b-1}\exp\left[-\left(\frac{x}{a}\right)^b\right],\ x \geq 0\ (a>0,\ b>0)$$

$$M(1.851{,}3.603;\ 1.673{,}0.532) = 0.985$$

Let $\Gamma_k = \Gamma(1+k/b)$, then

$$\gamma_1(b) = \frac{2\Gamma_1^3 - 3\Gamma_1\Gamma_2 + \Gamma_3}{(\Gamma_2-\Gamma_1^2)^{1.5}} \ (\gamma_1 \geq -1.14)$$

$$\bar{\gamma}_2(b) = \frac{\Gamma_4 - 3\Gamma_2^2 - 4\Gamma_1\Gamma_3 + 12\Gamma_1^2\Gamma_2 - 6\Gamma_1^4}{\left(\Gamma_2-\Gamma_1^2\right)^2} \ (\bar{\gamma}_2 \geq -0.289).$$

12. Edgeworth series distribution (Aliaga et al., 2003)

$$f_{ES}(x;\ \gamma_{1,},\bar{\gamma}_2) = \frac{\phi(x;0,1)}{\left[1+\frac{1}{3!}\gamma_{1,}(x^3-3x)+\frac{1}{4!}\bar{\gamma}_2(x^4-6x^2+3)\right]^{-1}},$$

$$x \in R\ (\gamma_1 \in R, \bar{\gamma}_2 \geq -2)$$

$$M(0{,}0;\ 0{,}1) = 1$$

The PDF formula is introduced in the Appendix.

13. Pearson distribution (Pearson, 1916)

Let

$$a = \frac{2\bar{\gamma}_2 - 3\gamma_1^2}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12},\ \ b = \frac{|\gamma_1|(\bar{\gamma}_2+6)}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, c = \frac{4\bar{\gamma}_2 - 3\gamma_1^2 + 12}{10\bar{\gamma}_2 - 5\gamma_1^2 + 12}, \quad (2)$$

$$\Delta = b^2 - 4ac$$

then

$$f_P(x;\gamma_1,\bar{\gamma}_2)=\begin{cases}\dfrac{exp\left[\dfrac{2ab-b}{a(2ax+b)}\right]}{C_2(2ax+b)^{1/a}} & \Delta=0\\ \dfrac{exp\left[\dfrac{b-2ab}{a\sqrt{4ac-b^2}}tan^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{C_4(ax^2+bx+c)^{1/(2a)}} & \Delta<0\\ \dfrac{\left(\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right)^{\frac{b-2ab}{2a\sqrt{b^2-4ac}}}}{C_8(ax^2+bx+c)^{1/(2a)}} & \Delta>0,\end{cases}$$

where C_2, C_4, C_8 are normalising constants given by

$$C_2=\int_{-\infty}^{\infty}\frac{exp\left[\dfrac{2ab-b}{a(2ax+b)}\right]}{(2ax+b)^{1/a}}dx, \tag{2}$$

$$C_4=\int_{-\infty}^{\infty}\frac{exp\left[\dfrac{b-2ab}{a\sqrt{4ac-b^2}}tan^{-1}\left(\dfrac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{(ax^2+bx+c)^{1/(2a)}}dx, \tag{3}$$

$$C_8=\int_{-\infty}^{\infty}\frac{\left(\dfrac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)^{\frac{b-2ab}{2a\sqrt{\Delta}}}}{C_8(ax^2+bx+c)^{1/(2a)}}dx, \tag{4}$$

$$M(0,0;\ 0,1)=1.$$

The PDF formula is introduced in the Appendix.

The 13 above-mentioned distributions are grouped in Table 3 according to different properties. Table 3 shows that most of the analysed distributions have an infinite domain and assume negative or positive skewness values. The normal distribution is a special case of six distributions.

Table 3. Asymmetric distributions with non-constant $\bar{\gamma}_2$ grouped by different properties. The numbers of distributions with the given properties are provided in brackets

Property	Distributions
Finite domain	B, SB (2)
Infinite domain	$\chi^2, G, GP, LCN, LOG, NM, SN, SU, W, ES, P$ (11)
$M(\boldsymbol{\theta}; \mu, \sigma) = 1$ for some $\boldsymbol{\theta}, \mu, \sigma$	GP, LCN, NM, SN, ES, P (6)
$\gamma_1 < 0$	(0)
$\gamma_1 > 0$	LOG, G, χ^2 (3)
$\gamma_1 < 0 \vee \gamma_1 > 0$	$B, GP, LCN, NM, SB, SN, SU, W, ES, P$ (10)
Unimodal	$B, \chi^2, G, GP, LOG, SB, SN, SU, W, ES, P$ (11)
Bimodal	LCN, NM (2)

Source: author's work.

4. Monte Carlo simulation

For alternatives numbered from 1 to 13 (see Section 3.2), 21 large-scale experiments are performed, each dedicated to one of the GoFTs (see Section 2). Each experiment involves generating 10^4 samples of size $n = 25$. The samples come from a given alternative. Each sample is tested for normality at significance level $\alpha = 0.05$. The values of the alternative parameters are determined to obtain appropriate γ_1 values. The power of tests (PoTs) are calculated for the given γ_1 values.

All calculations are performed in R software using the codes presented in Table 4. A research tool facilitating the Monte Carlo power simulation studies for GoFTs in R called the PowerR package (Lafaye de Micheaux & Tran, 2016) proved very helpful in the process. The 'statcompute()' function calculates the test statistic value and the p-value for the GoFT described by the 'stat.index' argument, the sample described by the argument as 'data', and the significance level described by the argument as 'level'. Thus, e.g. in the case of the ZA test, the calculations take the following form: statcompute(stat.index = 4, data = sample, level = 0.05). See Table 4 for more information.

Table 5 presents the generator formulas for all the alternatives described in Section 3.

Table 4. The R codes of the used GoFTs

GoFT	R codes	GoFT	R codes
AD	ad.test	CS	statcompute(stat.index = 26…)
SW	shapiro.test	AJB	ajb.norm.test
KT	kurtosis.norm.test	ZA	statcompute(stat.index = 4…)
AS	agostino.test	ZC	statcompute(stat.index = 3…)
SF	sf.test	β_3^2	statcompute(stat.index = 30…)
AP	dagoTest	H_n	author's function, see Appendix
RJ	author's function, see Appendix	X_{APD}	statcompute(stat.index = 36…)
T_{1n}	author's function, see Appendix	B_v	author's function, see Appendix
JB	jarque.test	$LF_{\bar{\alpha},\bar{\beta}}$	author's function, see Appendix
H1	statcompute(stat.index = 10…)	δ	author's function, see Appendix
CC	statcompute(stat.index = 19…)		

Source: author's work.

Table 5. Generator formulas for the analysed alternatives (A) in R

A	Generator	A	Generator
B	rbeta(n,a,b)	SB	rJohnsonSB(n,a,b,0,1)
χ^2	rchisq(n,k)	SN	rskewnorm(n,0,1,a)
G	rgamma(n,b,1/a)	SU	rJohnsonSU(n,c,d,0,b))
GP	rGP(n,a,b) see Appendix	W	rweibull(n,b,1.851)
LCN	rLCN(n,a,b) see Appendix	ES[a]	$rEdge(n, \gamma_1, \bar{\gamma}_2, x_l, x_u)$, see Appendix
LOG	rlnorm(n,0.103,b)	P	mom = c(0,1, γ_1, $\bar{\gamma}_2$) rpearson(n,moments=mom))
NM	rNM(n,a,b,ω) see Appendix		

a The quality of built-in function rCornishFisher(n,1, γ_1, $\bar{\gamma}_2$) was not satisfactory.
Source: author's work.

The simulation results for the alternatives are presented in alphabetical order in Tables 6–22. We assume that a GoFT detects negative or positive γ_1 if its power reaches at least 0.06. PoT values are marked in bold, while the highest average PoT values for positive and negative γ_1 are underlined.

Table 6. $B(a,b)$ distribution. PoT versus γ_1 and $M(a,b;0.5,0.105)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	–0.312	–0.306	–0.270	–0.338	–0.242	–0.233	–0.242	–0.338	–0.270	–0.306	-0.312
a	7.438	8.135	9.824	7.733	11.372	11.372	10.050	6.294	7.040	5.426	4.596
b	4.595	5.426	7.040	6.294	10.050	11.372	11.372	7.733	9.824	8.135	7.438
M	0.600	0.650	0.700	0.800	0.881	0.990	0.881	0.800	0.700	0.650	0.600
GoFT	PoT										
AD	0.058	0.052	0.050	0.044	0.045	0.042	0.046	0.045	0.046	0.052	**0.060**
SW	0.058	0.050	0.046	0.041	0.039	0.039	0.041	0.039	0.045	0.050	0.060
KT	0.038	0.032	0.036	0.029	0.032	0.033	0.032	0.031	0.033	0.033	0.038
AS	0.041	0.035	0.035	0.026	0.029	0.029	0.028	0.025	0.035	0.035	0.045
SF	0.046	0.038	0.038	0.031	0.034	0.034	0.035	0.030	0.037	0.042	0.049
AP	0.041	0.034	0.036	0.027	0.030	0.031	0.030	0.029	0.035	0.036	0.044
RJ	0.043	0.035	0.035	0.028	0.032	0.031	0.032	0.028	0.033	0.039	0.045
T_1	0.057	0.051	0.048	0.035	0.037	0.035	0.038	0.036	0.043	0.048	0.055
JB	0.034	0.027	0.030	0.022	0.027	0.026	0.025	0.022	0.029	0.029	0.038
H1	0.053	0.047	0.043	0.036	0.040	0.041	0.040	0.039	0.045	0.048	0.052
CC	0.040	0.035	0.035	0.026	0.030	0.030	0.030	0.027	0.035	0.039	0.048
CS	**0.060**	0.052	0.048	0.043	0.040	0.041	0.042	0.042	0.046	0.051	**0.061**
AJB	0.032	0.024	0.028	0.020	0.025	0.026	0.024	0.020	0.028	0.026	0.035
ZA	0.054	0.044	0.044	0.036	0.036	0.036	0.034	0.035	0.041	0.046	0.056
ZC	0.053	0.045	0.043	0.038	0.036	0.037	0.038	0.035	0.041	0.048	0.058
β_3^2	0.039	0.039	0.042	0.038	0.042	0.041	0.041	0.041	0.043	0.038	0.042
H_n	0.050	0.047	0.051	0.047	0.050	0.054	<u>**0.062**</u>	<u>**0.068**</u>	<u>**0.071**</u>	<u>**0.081**</u>	<u>**0.093**</u>
X_{APD}	0.051	0.043	0.043	0.035	0.037	0.036	0.038	0.036	0.040	0.044	0.049
B_v	**0.071**	**0.061**	**0.062**	0.054	0.051	0.049	0.052	0.055	0.060	**0.061**	**0.070**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	<u>**0.083**</u>	<u>**0.070**</u>	<u>**0.065**</u>	0.054	0.053	0.045	0.053	0.058	**0.063**	**0.074**	**0.081**
δ	0.042	0.041	0.041	0.038	0.041	0.042	0.050	0.052	0.055	**0.066**	**0.072**

Source: author's work.

Table 7. $\chi^2(k)$ distribution. PoT versus γ_1 and $M(k;\mu,\sigma)$ for $n=25$

γ_1	0.294	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.705	0.800
$\bar{\gamma}_2$	0.130	0.184	0.240	0.304	0.375	0.454	0.540	0.634	0.735	0.844	0.960
$0k$	92.550	65.310	50.000	39.510	32.000	26.450	22.220	18.930	16.330	14.220	12.500
μ	91.410	66.930	51.864	42.556	36.448	31.503	23.608	20.842	23.742	15.101	21.708
σ	13.503	12.750	12.639	11.790	10.592	10.663	13.538	13.895	10.950	15.337	12.699
M	0.973	0.9010	0.850	0.801	0.750	0.700	0.650	0.602	0.550	0.511	0.450
GoFT	PoT										
AD	**0.068**	**0.076**	**0.087**	**0.095**	**0.098**	**0.122**	**0.144**	**0.150**	**0.164**	**0.188**	**0.212**
SW	**0.072**	**0.087**	**0.099**	**0.109**	**0.121**	**0.143**	**0.167**	**0.181**	**0.203**	**0.221**	**0.256**
KT	**0.065**	**0.074**	**0.075**	**0.083**	**0.088**	**0.098**	**0.116**	**0.117**	**0.130**	**0.133**	**0.153**
AS	**0.080**	**0.095**	**0.104**	**0.117**	**0.133**	**0.152**	**0.179**	**0.191**	**0.209**	**0.233**	**0.259**
SF	**0.078**	**0.090**	**0.098**	**0.110**	**0.122**	**0.145**	**0.170**	**0.179**	**0.199**	**0.218**	**0.248**
AP	**0.074**	**0.088**	**0.092**	**0.106**	**0.113**	**0.129**	**0.152**	**0.158**	**0.174**	**0.192**	**0.213**
RJ	**0.073**	**0.085**	**0.094**	**0.106**	**0.114**	**0.140**	**0.163**	**0.172**	**0.191**	**0.209**	**0.237**
T_1	**0.081**	**0.091**	**0.106**	**0.120**	**0.138**	**0.157**	**0.187**	**0.201**	**0.223**	**0.253**	**0.291**
JB	**0.076**	**0.090**	**0.094**	**0.108**	**0.114**	**0.134**	**0.157**	**0.164**	**0.180**	**0.202**	**0.225**
H1	**0.070**	**0.079**	**0.090**	**0.100**	**0.106**	**0.124**	**0.150**	**0.154**	**0.180**	**0.192**	**0.216**
CC	**0.080**	**0.095**	**0.102**	**0.120**	**0.134**	**0.155**	**0.182**	**0.195**	**0.216**	**0.239**	**0.268**
CS	**0.072**	**0.085**	**0.097**	**0.107**	**0.120**	**0.141**	**0.164**	**0.179**	**0.202**	**0.219**	**0.255**
AJB	**0.074**	**0.086**	**0.090**	**0.103**	**0.108**	**0.126**	**0.147**	**0.152**	**0.166**	**0.186**	**0.205**
ZA	**0.074**	**0.088**	**0.101**	**0.110**	**0.124**	**0.150**	**0.174**	**0.188**	**0.211**	**0.233**	**0.265**
ZC	**0.076**	**0.088**	**0.097**	**0.108**	**0.119**	**0.146**	**0.167**	**0.182**	**0.205**	**0.222**	**0.254**
β_3^2	0.052	0.055	0.058	0.058	0.058	0.058	**0.064**	**0.062**	**0.064**	**0.066**	**0.073**
H_n	**0.096**	**0.106**	**0.120**	**0.127**	**0.139**	**0.166**	**0.189**	**0.204**	**0.217**	**0.251**	**0.274**
X_{APD}	**0.071**	**0.077**	**0.087**	**0.099**	**0.108**	**0.124**	**0.146**	**0.154**	**0.172**	**0.191**	**0.215**
B_v	**0.064**	**0.074**	**0.083**	**0.083**	**0.098**	**0.110**	**0.128**	**0.140**	**0.158**	**0.176**	**0.200**
$LF_{\bar{\alpha},\bar{\beta}}$	**0.084**	**0.096**	**0.104**	**0.122**	**0.126**	**0.148**	**0.165**	**0.172**	**0.180**	**0.206**	**0.222**
δ	**0.085**	**0.097**	**0.110**	**0.119**	**0.131**	**0.154**	**0.179**	**0.191**	**0.206**	**0.230**	**0.262**

Source: author's work.

Table 8. $\mathrm{G}(a, b)$ distribution. PoT versus γ_1 and $M(a, b; 4.815{,}0.543)$ for $n = 25$

γ_1	0.223	0.300	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750
$\bar{\gamma}_2$	0.075	0.135	0.184	0.240	0.304	0.375	0.454	0.540	0.634	0.735	0.844
a	0.060	0.107	0.140	0.182	0.228	0.277	0.325	0.410	0.452	0.577	0.590
b	80.436	44.444	32.653	25.000	19.753	16.000	13.223	11.111	9.467	8.163	7.111
M	0.979	0.850	0.750	0.700	0.650	0.600	**0.550**	**0.550**	**0.500**	**0.500**	0.450
GoFT	PoT										
AD	**0.063**	**0.069**	**0.077**	**0.083**	**0.101**	**0.107**	**0.120**	**0.135**	**0.153**	**0.171**	**0.187**
SW	**0.065**	**0.076**	**0.086**	**0.095**	**0.115**	**0.129**	**0.141**	**0.166**	**0.181**	**0.204**	**0.226**
KT	**0.061**	**0.063**	**0.068**	**0.079**	**0.085**	**0.097**	**0.097**	**0.111**	**0.115**	**0.129**	**0.136**
AS	**0.066**	**0.077**	**0.093**	**0.105**	**0.121**	**0.142**	**0.152**	**0.172**	**0.192**	**0.214**	**0.234**
SF	**0.069**	**0.077**	**0.087**	**0.097**	**0.117**	**0.129**	**0.144**	**0.161**	**0.180**	**0.201**	**0.221**
AP	**0.065**	**0.071**	**0.081**	**0.093**	**0.105**	**0.123**	**0.133**	**0.149**	**0.157**	**0.177**	**0.192**
RJ	**0.066**	**0.073**	**0.082**	**0.092**	**0.110**	**0.123**	**0.137**	**0.155**	**0.174**	**0.193**	**0.212**
T_1	**0.066**	**0.080**	**0.095**	**0.108**	**0.119**	**0.143**	**0.157**	**0.184**	**0.204**	**0.229**	**0.256**
JB	**0.066**	**0.073**	**0.083**	**0.094**	**0.109**	**0.127**	**0.136**	**0.153**	**0.164**	**0.184**	**0.200**
H1	**0.065**	**0.070**	**0.077**	**0.088**	**0.103**	**0.114**	**0.128**	**0.144**	**0.159**	**0.178**	**0.192**
CC	**0.067**	**0.077**	**0.094**	**0.106**	**0.122**	**0.143**	**0.156**	**0.177**	**0.199**	**0.219**	**0.241**
CS	**0.064**	**0.073**	**0.084**	**0.093**	**0.113**	**0.127**	**0.140**	**0.163**	**0.179**	**0.201**	**0.223**
AJB	**0.065**	**0.070**	**0.080**	**0.089**	**0.102**	**0.120**	**0.128**	**0.143**	**0.152**	**0.172**	**0.184**
ZA	**0.065**	**0.077**	**0.088**	**0.097**	**0.115**	**0.132**	**0.147**	**0.167**	**0.189**	**0.214**	**0.237**
ZC	**0.067**	**0.075**	**0.084**	**0.095**	**0.113**	**0.131**	**0.144**	**0.163**	**0.182**	**0.205**	**0.226**
β_3^2	0.055	0.051	0.053	0.057	0.059	**0.062**	**0.061**	**0.064**	**0.060**	**0.066**	**0.065**
H_n	**0.084**	**0.098**	**0.106**	**0.120**	**0.136**	**0.148**	**0.160**	**0.182**	**0.205**	**0.226**	**0.245**
X_{APD}	**0.064**	**0.068**	**0.075**	**0.088**	**0.101**	**0.113**	**0.123**	**0.142**	**0.154**	**0.175**	**0.187**
B_v	**0.062**	**0.068**	**0.069**	**0.079**	**0.095**	**0.102**	**0.116**	**0.131**	**0.145**	**0.159**	**0.178**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.076**	**0.086**	**0.101**	**0.108**	**0.121**	**0.136**	**0.136**	**0.153**	**0.178**	**0.187**	**0.202**
δ	**0.077**	**0.088**	**0.095**	**0.109**	**0.125**	**0.138**	**0.152**	**0.174**	**0.194**	**0.210**	**0.234**

Source: author's work.

Table 9. $\mathrm{GP}(a,2)$ distribution. PoT versus γ_1 and $M(a,2;0{,}0.707)$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.045	0.029	0.016	0.007	0.002	0	0.002	0.007	0.016	0.029	0.045
a	0.579	0.563	0.547	0.531	0.516	0.5	0.484	0.469	0.453	0.437	0.421
M	0.916	0.934	0.951	0.968	0.984	1	0.984	0.968	0.951	0.934	0.916
GoFT	PoT										
AD	**0.067**	**0.065**	0.057	0.057	0.049	0.051	0.051	0.050	0.058	0.060	**0.069**
SW	**0.068**	**0.067**	0.058	0.057	0.048	0.053	0.053	0.052	0.060	**0.062**	**0.067**
KT	0.057	0.056	0.054	0.054	0.048	0.051	0.052	0.051	0.053	0.058	0.054
AS	**0.068**	**0.062**	0.058	0.055	0.046	0.052	0.055	0.053	0.059	**0.063**	**0.065**
SF	**0.068**	**0.067**	**0.062**	0.057	0.049	0.056	0.055	0.051	0.060	**0.063**	**0.067**
AP	**0.064**	0.059	0.056	0.053	0.048	0.052	0.053	0.051	0.055	**0.064**	0.059
RJ	**0.065**	**0.061**	0.058	0.052	0.046	0.053	0.052	0.049	0.056	0.060	**0.064**
T_1	**0.070**	**0.065**	**0.060**	0.056	0.048	0.049	0.053	0.052	**0.060**	**0.064**	**0.068**
JB	**0.064**	0.058	0.055	0.053	0.047	0.051	0.054	0.050	0.055	**0.062**	**0.060**
H1	**0.065**	**0.062**	0.058	0.058	0.050	0.056	0.053	0.051	0.059	0.058	**0.065**
CC	**0.067**	**0.062**	0.057	0.055	0.047	0.050	0.055	0.050	0.059	**0.064**	**0.067**
CS	**0.066**	**0.066**	0.059	0.056	0.048	0.053	0.051	0.052	0.059	**0.062**	**0.067**
AJB	0.060	0.059	0.053	0.054	0.045	0.051	0.054	0.050	0.054	**0.061**	0.058
ZA	**0.067**	**0.066**	0.058	0.055	0.047	0.053	0.053	0.053	0.057	**0.062**	**0.065**
ZC	**0.067**	**0.063**	**0.060**	0.055	0.048	0.053	0.051	0.053	0.058	**0.063**	**0.064**
β_3^2	0.049	0.050	0.048	0.050	0.049	0.053	0.053	0.049	0.051	0.052	0.055
H_n	0.052	0.052	0.054	0.056	0.054	0.057	0.059	**0.066**	**0.079**	**0.084**	**0.096**
X_{APD}	**0.064**	0.060	0.057	0.056	0.047	0.054	0.050	0.051	0.056	**0.061**	**0.063**
B_v	0.059	**0.063**	0.059	0.053	0.050	0.056	0.050	0.053	0.055	**0.061**	**0.068**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.085**	**0.084**	**0.069**	**0.066**	0.059	0.049	0.055	0.060	**0.069**	**0.077**	**0.086**
δ	0.048	0.049	0.048	0.051	0.046	0.051	0.054	0.055	**0.069**	**0.073**	**0.086**

Source: author's work.

Table 10. $\mathrm{LCN}(a, \omega)$ distribution. PoT versus γ_1 and $M(a, \omega; \mu, \sigma)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.110	0.110	0.065	0.020	–0.048	0	–0.138	–0.116	–0.029	0.121	0.124
a	1.630	1.492	1.316	1.108	0.977	0	1.240	1.307	1.340	1.501	1.633
ω	0.845	0.866	0.855	0.823	0.660	1	0.413	0.337	0.237	0.127	0.147
μ	0.884	0.888	0.902	0.902	0.754	0	0.572	0.562	0.544	0.729	0.760
σ	0.990	0.976	0.915	0.915	1.155	1	1.218	1.320	1.393	1.332	1.516
M	0.786	0.821	0.868	0.917	0.961	1	0.971	0.930	0.880	0.800	0.791
GoFT	PoT										
AD	**0.069**	**0.061**	0.056	0.053	0.054	0.052	0.050	0.050	0.055	**0.061**	**0.065**
SW	**0.072**	**0.068**	**0.061**	0.052	0.049	0.052	0.048	0.047	0.054	**0.062**	**0.070**
KT	**0.063**	0.059	0.056	0.048	0.047	0.052	0.043	0.040	0.046	**0.062**	0.056
AS	**0.078**	**0.069**	**0.067**	0.051	0.047	0.051	0.043	0.041	0.051	**0.067**	**0.074**
SF	**0.075**	**0.071**	**0.065**	0.055	0.050	0.056	0.046	0.045	0.053	**0.069**	**0.074**
AP	**0.069**	**0.064**	0.059	0.048	0.047	0.049	0.044	0.039	0.050	**0.063**	**0.065**
RJ	**0.071**	**0.066**	0.060	0.051	0.047	0.051	0.043	0.041	0.050	**0.064**	**0.070**
T_1	**0.078**	**0.071**	**0.061**	0.052	0.048	0.053	0.047	0.047	0.052	**0.064**	**0.076**
JB	**0.074**	**0.065**	**0.061**	0.048	0.047	0.051	0.041	0.040	0.047	**0.068**	**0.068**
H1	**0.071**	**0.066**	**0.064**	0.052	0.051	0.054	0.047	0.049	0.054	**0.065**	**0.069**
CC	**0.077**	**0.068**	**0.063**	0.051	0.049	0.050	0.043	0.043	0.052	**0.068**	**0.075**
CS	**0.069**	**0.066**	0.059	0.051	0.050	0.052	0.049	0.048	0.054	0.060	**0.066**
AJB	**0.071**	**0.063**	0.060	0.049	0.047	0.050	0.041	0.040	0.047	**0.068**	**0.066**
ZA	**0.073**	**0.067**	**0.062**	0.052	0.049	0.053	0.044	0.046	0.052	**0.064**	**0.070**
ZC	**0.069**	**0.065**	**0.060**	0.050	0.049	0.051	0.047	0.044	0.052	**0.063**	**0.068**
β_3^2	0.056	0.054	0.053	0.050	0.047	0.052	0.046	0.046	0.052	0.058	0.053
H_n	0.046	0.043	0.046	0.049	0.048	0.051	0.058	**0.060**	**0.070**	**0.072**	**0.081**
X_{APD}	**0.068**	**0.062**	0.057	0.050	0.049	0.054	0.046	0.045	0.052	**0.062**	**0.064**
B_v	**0.065**	0.060	0.059	0.049	0.055	0.052	0.054	0.054	0.052	0.056	**0.061**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.088**	**0.074**	**0.067**	**0.062**	0.056	0.053	0.050	**0.063**	**0.069**	**0.075**	**0.086**
δ	0.047	0.052	0.048	0.047	0.048	0.050	0.048	0.056	**0.066**	**0.073**	**0.084**

Source: author's work.

Table 11. $LOG(0.103, b)$ distribution. PoT versus γ_1 and $M(0.103, b; 1.106, \sigma)$ for $n = 25$

γ_1	0.290	0.350	0.400	0.450	0.500	0.550	0.600	0.650	0.700	0.750	0.800
$\bar{\gamma}_2$	0.149	0.219	0.286	0.362	0.448	0.543	0.647	0.761	0.884	1.017	1.159
b	0.096	0.116	0.132	0.148	0.164	0.180	0.196	0.211	0.226	0.242	0.256
σ	0.106	0.117	0.119	0.133	0.148	0.163	0.178	0.169	0.155	0.140	0.126
M	0.974	0.950	**0.900**	**0.900**	**0.900**	**0.900**	**0.900**	0.844	0.772	0.699	0.628
GoFT	PoT										
AD	**0.069**	**0.083**	**0.090**	**0.093**	**0.110**	**0.117**	**0.136**	**0.152**	**0.166**	**0.196**	**0.198**
SW	**0.082**	**0.088**	**0.102**	**0.108**	**0.131**	**0.138**	**0.164**	**0.183**	**0.199**	**0.236**	**0.238**
KT	**0.066**	**0.076**	**0.080**	**0.083**	**0.097**	**0.104**	**0.114**	**0.125**	**0.132**	**0.148**	**0.155**
AS	**0.084**	**0.092**	**0.107**	**0.121**	**0.141**	**0.155**	**0.175**	**0.197**	**0.212**	**0.251**	**0.255**
SF	**0.083**	**0.092**	**0.102**	**0.112**	**0.133**	**0.144**	**0.164**	**0.183**	**0.199**	**0.235**	**0.238**
AP	**0.078**	**0.085**	**0.101**	**0.105**	**0.123**	**0.135**	**0.151**	**0.168**	**0.178**	**0.209**	**0.211**
RJ	**0.080**	**0.088**	**0.099**	**0.107**	**0.128**	**0.138**	**0.158**	**0.176**	**0.192**	**0.229**	**0.231**
T_1	**0.082**	**0.092**	**0.108**	**0.119**	**0.138**	**0.153**	***0.180***	***0.203***	***0.223***	***0.261***	***0.265***
JB	**0.079**	**0.085**	**0.102**	**0.109**	**0.127**	**0.138**	**0.155**	**0.173**	**0.186**	**0.220**	**0.222**
H1	**0.075**	**0.085**	**0.091**	**0.096**	**0.117**	**0.126**	**0.145**	**0.160**	**0.171**	**0.208**	**0.211**
CC	**0.083**	**0.092**	**0.106**	**0.121**	**0.140**	**0.155**	**0.177**	**0.202**	**0.217**	**0.256**	**0.259**
CS	**0.079**	**0.087**	**0.099**	**0.106**	**0.128**	**0.135**	**0.161**	**0.181**	**0.196**	**0.234**	**0.234**
AJB	**0.078**	**0.082**	**0.100**	**0.103**	**0.120**	**0.131**	**0.147**	**0.163**	**0.174**	**0.203**	**0.206**
ZA	**0.079**	**0.087**	**0.101**	**0.112**	**0.132**	**0.143**	**0.166**	**0.187**	**0.203**	**0.243**	**0.248**
ZC	**0.080**	**0.087**	**0.102**	**0.109**	**0.130**	**0.141**	**0.162**	**0.183**	**0.200**	**0.235**	**0.239**
β_3^2	0.056	0.060	**0.061**	0.058	**0.062**	**0.067**	**0.071**	**0.071**	**0.069**	**0.074**	**0.076**
H_n	**0.088**	**0.102**	**0.112**	**0.114**	**0.137**	**0.139**	**0.165**	**0.186**	**0.201**	**0.234**	**0.234**
X_{APD}	**0.076**	**0.081**	**0.092**	**0.095**	**0.115**	**0.126**	**0.142**	**0.157**	**0.171**	**0.206**	**0.207**
B_v	**0.066**	**0.076**	**0.081**	**0.084**	**0.098**	**0.108**	**0.122**	**0.136**	**0.151**	**0.175**	**0.179**
$LF_{\bar{\alpha},\bar{\beta}}$	**0.092**	**0.100**	**0.112**	**0.113**	**0.131**	**0.135**	**0.156**	**0.173**	**0.180**	**0.204**	**0.214**
δ	**0.086**	**0.104**	**0.115**	**0.116**	**0.138**	**0.146**	**0.176**	**0.193**	**0.205**	**0.241**	**0.244**

Source: author's work.

Table 12. $\mathrm{NM}_1(a, b, \omega)$ distribution. PoT versus γ_1 and $M(a, b, \omega; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.522	0.522	0.449	0.471	0.110	0	0.110	0.471	0.449	0.522	0.522
a	–0.739	–0.524	–0.371	–0.216	–0.194	0	0.194	0.216	0.371	0.524	0.739
b	2.033	1.827	1.620	1.516	1.211	1	1.211	1.516	1.620	1.827	2.033
ω	0.317	0.352	0.395	0.516	0.551	1	0.551	0.516	0.395	0.352	0.317
M	0.750	0.800	0.850	0.900	0.950	1	0.950	0.900	0.850	0.800	0.750
GoFT	PoT										
AD	**0.113**	**0.099**	**0.080**	**0.069**	0.057	0.050	0.055	**0.075**	**0.080**	**0.097**	**0.113**
SW	**0.114**	**0.099**	**0.086**	**0.082**	**0.061**	0.052	0.058	**0.084**	**0.083**	**0.099**	**0.112**
KT	**0.106**	**0.102**	**0.097**	**0.096**	**0.061**	0.052	0.060	**0.101**	**0.092**	**0.098**	**0.111**
AS	**0.120**	**0.108**	**0.098**	**0.095**	**0.063**	0.052	**0.064**	**0.097**	**0.093**	**0.107**	**0.119**
SF	**0.131**	**0.122**	**0.100**	**0.097**	**0.065**	0.055	**0.066**	**0.102**	**0.100**	**0.118**	**0.135**
AP	**0.111**	**0.106**	**0.096**	**0.095**	**0.064**	0.054	**0.062**	**0.097**	**0.095**	**0.102**	**0.113**
RJ	**0.124**	**0.116**	**0.096**	**0.091**	**0.061**	0.051	**0.062**	**0.098**	**0.095**	**0.112**	**0.127**
T_1	**0.108**	**0.096**	**0.081**	**0.076**	0.059	0.051	**0.061**	**0.083**	**0.079**	**0.094**	**0.106**
JB	**0.121**	**0.117**	**0.104**	**0.103**	**0.066**	0.053	**0.063**	**0.107**	**0.103**	**0.114**	**0.127**
H1	**0.124**	**0.116**	**0.095**	**0.087**	**0.063**	0.053	**0.064**	**0.093**	**0.090**	**0.111**	**0.125**
CC	**0.119**	**0.109**	**0.097**	**0.091**	**0.061**	0.052	**0.064**	**0.096**	**0.093**	**0.110**	**0.122**
CS	**0.109**	**0.094**	**0.082**	**0.079**	0.060	0.053	0.058	**0.081**	**0.080**	**0.093**	**0.108**
AJB	**0.121**	**0.118**	**0.107**	**0.103**	**0.067**	0.054	**0.062**	**0.110**	**0.103**	**0.111**	**0.129**
ZA	**0.109**	**0.097**	**0.089**	**0.086**	**0.061**	0.053	0.060	**0.086**	**0.084**	**0.098**	**0.110**
ZC	**0.106**	**0.096**	**0.084**	**0.084**	**0.061**	0.053	0.058	**0.085**	**0.084**	**0.095**	**0.108**
β_3^2	**0.098**	**0.101**	**0.091**	**0.086**	0.058	0.052	0.059	**0.088**	**0.080**	**0.097**	**0.103**
H_n	**0.074**	**0.064**	0.057	0.053	0.053	0.053	0.055	**0.070**	**0.080**	**0.102**	**0.126**
X_{APD}	**0.119**	**0.108**	**0.093**	**0.087**	**0.062**	0.053	0.059	**0.095**	**0.089**	**0.101**	**0.119**
B_v	**0.078**	**0.074**	**0.065**	0.058	0.054	0.049	0.054	0.060	**0.065**	**0.076**	**0.081**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.128**	**0.107**	**0.085**	**0.068**	0.059	0.049	0.058	**0.074**	**0.083**	**0.104**	**0.128**
δ	**0.084**	**0.074**	**0.065**	0.060	0.054	0.051	0.059	**0.080**	**0.087**	**0.113**	**0.130**

Source: author's work.

Table 13. $\mathrm{NM}_2(a,b,\omega)$ distribution. PoT versus γ_1 and $M(a,b,\omega;0{,}1) = 0.95$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.546	0.425	0.350	0.134	0.130	0	0.130	0.134	0.350	0.425	0.546
a	−0.621	−0.522	−0.396	−0.398	−0.180	0	0.180	0.398	0.396	0.522	0.621
b	1.474	1.418	1.384	1.219	1.230	1	1.230	1.219	1.384	1.418	1.474
ω	0.805	0.779	0.743	0.693	0.564	1	0.564	0.693	0.743	0.779	0.805
M	0.950	0.950	0.950	0.950	0.950	1	0.950	0.950	0.950	0.950	0.950
GoFT	PoT										
AD	**0.081**	**0.074**	**0.079**	0.059	0.057	0.050	0.056	**0.061**	**0.067**	**0.074**	**0.080**
SW	**0.097**	**0.088**	**0.088**	**0.067**	0.059	0.051	0.057	**0.063**	**0.075**	**0.081**	**0.094**
KT	**0.105**	**0.091**	**0.091**	**0.068**	0.058	0.051	0.058	**0.067**	**0.083**	**0.092**	**0.103**
AS	**0.109**	**0.100**	**0.097**	**0.070**	**0.064**	0.051	0.060	**0.068**	**0.085**	**0.094**	**0.104**
SF	**0.110**	**0.099**	**0.099**	**0.073**	**0.063**	0.053	**0.063**	**0.069**	**0.087**	**0.094**	**0.107**
AP	**0.108**	**0.097**	**0.095**	**0.068**	**0.063**	0.051	0.057	**0.067**	**0.084**	**0.094**	**0.107**
RJ	**0.106**	**0.094**	**0.095**	**0.070**	0.060	0.049	0.060	**0.065**	**0.082**	**0.089**	**0.102**
T_1	**0.092**	**0.084**	**0.084**	**0.064**	**0.062**	0.047	0.057	**0.063**	**0.073**	**0.079**	**0.090**
JB	**0.115**	**0.102**	**0.100**	**0.071**	**0.064**	0.052	**0.061**	**0.069**	**0.089**	**0.101**	**0.112**
H1	**0.095**	**0.084**	**0.088**	**0.066**	**0.062**	0.053	**0.061**	**0.064**	**0.074**	**0.086**	**0.094**
CC	**0.108**	**0.099**	**0.097**	**0.070**	**0.062**	0.049	0.060	**0.067**	**0.082**	**0.092**	**0.103**
CS	**0.093**	**0.085**	**0.084**	**0.064**	0.058	0.051	0.055	**0.062**	**0.074**	**0.078**	**0.092**
AJB	**0.115**	**0.101**	**0.100**	**0.072**	**0.063**	0.052	**0.061**	**0.070**	**0.090**	**0.101**	**0.112**
ZA	**0.100**	**0.090**	**0.086**	**0.068**	0.059	0.051	0.059	**0.067**	**0.079**	**0.084**	**0.100**
ZC	**0.100**	**0.093**	**0.089**	**0.068**	**0.060**	0.051	0.057	**0.064**	**0.079**	**0.083**	**0.098**
β_3^2	**0.084**	**0.078**	**0.078**	**0.062**	0.057	0.054	0.057	**0.061**	**0.072**	**0.080**	**0.084**
H_n	**0.082**	**0.079**	**0.085**	**0.066**	**0.064**	0.052	0.058	**0.066**	**0.069**	**0.078**	**0.083**
X_{APD}	**0.099**	**0.088**	**0.091**	**0.067**	0.059	0.048	0.058	**0.063**	**0.074**	**0.089**	**0.097**
B_v	**0.068**	**0.067**	**0.066**	0.058	0.056	0.051	0.052	0.056	**0.060**	**0.064**	**0.069**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.044	0.041	0.041	0.032	0.029	0.050	0.056	**0.066**	**0.070**	**0.079**	**0.085**
δ	**0.091**	**0.085**	**0.092**	**0.069**	**0.068**	0.050	0.060	**0.066**	**0.072**	**0.087**	**0.089**

Source: author’s work.

Table 14. $SB(a,b)$ distribution. PoT versus γ_1 and $M(a,b;0,0.093)$ for $n=25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$						−0.038					
a	−0.810	−0.724	−0.599	−0.463	−0.262	0	0.262	0.463	0.599	0.724	0.810
b	1.913	2.055	2.187	2.396	2.576	2.669	2.576	2.396	2.187	2.055	1.913
M	0.543	0.540	0.534	0.526	0.514	0.500	0.486	0.474	0.466	0.460	0.457
GoFT						PoT					
AD	**0.061**	0.051	0.049	0.047	0.045	0.043	0.044	0.044	0.047	0.055	**0.061**
SW	**0.060**	0.048	0.047	0.043	0.039	0.038	0.041	0.043	0.046	0.053	0.058
KT	0.038	0.036	0.035	0.033	0.034	0.032	0.033	0.032	0.037	0.037	0.036
AS	0.045	0.038	0.037	0.035	0.030	0.030	0.030	0.034	0.034	0.041	0.045
SF	0.048	0.041	0.038	0.037	0.034	0.035	0.035	0.036	0.038	0.045	0.047
AP	0.041	0.039	0.036	0.034	0.030	0.031	0.032	0.034	0.036	0.041	0.041
RJ	0.044	0.038	0.035	0.034	0.031	0.032	0.031	0.033	0.035	0.041	0.043
T_1	0.059	0.047	0.044	0.042	0.037	0.036	0.039	0.043	0.044	0.052	0.058
JB	0.035	0.032	0.031	0.029	0.026	0.029	0.027	0.028	0.030	0.033	0.035
H1	0.053	0.047	0.045	0.042	0.039	0.039	0.040	0.040	0.045	0.050	0.051
CC	0.044	0.038	0.036	0.035	0.031	0.031	0.031	0.034	0.036	0.043	0.047
CS	**0.061**	0.050	0.048	0.045	0.039	0.039	0.041	0.044	0.049	0.055	**0.061**
AJB	0.033	0.030	0.028	0.029	0.025	0.026	0.027	0.027	0.028	0.031	0.032
ZA	0.054	0.044	0.044	0.041	0.036	0.035	0.038	0.041	0.044	0.051	0.054
ZC	0.057	0.047	0.043	0.041	0.035	0.035	0.038	0.042	0.044	0.052	0.056
β_3^2	0.038	0.040	0.042	0.042	0.041	0.041	0.038	0.039	0.044	0.045	0.038
H_n	0.045	0.038	0.042	0.043	0.047	0.048	0.053	0.056	**0.065**	**0.079**	**0.086**
X_{APD}	0.052	0.045	0.041	0.039	0.037	0.034	0.039	0.040	0.042	0.049	0.050
B_v	**0.070**	**0.061**	0.060	0.056	0.051	0.052	0.053	0.054	0.060	**0.066**	**0.070**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.081**	**0.072**	**0.065**	0.058	0.052	0.045	0.053	0.056	**0.064**	**0.078**	**0.085**
δ	0.044	0.039	0.040	0.040	0.043	0.044	0.048	0.050	0.059	**0.069**	**0.073**

Source: author's work.

Table 15. $SN(a)$ distribution. PoT versus γ_1 and $M(a;\mu,\sigma)$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.138	0.102	0.070	0.041	0.016	0	0.016	0.041	0.070	0.102	0.138
a	−1.349	−1.199	−1.043	−0.871	−0.659	0	0.659	0.871	1.043	1.199	1.349
μ	−0.252	−0.231	−0.262	−0.299	−0.339	0	0.476	0.677	0.825	0.879	1
σ	1.012	1.018	0.968	0.905	0.967	1	0.954	0.955	0.771	0.720	0.500
M	0.800	0.810	0.850	0.900	0.950	1	0.965	0.910	0.863	0.840	0.692
GoFT	PoT										
AD	**0.065**	**0.063**	0.052	0.052	0.050	0.053	0.053	0.059	0.059	**0.062**	**0.066**
SW	**0.071**	**0.066**	0.058	0.054	0.049	0.053	0.054	0.058	0.060	**0.067**	**0.073**
KT	**0.062**	**0.061**	0.055	0.054	0.048	0.049	0.053	0.054	0.056	0.060	**0.065**
AS	**0.075**	**0.069**	**0.061**	0.055	0.048	0.049	0.054	0.058	0.060	**0.070**	**0.077**
SF	**0.075**	**0.069**	**0.062**	0.056	0.051	0.052	0.058	**0.061**	**0.063**	**0.071**	**0.076**
AP	**0.070**	**0.065**	0.057	0.053	0.048	0.049	0.052	0.053	0.057	**0.065**	**0.071**
RJ	**0.071**	**0.065**	0.057	0.051	0.048	0.049	0.054	0.056	0.060	**0.066**	**0.072**
T_1	**0.074**	**0.068**	0.058	0.055	0.048	0.049	0.056	0.057	0.059	**0.067**	**0.075**
JB	**0.072**	**0.065**	0.059	0.053	0.048	0.049	0.052	0.054	0.059	**0.066**	**0.074**
H1	**0.069**	**0.067**	0.054	0.055	0.049	0.052	0.057	0.059	**0.061**	**0.067**	**0.070**
CC	**0.074**	**0.067**	0.060	0.054	0.048	0.048	0.055	0.057	0.060	**0.070**	**0.077**
CS	**0.071**	**0.065**	0.057	0.052	0.048	0.051	0.052	0.058	0.060	**0.065**	**0.071**
AJB	**0.069**	**0.065**	0.059	0.053	0.050	0.047	0.054	0.054	0.059	**0.066**	**0.073**
ZA	**0.071**	**0.068**	0.058	0.054	0.049	0.050	0.052	0.056	0.058	**0.067**	**0.075**
ZC	**0.071**	**0.065**	0.057	0.054	0.049	0.050	0.053	0.055	0.058	**0.066**	**0.075**
β_3^2	0.053	0.058	0.049	0.055	0.052	0.052	0.053	0.053	0.055	0.053	0.053
H_n	0.047	0.049	0.045	0.048	0.048	0.053	0.058	**0.065**	**0.068**	**0.074**	**0.079**
X_{APD}	**0.069**	**0.065**	0.054	0.053	0.048	0.051	0.056	0.057	0.058	**0.065**	**0.067**
B_v	**0.063**	**0.062**	0.053	0.054	0.050	0.054	0.055	0.054	0.055	0.060	0.060
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.086**	**0.072**	**0.066**	**0.062**	0.055	0.051	0.054	**0.067**	**0.069**	**0.072**	**0.083**
δ	0.050	0.051	0.045	0.046	0.048	0.051	0.055	**0.063**	**0.065**	**0.074**	**0.081**

Source: author's work.

Table 16. $SU(b,c,d)$ distribution. PoT versus γ_1 and $M(b,c,d;\mu,\sigma)$ for $n=25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	–2.470	–2.010	–1.340	–0.646	–0.119	0.085	–0.119	–0.646	–1.340	–2.010	–2.470
b	0.529	0.619	0.738	0.906	1.192	1.375	1.192	0.906	0.738	0.619	0.529
c	9.322	7.823	5.909	3.972	2.038	0	–2.038	–3.972	–5.909	–7.820	–9.320
d	9.256	9.936	10.381	10.721	11.041	11.129	11.041	10.721	10.381	9.936	9.256
μ	–0.570	–0.500	–0.412	–0.321	–0.207	0	0.207	0.321	0.412	0.497	0.569
σ	0.104	0.092	0.087	0.097	0.111	0.124	0.111	0.097	0.087	0.092	0.104
M	0.750	0.800	0.851	0.900	0.950	0.998	0.950	0.900	0.851	0.800	0.750
GoFT	PoT										
AD	**0.066**	**0.066**	0.050	0.054	0.051	0.051	0.052	0.056	0.055	**0.063**	**0.064**
SW	**0.076**	**0.068**	0.055	0.055	0.048	0.052	0.051	0.058	0.057	**0.064**	**0.069**
KT	**0.069**	**0.061**	0.058	0.056	0.052	0.053	0.052	0.057	0.055	**0.061**	0.059
AS	**0.077**	**0.067**	0.059	0.059	0.054	0.055	0.052	0.059	0.056	**0.067**	**0.075**
SF	**0.079**	**0.067**	**0.061**	0.059	0.051	0.057	0.052	**0.063**	0.059	**0.067**	**0.071**
AP	**0.074**	**0.067**	0.059	0.058	0.055	0.055	0.052	0.060	0.055	**0.063**	**0.070**
RJ	**0.075**	**0.063**	0.056	0.055	0.046	0.054	0.049	0.059	0.056	**0.063**	**0.068**
T_1	**0.076**	**0.065**	0.055	0.058	0.052	0.054	0.047	0.057	0.055	**0.065**	**0.075**
JB	**0.076**	**0.067**	0.059	0.060	0.054	0.057	0.053	0.060	0.056	**0.065**	**0.070**
H1	**0.069**	**0.066**	0.058	0.055	0.050	0.056	0.052	0.058	0.056	**0.064**	**0.069**
CC	**0.078**	**0.066**	0.058	0.057	0.052	0.055	0.051	0.060	0.055	**0.068**	**0.073**
CS	**0.076**	**0.068**	0.054	0.054	0.048	0.052	0.051	0.057	0.056	**0.063**	**0.069**
AJB	**0.074**	**0.066**	0.058	0.060	0.055	0.057	0.053	0.060	0.055	**0.064**	**0.069**
ZA	**0.078**	**0.069**	0.056	0.054	0.049	0.053	0.049	0.057	0.057	**0.063**	**0.071**
ZC	**0.076**	**0.068**	0.057	0.056	0.050	0.051	0.049	0.059	0.056	**0.063**	**0.070**
β_3^2	0.054	0.058	0.054	0.055	0.050	0.055	0.054	0.056	0.055	0.055	0.055
H_n	0.046	0.047	0.042	0.045	0.047	0.051	0.057	**0.061**	**0.063**	**0.073**	**0.078**
X_{APD}	**0.070**	**0.062**	0.055	0.055	0.048	0.053	0.051	0.060	0.054	**0.064**	**0.066**
B_v	**0.065**	0.060	0.054	0.053	0.049	0.052	0.050	0.055	0.055	0.059	**0.063**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.086**	**0.076**	**0.066**	**0.061**	0.057	0.052	0.057	**0.062**	**0.065**	**0.073**	**0.080**
δ	0.051	0.045	0.045	0.049	0.049	0.052	0.058	**0.061**	**0.068**	**0.072**	**0.082**

Source: author's work.

Table 17. $W(1.851, b)$ distribution. PoT versus γ_1 and $M(1.851, b; 1.673, 0.532)$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	−0.124	−0.173	−0.214	−0.245	−0.269	−0.283	−0.289	−0.287	−0.276	−0.258	−0.230
b	4.971	4.634	4.334	4.064	3.822	3.602	3.403	3.222	3.056	2.905	2.766
M	0.849	0.880	0.911	0.940	0.968	0.985	0.974	0.951	0.927	0.904	0.882
GoFT	PoT										
AD	**0.065**	0.058	0.053	0.049	0.044	0.041	0.044	0.043	0.047	0.052	0.058
SW	**0.064**	0.055	0.052	0.047	0.041	0.038	0.038	0.039	0.043	0.053	0.060
KT	0.044	0.040	0.038	0.031	0.033	0.029	0.034	0.034	0.032	0.039	0.043
AS	0.059	0.051	0.041	0.033	0.029	0.029	0.028	0.029	0.034	0.042	0.053
SF	**0.061**	0.051	0.045	0.038	0.034	0.032	0.032	0.034	0.036	0.044	0.051
AP	0.048	0.045	0.040	0.031	0.031	0.029	0.030	0.032	0.034	0.041	0.049
RJ	0.056	0.048	0.042	0.034	0.033	0.029	0.030	0.030	0.033	0.041	0.048
T_1	**0.067**	**0.060**	0.051	0.046	0.040	0.035	0.036	0.037	0.042	0.050	0.060
JB	0.049	0.044	0.035	0.027	0.027	0.025	0.023	0.026	0.028	0.035	0.046
H1	0.058	0.052	0.049	0.044	0.039	0.038	0.039	0.041	0.040	0.047	0.056
CC	0.057	0.050	0.041	0.034	0.029	0.029	0.027	0.030	0.035	0.043	0.054
CS	**0.065**	0.055	0.054	0.049	0.042	0.040	0.041	0.042	0.045	0.055	0.060
AJB	0.046	0.042	0.033	0.025	0.026	0.023	0.023	0.026	0.027	0.034	0.042
ZA	**0.063**	0.054	0.049	0.043	0.038	0.035	0.036	0.037	0.037	0.047	0.055
ZC	**0.060**	0.052	0.048	0.043	0.039	0.035	0.037	0.039	0.041	0.049	0.058
β_3^2	0.042	0.043	0.043	0.042	0.041	0.038	0.041	0.041	0.038	0.041	0.041
H_n	0.048	0.043	0.043	0.043	0.043	0.045	0.053	0.055	0.058	**0.073**	**0.078**
X_{APD}	0.056	0.050	0.046	0.042	0.038	0.036	0.039	0.039	0.040	0.047	0.053
B_v	**0.064**	0.059	0.058	**0.065**	0.054	0.050	0.053	0.053	0.054	**0.062**	**0.067**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.090**	**0.078**	**0.065**	**0.063**	0.052	0.039	0.049	0.052	0.058	**0.069**	**0.077**
δ	0.047	0.041	0.044	0.041	0.041	0.041	0.048	0.049	0.054	**0.063**	**0.072**

Source: author's work.

Table 18. $ES_1(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.250	0.200	0.150	0.100	0.050	0	0.050	0.100	0.150	0.200	0.250
M	0.966	0.973	0.979	0.986	0.993	1	0.993	0.986	0.979	0.973	0.966
GoFT	PoT										
AD	**0.073**	**0.068**	**0.060**	0.056	0.050	0.052	0.052	0.057	0.060	**0.064**	**0.071**
SW	**0.081**	**0.072**	**0.062**	0.058	0.053	0.051	0.057	0.056	**0.063**	**0.070**	**0.082**
KT	**0.075**	**0.071**	**0.063**	0.057	0.057	0.049	0.051	0.060	**0.064**	**0.064**	**0.080**
AS	**0.087**	**0.080**	**0.064**	0.057	0.055	0.053	0.057	**0.062**	**0.066**	**0.079**	**0.085**
SF	**0.089**	**0.077**	**0.067**	**0.063**	0.058	0.053	0.060	**0.061**	**0.068**	**0.077**	**0.088**
AP	**0.083**	**0.076**	**0.065**	0.060	0.054	0.054	0.053	**0.061**	**0.065**	**0.073**	**0.083**
RJ	**0.083**	**0.072**	**0.063**	0.059	0.054	0.049	0.056	0.057	**0.064**	**0.073**	**0.082**
T_1	**0.080**	**0.074**	0.060	0.054	0.050	0.051	0.058	0.059	**0.062**	**0.073**	**0.076**
JB	**0.086**	**0.079**	**0.068**	0.059	0.056	0.054	0.055	**0.062**	**0.068**	**0.077**	**0.087**
H1	**0.080**	**0.074**	**0.062**	0.058	0.052	0.055	0.057	0.059	**0.065**	**0.071**	**0.081**
CC	**0.088**	**0.081**	**0.063**	0.057	0.054	0.053	0.058	**0.061**	**0.065**	**0.076**	**0.085**
CS	**0.078**	**0.071**	**0.061**	0.058	0.051	0.050	0.055	0.055	**0.060**	**0.067**	**0.079**
AJB	**0.085**	**0.080**	**0.067**	0.060	0.056	0.053	0.055	**0.063**	**0.067**	**0.075**	**0.089**
ZA	**0.080**	**0.073**	**0.061**	0.056	0.054	0.048	0.059	0.056	**0.063**	**0.072**	**0.082**
ZC	**0.078**	**0.073**	**0.060**	0.057	0.051	0.049	0.055	0.055	**0.065**	**0.072**	**0.081**
β_3^2	**0.063**	**0.062**	0.058	0.055	0.054	0.050	0.049	0.059	0.057	0.058	**0.069**
H_n	0.050	0.050	0.047	0.047	0.046	0.053	0.057	**0.060**	**0.065**	**0.073**	**0.083**
X_{APD}	**0.078**	**0.070**	**0.062**	0.059	0.051	0.050	0.054	0.055	**0.063**	**0.067**	**0.077**
B_v	**0.063**	**0.065**	0.057	0.054	0.051	0.055	0.051	0.053	0.055	0.060	**0.065**
$LF_{\bar{\alpha},\bar{\beta}}$	**0.085**	**0.079**	**0.069**	0.058	0.055	0.051	0.058	**0.066**	**0.066**	**0.079**	**0.085**
δ	0.052	0.050	0.050	0.049	0.048	0.050	0.056	**0.061**	**0.067**	**0.078**	**0.088**

Source: author's work.

Table 19. $ES_1(\gamma_1, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_1, \bar{\gamma}_2 = 0; 0{,}1)$ for $n = 25$

γ_1	−0.205	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$						0					
M	0.969	0.975	0.981	0.987	0.994	1	0.994	0.987	0.981	0.975	0.969
GoFT	PoT										
AD	**0.063**	0.051	0.054	0.050	0.047	0.050	0.049	0.052	0.052	0.058	**0.061**
SW	**0.065**	0.056	0.054	0.053	0.048	0.051	0.049	0.052	0.053	0.060	**0.064**
KT	0.051	0.049	0.048	0.051	0.046	0.051	0.048	0.047	0.050	0.051	0.053
AS	**0.063**	0.058	0.056	0.055	0.049	0.051	0.050	0.051	0.053	**0.062**	**0.065**
SF	**0.065**	0.055	0.057	0.056	0.049	0.053	0.051	0.052	0.055	0.060	**0.066**
AP	0.056	0.053	0.053	0.054	0.045	0.051	0.049	0.048	0.050	0.056	0.060
RJ	0.059	0.053	0.053	0.053	0.045	0.049	0.048	0.048	0.051	0.056	**0.061**
T_1	<u>**0.067**</u>	<u>0.058</u>	<u>0.056</u>	<u>0.054</u>	0.049	0.047	0.048	0.052	0.054	**0.062**	**0.069**
JB	0.057	0.053	0.054	0.053	0.046	0.052	0.049	0.049	0.050	0.057	0.059
H1	**0.065**	0.053	0.056	0.053	0.050	0.053	0.052	0.052	0.055	0.058	**0.064**
CC	**0.062**	0.057	0.055	0.054	0.047	0.049	0.050	0.052	0.053	**0.062**	**0.066**
CS	**0.064**	0.055	0.054	0.052	0.048	0.051	0.048	0.051	0.052	0.058	**0.062**
AJB	0.056	0.052	0.053	0.053	0.047	0.052	0.048	0.047	0.051	0.055	0.059
ZA	**0.065**	0.055	0.053	0.052	0.047	0.051	0.047	0.048	0.052	0.057	**0.061**
ZC	**0.064**	0.056	0.053	0.053	0.047	0.051	0.047	0.049	0.053	0.056	0.059
β_3^2	0.047	0.044	0.048	0.050	0.048	0.054	0.052	0.048	0.048	0.047	0.047
H_n	0.043	0.041	0.046	0.044	0.047	0.052	0.054	0.057	**0.061**	**0.070**	**0.076**
X_{APD}	0.060	0.052	0.054	0.052	0.045	0.048	0.048	0.047	0.051	0.054	0.059
B_v	0.060	0.052	0.053	0.051	0.050	0.051	0.049	0.051	0.056	0.056	0.059
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.033	0.036	0.041	0.040	0.046	0.050	0.055	<u>0.055</u>	<u>**0.061**</u>	<u>**0.075**</u>	<u>**0.075**</u>
δ	0.046	0.039	0.045	0.044	0.046	0.050	0.055	0.056	**0.062**	**0.068**	**0.077**

Source: author's work.

Table 20. $P_1(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2 > 0; 0{,}1)$ for $n = 25$

γ_1	–0.250	–0.200	–0.150	–0.100	–0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	0.250	0.200	0.150	0.100	0.050	0	0.050	0.100	0.150	0.200	0.250
M	0.969	0.975	0.981	0.987	0.993	1	0.993	0.987	0.981	0.975	0.969
GoFT	PoT										
AD	**0.070**	**0.065**	**0.061**	0.056	0.054	0.052	0.052	0.056	**0.061**	**0.068**	**0.071**
SW	**0.078**	**0.070**	**0.065**	0.056	0.054	0.051	0.057	0.058	**0.064**	**0.074**	**0.078**
KT	**0.073**	**0.067**	**0.063**	0.055	0.055	0.049	0.055	**0.062**	**0.065**	**0.069**	**0.079**
AS	**0.082**	**0.073**	**0.068**	0.058	0.055	0.053	0.055	**0.062**	**0.069**	**0.078**	**0.086**
SF	**0.086**	**0.076**	**0.069**	0.057	0.057	0.053	0.058	**0.063**	**0.068**	**0.077**	**0.087**
AP	**0.079**	**0.072**	**0.067**	0.056	0.055	0.054	0.056	**0.062**	**0.070**	**0.075**	**0.084**
RJ	**0.081**	**0.071**	**0.065**	0.055	0.053	0.049	0.054	0.059	**0.064**	**0.073**	**0.081**
T_1	**0.081**	**0.071**	**0.061**	0.059	0.055	0.051	0.055	0.058	**0.064**	**0.077**	**0.077**
JB	**0.080**	**0.074**	**0.067**	0.057	0.056	0.054	0.055	**0.064**	**0.071**	**0.076**	**0.088**
H1	**0.076**	**0.069**	**0.064**	0.057	0.054	0.055	0.057	0.056	**0.063**	**0.071**	**0.078**
CC	**0.081**	**0.073**	**0.067**	0.058	0.054	0.053	0.055	**0.062**	**0.067**	**0.078**	**0.083**
CS	**0.077**	**0.069**	**0.063**	0.055	0.053	0.050	0.057	0.057	**0.063**	**0.071**	**0.076**
AJB	**0.077**	**0.075**	**0.066**	0.058	0.055	0.053	0.055	**0.065**	**0.071**	**0.075**	**0.089**
ZA	**0.081**	**0.070**	**0.066**	0.056	0.054	0.048	0.057	0.058	**0.065**	**0.077**	**0.080**
ZC	**0.079**	**0.070**	**0.064**	0.055	0.053	0.049	0.059	**0.061**	**0.065**	**0.075**	**0.081**
β_3^2	**0.060**	0.060	0.058	0.054	0.053	0.050	0.054	0.056	0.055	**0.061**	**0.065**
H_n	0.049	0.050	0.049	0.047	0.052	0.053	0.055	**0.064**	**0.065**	**0.076**	**0.081**
X_{APD}	**0.074**	**0.069**	**0.062**	0.056	0.054	0.050	0.056	0.058	**0.063**	**0.072**	**0.077**
B_v	**0.067**	**0.063**	0.057	0.053	0.054	0.055	0.054	0.053	0.056	**0.066**	**0.064**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**0.081**	**0.076**	**0.071**	**0.065**	0.057	0.051	0.055	**0.060**	**0.070**	**0.077**	**0.084**
δ	0.053	0.050	0.051	0.049	0.051	0.050	0.054	**0.063**	**0.068**	**0.079**	**0.085**

Source: author's work.

Table 21. $P_2(\gamma_{1,},\bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,},\bar{\gamma}_2 = 0; 0{,}1)$ for $n = 25$

γ_1 $\bar{\gamma}_2$ M	−0.250 0.967	−0.200 0.974	−0.150 0.981	−0.100 0.987	−0.050 0.994	0 0 1	0.050 0.994	0.100 0.987	0.150 0.981	0.200 0.974	0.250 0.967
GoFT	PoT										
AD	0.058	**0.060**	0.054	0.049	0.050	0.052	0.049	0.051	0.052	0.056	**0.065**
SW	**0.062**	0.060	0.057	0.050	0.051	0.051	0.051	0.051	0.052	0.058	**0.068**
KT	0.051	0.051	0.053	0.048	0.053	0.049	0.048	0.050	0.046	0.053	0.055
AS	**0.062**	0.059	0.056	0.050	0.052	0.053	0.046	0.051	0.050	0.059	**0.067**
SF	**0.060**	**0.065**	0.058	0.052	0.055	0.053	0.052	0.052	0.053	0.057	**0.067**
AP	0.058	0.055	0.055	0.049	0.051	0.054	0.049	0.049	0.049	0.056	**0.063**
RJ	0.057	0.059	0.054	0.048	0.051	0.049	0.049	0.048	0.050	0.054	**0.063**
T_1	**0.064**	**0.062**	0.057	0.051	0.050	0.051	0.046	0.050	0.051	**0.061**	**0.069**
JB	0.057	0.055	0.054	0.048	0.051	0.054	0.047	0.052	0.048	0.056	**0.062**
H1	0.058	**0.060**	0.058	0.050	0.055	0.055	0.048	0.054	0.051	0.056	**0.065**
CC	**0.061**	0.060	0.054	0.049	0.052	0.053	0.046	0.052	0.050	0.060	**0.068**
CS	**0.064**	**0.061**	0.056	0.049	0.051	0.050	0.050	0.050	0.052	0.058	**0.068**
AJB	0.053	0.054	0.053	0.046	0.052	0.053	0.049	0.051	0.046	0.055	0.059
ZA	**0.061**	0.058	0.054	0.050	0.049	0.048	0.049	0.049	0.048	0.059	**0.067**
ZC	0.059	0.060	0.057	0.050	0.051	0.049	0.050	0.049	0.050	0.060	**0.066**
β_3^2	0.046	0.048	0.051	0.048	0.052	0.050	0.051	0.052	0.048	0.047	0.048
H_n	0.042	0.048	0.044	0.047	0.047	0.053	0.051	0.058	**0.061**	**0.072**	**0.080**
X_{APD}	0.057	0.057	0.056	0.048	0.052	0.050	0.051	0.051	0.050	0.056	**0.062**
B_v	0.058	0.057	0.055	0.051	0.051	0.055	0.051	0.053	0.051	0.060	**0.064**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	**<u>0.079</u>**	**<u>0.077</u>**	**<u>0.068</u>**	<u>0.059</u>	0.053	0.051	0.049	<u>0.058</u>	**<u>0.062</u>**	**<u>0.074</u>**	**<u>0.082</u>**
δ	0.041	0.048	0.044	0.044	0.047	0.050	0.049	0.056	0.060	**0.070**	**0.081**

Source: author's work.

Table 22. $P_3(\gamma_{1,}, \bar{\gamma}_2)$ distribution. PoT versus γ_1 and $M(\gamma_{1,}, \bar{\gamma}_2 < 0; 0{,}1)$ for $n = 25$

γ_1	−0.250	−0.200	−0.150	−0.100	−0.050	0	0.050	0.100	0.150	0.200	0.250
$\bar{\gamma}_2$	−0.250	−0.200	−0.150	−0.100	−0.050	0	−0.050	−0.100	−0.150	−0.200	−0.250
M	0.957	0.967	0.977	0.985	0.993	1	0.993	0.985	0.977	0.967	0.957
GoFT	PoT										
AD	0.057	0.050	0.051	0.051	0.049	0.052	0.047	0.052	0.050	0.051	0.059
SW	0.056	0.049	0.052	0.050	0.048	0.051	0.048	0.051	0.048	0.050	**0.061**
KT	0.036	0.039	0.042	0.046	0.041	0.049	0.045	0.043	0.041	0.038	0.044
AS	0.044	0.043	0.048	0.046	0.046	0.053	0.045	0.043	0.041	0.042	0.053
SF	0.048	0.044	0.047	0.049	0.049	0.053	0.048	0.047	0.044	0.045	0.053
AP	0.040	0.042	0.044	0.042	0.045	0.054	0.045	0.043	0.040	0.041	0.050
RJ	0.045	0.040	0.044	0.045	0.045	0.049	0.044	0.044	0.040	0.042	0.050
T_1	0.058	0.049	0.051	0.050	0.050	0.051	0.048	0.048	0.046	0.049	0.059
JB	0.035	0.036	0.043	0.042	0.045	0.054	0.045	0.041	0.038	0.036	0.045
H1	0.051	0.047	0.049	0.050	0.049	0.055	0.047	0.052	0.045	0.048	0.057
CC	0.044	0.040	0.048	0.045	0.046	0.053	0.045	0.044	0.042	0.045	0.054
CS	0.057	0.051	0.053	0.051	0.048	0.050	0.048	0.052	0.048	0.052	**0.061**
AJB	0.032	0.036	0.041	0.042	0.044	0.053	0.043	0.041	0.037	0.035	0.041
ZA	0.056	0.047	0.049	0.047	0.048	0.048	0.047	0.050	0.047	0.047	0.058
ZC	0.052	0.049	0.050	0.048	0.047	0.049	0.045	0.048	0.046	0.049	0.060
β_3^2	0.042	0.042	0.042	0.048	0.047	0.050	0.047	0.048	0.043	0.041	0.046
H_n	0.043	0.041	0.042	0.048	0.044	0.053	0.055	0.059	**0.063**	**0.070**	**0.079**
X_{APD}	0.050	0.044	0.046	0.047	0.046	0.050	0.044	0.046	0.043	0.047	0.053
B_v	**0.068**	0.059	0.052	0.054	0.053	0.055	0.052	0.055	0.053	0.059	**0.069**
$\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$	0.033	0.033	0.036	0.041	0.042	0.051	0.053	**0.061**	**0.063**	**0.068**	**0.081**
δ	0.044	0.037	0.044	0.043	0.044	0.050	0.052	0.058	0.060	**0.062**	**0.075**

Source: author's work.

Tables 6–22 show that when alternatives are asymmetric with non-constant γ_1, the GoFT for normality detects positive or negative γ_1 differently, depending on the alternative. For distribution B, three and six analysed GoFTs detect $\gamma_1 \le -0.25$ and $\gamma_1 \ge 0.25$, respectively. For χ^2, all the analysed GoFTs, except β_3^2, detect $\gamma_1 \ge 0.294$. For G, all the analysed GoFTs, except β_3^2, detect $\gamma_1 \ge 0.223$. For LOG, all analysed GoFTs, except β_3^2, detect $\gamma_1 \ge 0.29$. For GP, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ and $\mathrm{H_n}$ tests detect $\gamma_1 \le -0.1$ and $\gamma_1 \ge 0$, respectively. For LCN, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ test detects $\gamma_1 \le -0.1$ or $\gamma_1 \ge 0.1$. For NM (see Table 12), thirteen and nine GoFTs detect $\gamma_1 \le -0.1$ and $\gamma_1 \ge 0.1$, respectively. For NM with $M = 0.95$ (see Table 13), eleven and three GoFTs detect $\gamma_1 \le -0.1$ and $\gamma_1 \ge 0.1$, respectively. For SB, $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ and $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, H_n$, the GoFTs detect $\gamma_1 \le -0.15$ and $\gamma_1 \ge 0.15$, respectively. For SN and SU, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}$ GoFT detects $|\gamma_1| \ge 0.1$. For W, the $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, B_V$ tests detect $\gamma_1 \le -0.1$ and $\mathrm{LF}_{\bar{\alpha},\bar{\beta}}, B_V$ detects $\gamma_1 \ge 0.2$. For ES (see Table 18), only the S_F tests detect $\gamma_1 \le -0.1$ and most tests detect $\gamma_1 \ge 0.15$. For ES with $\bar{\gamma}_2 = 0$ (see Table 19), only 10 tests detect $\gamma_1 \le$

-0.25 and the $H_n, \mathrm{LF}_{\overline{\alpha},\overline{\beta}}, \delta$ tests detect $\gamma_1 \geq 0.15$. For P (see Table 20), the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ GoFT detects $|\gamma_1| \geq 0.1$. For P (see Table 21), the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ GoFT detects $|\gamma_1| \geq 0.15$. For P (see Table 22), only the B_v test detects $\gamma_1 \leq -0.25$ and the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test detects $\gamma_1 \geq 0.1$. As shown in Table 23, the H_n test best detects $\gamma_1 > 0$ for seven alternatives; the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 < 0$ and $\gamma_1 > 0$ for nine and eight alternatives, respectively. The JB, AJB tests best detect $\gamma_1 \neq 0$ for two alternatives. The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n tests best detect $\gamma_1 > 0$ for 13 alternative cases out of 17 (except LOG, NM_1, NM_2 and P_1). The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 > 0$ for B, GP, LCN, SB, SN, SU, W, P_1, P_2. The JB, AJB tests best detect $\gamma_1 \neq 0$ for NM_1, NM_2 and $\gamma_1 > 0$ for the P_1 alternative. See Table 23 for more details.

Table 23. Summary of the results from Tables 6–22 for the analysed alternatives (A). The symbol in bold denotes $\bar{\gamma}_2 > 0$.

A	$\gamma_1 < 0$	$\gamma_1 > 0$	**A**	$\gamma_1 < 0$	$\gamma_1 > 0$
B	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n	SN	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
χ^2	n/a	H_n	SU	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, \delta$
	n/a	H_n	W	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n
GP	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	H_n	ES_1	SF	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
LCN	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, H_n$	ES_2	T_{1n}	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
LOG	n/a	T_{1n}	P_1	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, SF$	SF, JB, AJB
NM_1	SF, JB, AJB	SF, JB, AJB	P_2	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
NM_2	JB, AJB	JB, AJB	P_3	B_v	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$
SB	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$	$\mathrm{LF}_{\overline{\alpha},\overline{\beta}}, H_n$			

Source: author's work.

5. Summary and conclusions

The article contributes to the expansion of knowledge on GoFTs for normality. The study considers situations where the alternatives are asymmetric with non-constant skewness. At first, GoFTs were assessed with respect to their ability to detect samples for two reasons:

- they come from general populations where the alternatives with skewness values are close to zero or where the lowest possible skewness values occur, and
- the value of the normal-alternative similarity measure is close to unity.

Having already assessed the abilities of GoFTs, 21 of them were selected as a set of GoFTs to be applied to detect asymmetric alternatives with non-constant skewness.

Subsequently, a set of 13 alternatives was formed. These were distinguished as useful in deviation-from-normality-oriented Monte Carlo studies. Among them were alternatives of only negative skewness, only positive skewness or both negative and positive skewness. The alternatives in question fall into two categories: monolithic and compound distributions.

When describing a given distribution, the main emphasis was placed on defining formulas for skewness and its range. The (global) values of the similarity measure of the alternative to the normal distribution were determined.

The Monte Carlo study revealed that when alternatives are asymmetric with non-constant γ_1, GoFTs for normality detect positive or negative γ_1 differently, depending on the alternative. The H_n test best detects $\gamma_1 > 0$ for seven alternatives; the $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 < 0$ and $\gamma_1 > 0$ for nine and eight alternatives, respectively. The JB, AJB tests best detect $\gamma_1 \neq 0$ for two alternatives.

The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n tests best detect $\gamma_1 > 0$ in 13 alternative cases out of 17 (except the LOG, NM_1, NM_2 and P_1 alternatives). The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ test best detects $\gamma_1 > 0$ for $B, GP, LCN, SB, SN, SU, W, P_1, P_2$. The JB, AJB tests best detect $\gamma_1 \neq 0$ for NM_1, NM_2 and $\gamma_1 > 0$ for alternative P_1.

The $\mathrm{LF}_{\overline{\alpha},\overline{\beta}}$ and H_n GoFTs best detect asymmetric distributions that deviate from normality due to small skewness, equal to even 0.05.

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Appendix

ES distribution

The Edgeworth series (ES) distribution is defined as:

$$f_{ES}(x) = \phi(x;0,1)\left[1+\sum_{i=3}^{\infty}\frac{1}{i!}\chi_i H_i(x)\right], \tag{A1}$$

where χ_i $(i = 3,4,\dots)$ are cumulants and $H_i(x)$ $(i = 3,4,\dots)$ are the probabilist's Hermite polynomials defined by recurrence relations

$$H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1, \dots, H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

For the purposes of the simulation, we need the first three terms of the series. Then (A1) takes the following form:

$$f_{ES}(x) = \phi(x;0,1)\left(1+\frac{1}{3!}\chi_3 H_3(x)+\frac{1}{4!}\chi_4 H_4(x)\right), \tag{A2}$$

where $H_3(x) = x^3 - 3x, H_4(x) = x^4 - 6x^2 + 3$ and $\chi_3 = \gamma_{1,}\ \chi_4 = \bar{\gamma}_2$. The PDF of the ES distribution based on (A2), is given by:

$$f_{ES}(x;\gamma_{1,},\bar{\gamma}_2) = \phi(x;0,1)\left(1+\tfrac{1}{3!}\gamma_{1,}(x^3-3x)+\tfrac{1}{4!}\bar{\gamma}_2(x^4-6x^2+3)\right).$$

P distribution

The P distribution is defined as:

$$f_P(x;\gamma_{1,},\bar{\gamma}_2) = exp\left[-\int\frac{x+b}{ax^2+bx+c}dx\right], \tag{A3}$$

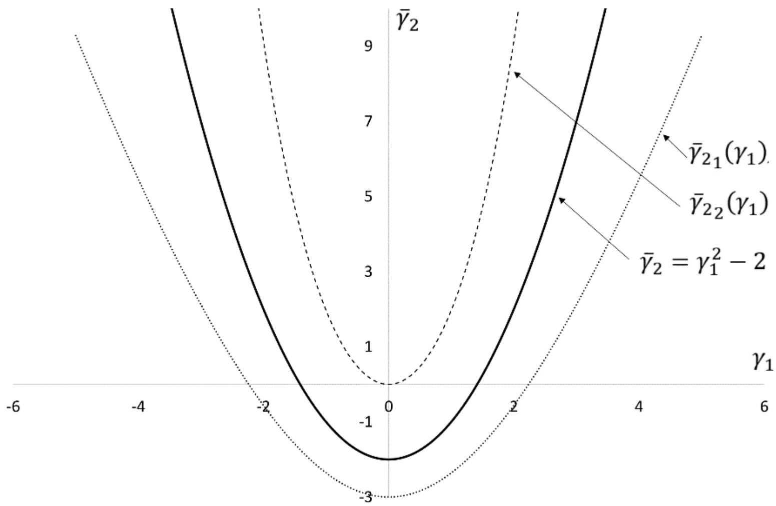
where a, b, c are given by (2). Let us consider three cases determined by the sign of the discriminant (and hence the number of real roots) of $ax^2 + bx + c$.

Case 1. $\Delta = 0 \Leftrightarrow b^2 = 4ac$. When solving this equation, we obtain the following:

$$\bar{\gamma}_{2_1} = \frac{6\sqrt{(\gamma_1^2+4)^3}-42\gamma_1^2+48}{\gamma_1^2-32},\quad \bar{\gamma}_{2_2} = \frac{-6\sqrt{(\gamma_1^2+4)^3}-42\gamma_1^2+48}{\gamma_1^2-32}.$$

The figure shows that the graph of function $\bar{\gamma}_{2_1}(\gamma_1)$ is located outside Malakhov's area $\bar{\gamma}_2 \geq \gamma_1^2 - 2$ (Malachov, 1978).

Figure. Excess kurtosis as a function of skewness when $b^2 = 4ac$



Source: author's work.

When substituting $\bar{\gamma}_2$ to (2), we obtain the following:

$$a = \frac{\gamma_1^4 - 4\gamma_1^2 + 4\sqrt{(\gamma_1^2+4)^3} - 32}{4\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}, \quad b = \frac{|\gamma_1|\left[6\gamma_1^2 + \sqrt{(\gamma_1^2+4)^3} + 24\right]}{2\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}, \quad c = \frac{\gamma_1^4 + 20\gamma_1^2 + 8\sqrt{(\gamma_1^2+4)^3} + 64}{4\left[\gamma_1^4 + 2\gamma_1^2 + 5\sqrt{(\gamma_1^2+4)^3} - 8\right]}.$$

The integral in (A3) can be written as:

$$\int \frac{x+b}{ax^2+bx+c} dx = 2\int \frac{dx}{2ax+b} + 2b \int \frac{2a-1}{(2ax+b)^2} dx,$$

then

$$\int \frac{x+b}{ax^2+bx+c} dx = \frac{ln(2ax+b)}{a} + \frac{b-2ab}{a(2ax+b)} + C_1.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P(x;\gamma_{1,},\bar{\gamma}_2)=\frac{1}{C_2(2ax+b)^{1/a}}exp\left[\frac{2ab-b}{a(2ax+b)}\right],$$

where C_2 is given by (3).

Case 2. $\Delta < 0 \Leftrightarrow b^2 < 4ac$.

The integral in (A3) can be written as:

$$\int\frac{x+b}{ax^2+bx+c}dx=\frac{1}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\frac{2ab-b}{2a}\int\frac{dx}{ax^2+bx+c},$$

then

$$\int\frac{x+b}{ax^2+bx+c}dx=\frac{ln(ax^2+bx+c)}{2a}+\frac{2ab-b}{a\sqrt{4ac-b^2}}tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C_3.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P(x;\gamma_{1,},\bar{\gamma}_2)=\frac{exp\left[\frac{b-2ab}{a\sqrt{4ac-b^2}}tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)\right]}{C_4(ax^2+bx+c)^{1/(2a)}},$$

where C_4 is given by (4).

Case 3. $\Delta > 0 \Leftrightarrow b^2 > 4ac$.

The integral in (A3) can be written as

$$\int\frac{x+b}{ax^2+bx+c}dx=\frac{1}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\frac{2ab-b}{2a}\int\frac{dx}{ax^2+bx+c}= \\ =I_1+I_2, \qquad \text{(A4)}$$

where

$$I_1=\frac{1}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx=\frac{ln(ax^2+bx+c)}{2a}+C_5,$$

$$I_2=\frac{2ab-b}{2a}\int\frac{dx}{ax^2+bx+c}.$$

Since

$$\frac{1}{ax^2+bx+c}=\frac{2a}{\sqrt{\Delta}}\left(\frac{1}{2ax+b-\sqrt{\Delta}}-\frac{1}{2ax+b+\sqrt{\Delta}}\right),$$

then

$$I_2=\frac{2ab-b}{2a\sqrt{\Delta}}ln\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)+C_6.$$

The integral in (A3), based on (A4), is given by:

$$\int\frac{x+b}{ax^2+bx+c}dx=\frac{ln(ax^2+bx+c)}{2a}+\frac{2ab-b}{2a\sqrt{\Delta}}ln\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)+C_7.$$

The PDF of the P distribution based on (A3) is given by:

$$f_P(x;\gamma_{1,},\bar{\gamma}_2)=\frac{\left(\frac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right)^{\frac{b-2ab}{2a\sqrt{\Delta}}}}{C_8(ax^2+bx+c)^{1/(2a)}},$$

where C_8 is given by (5).

R codes

```
h=function(x) ((x-1)/(x+1))^2
Hn=function(x) {
   x=sort((x-mean(x))/sd(x))
   n=length(x)
   Fn=1+1:n/n
   F1=pnorm(x,0,1)+1
   return(mean(h(F1/Fn))) }

Fn=function(i,n,a,b) ((i - a)/(n - a - b + 1))
LF=function(x,alfa,beta) {
   x=sort(x); n=length(x)
   F=pnorm(x, mean(x), sd(x))
   return(max(abs(Fn(1:n,n,alfa,beta)-F))) }

RJ=function(x) {}
   x=sort(x); n=length(x)
   z=qnorm(Fn1(1:n,n,3/8,3/8),0,1); s1=sum(x*z); s2=sum(z*z)
   return(s1/sqrt(s2*(n-1)*var(x))) }
```

```
W1=function(u) qnorm(u)^2-1
T1n=function(x) {
   x=sort(x); n=length(x)
   if (n==25) A1=-0.2114 else A1=-0.1297
   if (n==25) B1=0.2323 else B1=0.34
   s=sd(x)*sqrt((n-1)/n)
   Fn=1:n/(n+1)
   Cn=sum((W1(Fn)-A1)*x)/sqrt(n)
   return(Cn^2/s^2/B1) }

TestSigma=function(x) {
   x=sort(x); Ft=pnorm(x,mean(x),sd(x))
   n=length(x); Fn=1:n/n
   licz=sum((abs(Ft-Fn))); mian=0
   for (i in 1:n) {
      mian=mian+max(Ft[i],Fn[i]) }
   return(licz/mian) }

Bv=function(x) {
   x=sort(x); n=length(x)
   mi=mean(x); sdev=sd(x)*sqrt((n-1)/n)
   if (n==25) m=5 else m=15; s=0
   for (i in 1:n) {
      up=i-m; if (up$\mathrm{<}$1) up=1
      uk=i+m; if(uk$\mathrm{>}$n) uk=n
      a=2*m/(x[uk]-x[up])/n
      b=exp(-0.5*((x[i]-mi)/sdev)^2)/sdev/sqrt(2*pi)
      s=s+((a-b)/(a+b))^2 }
   return(s/n) }
rGP=function(n,a,b) {
   x=numeric(n)
   for (i in 1:n){
      W=rgamma(1,1/b)
      d=dG(a,b)
      V=(W/d)^(1/b)
      x[i]=ifelse(runif(1,0,1)<1-a,(1-a)*V,-a*V) }
   return(sort(x)) }
```

```
rLCN=function(n,a,c) {
   x=numeric(n)
   for (i in 1:n){
      x[i]=ifelse(runif(1,0,1)<c,rnorm(1,a,1),rnorm(1,0,1)) }
   return(sort(x)) }

rNM=function(n,a,b,c) {
   x=numeric(n)
   for (i in 1:n){
      x[i]=ifelse(runif(1,0,1)<c,rnorm(1,0,1),rnorm(1,a,b)) }
   return(sort(x)) }

dEdge=function(x,a,b){
   return(dnorm(x,0,1)*(1+a*(x^3-3*x)/6+b*(x^4-6*x^2+3)/24)) }
rEdge=function(n,a,b,xl,xu){ #with support (xl,xu)
   wyn=numeric(n)
   e=optimize(function(x)
dEdge(x,a,b),interval=c(xl,xu), maximum=1)$maximum
   d=dEdge(e,a,b)
   for (i in 1:n){
      R1 = runif(1,xlow,xup)
      R2 = runif(1,0,d)
      w = dEdge(R1,a,b)
      while(w<R2){
      R1 = runif(1,xlow,xup)
      R2 = runif(1,0,d)
      w = dEdge(R1,a,b) }
      wyn[i]=R1 }
   return(sort(wyn)) }
```