STATISTICS IN TRANSITION new series, March 2023
Vol. 24, No. 2, pp. 141–157, DOI 10.59170/stattrans-2023-025
Received – 29.06.2020; accepted – 29.07.2021

On some efficient classes of estimators using auxiliary attribute

Shashi Bhushan[1], **Anoop Kumar**[2]

ABSTRACT

This paper considers some efficient classes of estimators for the estimation of population mean using known population proportion. The usual mean estimator, classical ratio, and regression estimators suggested by Naik and Gupta (1996) and Abd-Elfattah *et al.* (2010) estimators are identified as the members of the suggested class of estimators. The expressions of bias and mean square errors are derived up to first-order approximation. The proposed estimators were put to test against various other competing estimators till date. It has been found both theoretically and empirically that the suggested classes of estimators dominate the existing estimators.

Key words: Bias, Mean square error, Efficiency, Auxiliary attribute.

1. Introduction

In sample survey methodology, it is well known that the information on the auxiliary variable helps to meliorate the efficiency of the estimators. Literature comprises several improved and modified ratio, regression, product and exponential type estimators in this dimension. Some contemporary relevant studies in this direction, namely, Zaman and Kadilar (2020), Bhushan and Kumar (2020, 2022), Bhushan et al. (2020a, b, c, d, e), Bhushan et al. (2021), Zaman and Kadilar (2021a, b) can be viewed. However, many times, in real life situations, the variable of interest may be associated with some qualitative auxiliary characteristics that might be easily available. For example:

(i). The height of the person (y) may rely on sex ϕ i.e. the person is male or female.

(ii). The amount of yield of paddy crop (y) may rely on a certain variety of paddy (ϕ).

(iii). The amount of milk produce (y) may depend on a certain breed of buffalo (ϕ).

(iv). The use of drugs (y) may depend on the sex (ϕ).

Furthermore, if measuring a quantitative variable is expensive, then such an auxiliary attribute may be considered, which can be constructed from the auxiliary variable and is highly associated with the variable of interest. For example:

(i). The yield of crop (y) may depend on large/small land holdings (ϕ).

[1]Department of Statistics, University of Lucknow, Lucknow, India, 226007. E-mail: bhushan_s@lkouniv.ac.in, ORCID: https://orcid.org/0000-0002-3931-888X.

[2]Corresponding author. Department of Statistics, Amity University, Lucknow, India, 226028. E-mail: anoop.asy@gmail.com, ORCID: https://orcid.org/0000-0003-2775-6548.

(ii). The tax paid by a company (y) may depend on its turn over (ϕ) which can be converted into large/small company.

(iii). The family expenditure (y) may depend on the household size (ϕ) which can be classified large/small household.

Thus, by considering the advantage of bi-serial correlation (ρ), Naik and Gupta (1996) suggested the classical ratio, product and regression estimators using the knowledge on auxiliary attribute. Later on, Jhajj *et al.* (2006) considered a general class of estimators for population mean using auxiliary attribute. Singh *et al.* (2007) developed exponential type ratio and product estimators of population mean using auxiliary attribute. Abd-Elfattah *et al.* (2010) introduced some modified estimators of population mean based on attribute. Grover and Kaur (2011) mooted an exponential type estimators of population mean using auxiliary attribute. Singh and Solanki (2012) investigated a general class of estimators of population mean using auxiliary attribute. Koyuncu (2012) suggested an efficient estimators of population mean based on auxiliary attribute. Zaman and Toksoy (2019) suggested ratio type estimators of population mean utilizing bivariate auxiliary attribute. Following Bahl and Tuteja (1991), Zaman and Kadilar (2019) suggested attribute based exponential ratio estimator of population mean. Yadav and Zaman (2020) extended the work of Zaman and Kadilar (2019) utilizing auxiliary attribute. Zaman (2019a) suggested improved ratio estimator of population mean utilizing skewness as an auxiliary attribute. Zaman (2019b) investigated an efficient class of estimator in stratified sampling based on attribute whereas following Ozel (2016), Zaman (2020) suggested a ratio cum exponential ratio type estimator using attribute. Further, Bhushan and Gupta (2020) developed logarithmic type estimators of population mean based on attribute. Recently, Zaman (2021) addressed an efficient exponential estimator of population mean in stratified random sampling.
This paper addresses the problem of estimating the population mean using information on an auxiliary attribute. The rest of the paper is drafted in the following sections. In Section 2, we have discussed the existing estimators along with their properties. In Section 3, we have suggested some efficient classes of estimators with their properties. The theoretical conditions are derived in Section 4 whereas an empirical study is conducted in Section 5. The final conclusion is made in Section 6.

2. Existing estimators

Consider a finite population $U = (U_1, U_2, \ldots, U_N)$ based on N units from which a sample s of size n is measured using the simple random sampling without replacement ($SRSWOR$) scheme. Let y_i and ϕ_i be the total number of units of the study variable y and the auxiliary attribute ϕ for unit i of the population U. It is to be noted that ϕ_i=1 if the unit i owns the attribute ϕ and $\phi_i = 0$, otherwise. Let $X = \sum_{i=1}^{N} \phi_i$ and $x = \sum_{i=1}^{n} \phi_i$ be the total number of units in the population U and sample s, respectively, possessing attribute ϕ whereas $P = X/N$ and $p = x/n$, respectively, denote the populatiton proportion and sample proportion having attribute ϕ. Let $\bar{Y} = N^{-1} \sum_{i=1}^{N} y_i$ and $\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$, respectively, be the population mean and sample mean of study variable y, $S_y^2 = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2$ and $s_y^2 = (n-1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2$, respectively, be the population mean square and sample mean square of study variable y, $S_\phi^2 = (N-1)^{-1}\sum_{i=1}^{N}(\phi_i - P)^2$ and $s_\phi^2 = (n-1)^{-1}\sum_{i=1}^{n}(\phi_i - p)^2$, respectively, be the population mean square and sample mean square of auxiliary attribute ϕ, $S_{y\phi} = (N-1)^{-1}(\sum_{i=1}^{N} y_i\phi_i - NP\bar{Y})$ and $s_{y\phi} = (n-1)^{-1}(\sum_{i=1}^{n} y_i\phi_i - np\bar{y})$ respectively be the population covariance and sample covariance between study variable y and attribute ϕ, $C_y = S_y/\bar{Y}$ and $C_\phi = S_\phi/P$, respectively, be the population coefficient of variation of study variable y and attribute ϕ and $\rho = S_{y\phi}/(S_y S_\phi)$ be the point biserial correlation coefficient between study variable y and attribute ϕ.

To obtain the bias and mean square error (MSE) of various estimators, let us assume that, $\bar{y} = \bar{Y}(1+e_0)$ and $p = P(1+e_1)$ such that $E(e_0) = E(e_1) = 0$ and $E(e_0^2) = \gamma C_y^2$, $E(e_1^2) = \gamma C_\phi^2$, $E(e_0 e_1) = \gamma\rho C_y C_\phi$, where $\gamma = (N-n)/Nn$.

The usual mean estimator is given by

$$t_m = \bar{y} \tag{2.1}$$

Naik and Gupta (1996) suggested the classical ratio and regression estimator for population mean $\bar{Y}$ using information on the auxiliary attribute as

$$t_r = \bar{y}\left(\frac{P}{p}\right) \tag{2.2}$$

$$t_{lr} = \bar{y} + \beta_\phi(P-p) \tag{2.3}$$

where $\beta_\phi = S_{y\phi}/S_\phi^2$ is the regression coefficient of y on ϕ.

Following Hansen *et al.* (1954), Srivastava (1967), Walsh (1970) and Bhushan and Gupta (2019), one may introduce the following class of estimators based on the auxiliary attribute as

$$t_1 = \bar{y} + \theta(p^* - P^*) \tag{2.4}$$

$$t_2 = \bar{y}\left(\frac{P^*}{p^*}\right)^{\Lambda} \tag{2.5}$$

$$t_3 = \bar{y}\left\{\frac{P^*}{P^* + \Delta(p^* - P^*)}\right\} \tag{2.6}$$

$$t_4 = \bar{y}\left\{1 + \log\left(\frac{p^*}{P^*}\right)\right\}^{\daleth} \tag{2.7}$$

where θ, Λ, Δ and $\daleth$ are suitably chosen scalars to be determined later. Furthermore, $p^* = \eta p + \lambda$ and $P^* = \eta P + \lambda$ such that η and λ are any real values or function of some known parameters of the auxiliary attribute ϕ, namely, population standard deviation S_ϕ, population coefficient of variation C_ϕ, population coefficient of skewness $\beta_1(\phi)$, population coefficient of kurtosis $\beta_2(\phi)$ and population point biserial correlation coefficient ρ between study variable y and attribute ϕ.

Jhajj *et al.* (2006) considered the class of estimators for population mean $\bar{Y}$ as

$$t_J = h(\bar{y}, u) \tag{2.8}$$

where $u = p/P$ and $h(\bar{y}, u)$ are the function of $(\bar{y}, u)$ such that $h(\bar{y}, 1) = \bar{Y}, \forall \bar{Y}$ and the function $h(\bar{y}, u)$ satisfies certain regularity conditions as mentioned in Jhajj *et al.* (2006).
On the lines of Bahl and Tuteja (1991), Singh *et al.* (2007) suggested ratio and product exponential type estimators as

$$t_{sr} = \bar{y} \exp\left(\frac{P-p}{P+p}\right) \tag{2.9}$$

$$t_{sp} = \bar{y} \exp\left(\frac{p-P}{p+P}\right) \tag{2.10}$$

Singh *et al.* (2008) envisaged a class of estimator using information on auxiliary attribite as

$$t_s = [\bar{y} + \beta_\phi(P-p)]\left(\frac{P^*}{p^*}\right) \tag{2.11}$$

Some members of the estimator t_s are given in Table 1 for ready reference.
Abd-Elfattah *et al.* (2010) also suggested the following classes of estimators for the population mean using information on auxiliary attribute as

$$t_{a_1} = m_1\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P}{p}\right) + m_2\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right) \tag{2.12}$$

$$t_{a_2} = m_1\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P}{p}\right) + m_2\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P+C_\phi}{p+C_\phi}\right) \tag{2.13}$$

$$t_{a_3} = m_1\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P}{p}\right) + m_2\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{\beta_2(\phi)P+C_\phi}{\beta_2(\phi)p+C_\phi}\right) \tag{2.14}$$

$$t_{a_4} = m_1\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{P}{p}\right) + m_2\{\bar{y} + \beta_\phi(P-p)\}\left(\frac{C_\phi P+\beta_2(\phi)}{C_\phi p+\beta_2(\phi)}\right) \tag{2.15}$$

where m_1 and m_2 are weights such that $m_1 + m_2 = 1$.
Furthermore, Abd-Elfattah *et al.* (2010) envisaged the following class of estimators as

$$t_{a_5} = \bar{y}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right) \tag{2.16}$$

$$t_{a_6} = \bar{y}\left(\frac{P+C_\phi}{p+C_\phi}\right) \tag{2.17}$$

$$t_{a_7} = \bar{y}\left(\frac{\beta_2(\phi)P+C_\phi}{\beta_2(\phi)p+C_\phi}\right) \tag{2.18}$$

$$t_{a_8} = \bar{y}\left(\frac{C_\phi P+\beta_2(\phi)}{C_\phi p+\beta_2(\phi)}\right) \tag{2.19}$$

$$t_{a_9} = \bar{y}\left(\frac{P+\rho}{p+\rho}\right) \tag{2.20}$$

Following Kadilar and Cingi (2006), one can define some more estimators like the ratio type estimators $t_{a_i},\ i = 5, 6, ..., 9$ as

$$t_{a_{10}} = \bar{y}\left(\frac{C_\phi P + \rho}{C_\phi p + \rho}\right) \tag{2.21}$$

$$t_{a_{11}} = \bar{y}\left(\frac{\rho P + C_\phi}{\rho p + C_\phi}\right) \tag{2.22}$$

$$t_{a_{12}} = \bar{y}\left(\frac{\beta_2(\phi) P + \rho}{\beta_2(\phi) p + \rho}\right) \tag{2.23}$$

$$t_{a_{13}} = \bar{y}\left(\frac{\rho P + \beta_2(\phi)}{\rho p + \beta_2(\phi)}\right) \tag{2.24}$$

$$t_{a_{14}} = \bar{y}\left(\frac{S_\phi P + \beta_2(\phi)}{S_\phi p + \beta_2(\phi)}\right) \tag{2.25}$$

Solanki and Singh (2013) suggested the exponential type estimator as

$$t_{ss} = \bar{y}\exp\left\{\frac{\alpha(P - p)}{(P + p)}\right\} \tag{2.26}$$

where α is a suitably chosen scalar.
Zaman (2019a) introduced some improved general class of estimators based on the coefficient of skeness of auxiliary attribte as

$$t_{z_1} = \omega\bar{y}\left(\frac{P + \beta_1(\phi)}{p + \beta_1(\phi)}\right) + (1 - \omega)\bar{y}\left(\frac{\beta_2(\phi) P + \beta_1(\phi)}{\beta_2(\phi) p + \beta_1(\phi)}\right) \tag{2.27}$$

$$t_{z_2} = \omega\bar{y}\left(\frac{P + \beta_1(\phi)}{p + \beta_1(\phi)}\right) + (1 - \omega)\bar{y}\left(\frac{\beta_1(\phi) P + \beta_2(\phi)}{\beta_1(\phi) p + \beta_2(\phi)}\right) \tag{2.28}$$

$$t_{z_3} = \omega\bar{y}\left(\frac{P + \beta_1(\phi)}{p + \beta_1(\phi)}\right) + (1 - \omega)\bar{y}\left(\frac{C_\phi P + \beta_1(\phi)}{C_\phi p + \beta_1(\phi)}\right) \tag{2.29}$$

$$t_{z_4} = \omega\bar{y}\left(\frac{P + \beta_1(\phi)}{p + \beta_1(\phi)}\right) + (1 - \omega)\bar{y}\left(\frac{\beta_1(\phi) P + C_\phi}{\beta_1(\phi) p + C_\phi}\right) \tag{2.30}$$

where ω is a suitably chosen scalar.
Zaman and Kadilar (2019) suggested a family of ratio exponential estimator as

$$t_{zk} = \bar{y}\exp\left[\frac{(\eta P + \lambda) - (\eta p + \lambda)}{(\eta P + \lambda) + (\eta p + \lambda)}\right] \tag{2.31}$$

where η and λ are same as defined earlier.
Following Zaman and Kadilar (2019), Yadav and Zaman (2020) introduced a general class of estimators of population mean using the auxiliary attribute as

$$t_{yz} = k_1\bar{y} + k_2\bar{y}\exp\left[\frac{(\eta P + \lambda) - (\eta p + \lambda)}{(\eta P + \lambda) + (\eta p + \lambda)}\right] \tag{2.32}$$

where k_1 and k_2 are suitably chosen scalars such that $k_1+k_2=1$. Furthermore, some members of the estimator t_{zk} and t_{yz} are given in Table 1 for ready reference.
Following Ozel (2016), Zaman (2020) suggested exponential ratio type estimator using auxiliary attribute as

$$t_{z_5}=\bar{y}\left(\frac{p}{P}\right)^{\theta}\exp\left[\frac{(\eta P+\lambda)-(\eta p+\lambda)}{(\eta P+\lambda)+(\eta p+\lambda)}\right] \tag{2.33}$$

where θ is a suitably chosen scalar and η and λ are the same as defined earlier.
Bhushan and Gupta (2020) suggested the following family of estimators for the estimation of population mean $\bar{Y}$ as

$$t_{bg}=\left[w_1\bar{y}+w_2\left(\frac{p}{P}\right)\right]\left[1+\alpha\log\left(\frac{p^*}{P^*}\right)\right] \tag{2.34}$$

where w_1, w_2 and α are suitably chosen scalars. In addition, some members of the estimator t_{bg} are given in Table 1 for ready reference.
We would like to note that the minimum MSE of the estimators $t_i,\ i=1,2,3,4$ envisaged on the lines of Hansen *et al.* (1954), Srivastava (1967), Walsh (1970) and Bhushan and Gupta (2019), Jhajj *et al.* (2006) estimator t_J, Abd-Elfattah *et al.* (2010) estimators $t_{a_i},\ i=1,2,3,4$, Solanki and Singh (2013) estimator t_{ss}, Zaman (2019a) estimators $t_{z_i},\ i=1,2,3,4$, Yadav and Zaman (2020) estimators t_{yz} and Zaman (2020) estimator t_z attain the minimum MSE of the classical regression estimator t_{lr}.
The MSE of the above estimators are given in Appendix A for quick reference.

3. Proposed Estimators

Adapting the procedure of Kadilar and Cingi (2006), we introduce the following classes of estimators by combining the difference estimator given in (2.4) with the log type, Srivastava and Walsh type estimator given in (2.7), (2.5) and (2.6) and Srivastava and Walsh type estimators given in (2.5) and (2.6).

$$t_{k_1}=\zeta_1\{\bar{y}+\theta(p^*-P^*)\}+\psi_1\bar{y}\left\{1+\log\left(\frac{p^*}{P^*}\right)\right\}^{\daleth} \tag{3.1}$$

$$t_{k_2}=\zeta_2\{\bar{y}+\theta(p^*-P^*)\}+\psi_2\bar{y}\left(\frac{P^*}{p^*}\right)^{\Lambda} \tag{3.2}$$

$$t_{k_3}=\zeta_3\{\bar{y}+\theta(p^*-P^*)\}+\psi_3\bar{y}\left(\frac{P^*}{P^*+\Delta(p^*-P^*)}\right) \tag{3.3}$$

$$t_{k_4}=\zeta_4\bar{y}\left(\frac{P^*}{p^*}\right)^{\Lambda}+\psi_4\bar{y}\left(\frac{P^*}{P^*+\Delta(p^*-P^*)}\right) \tag{3.4}$$

where ζ_i, ψ_i, $i=1,2,3,4$, θ, Λ, Δ and $\daleth$ are suitably chosen scalars. The proposed classes of estimators are contemporary and novel in nature as they proposed to utilize the relationship of a quantitative variable with qualitative characteristics i.e., attribute. Moreover, the proposed classes explore various linear and non-linear relationships including exponential and logarithmic relationships. Furthermore, these proposed classes of estimators are general in nature and possess various prominent classes of estimators as their special cases like linear regression estimator, logarithmic type, Srivastava, and Walsh type estimators and their linear combinations. Further, the classes of estimators $t_{k_i},\ i = 1,2,3,4$ are reduced into some existing estimators for different values of scalars as:

(i). for (ζ_1, ψ_1, θ, $\daleth$)=(1, 0, 0, 0); $t_{k_1} \rightarrow t_m$

(ii). for (ζ_2, ψ_2, θ, Λ, η, λ)=(1, 0, -β_ϕ, 0, 1, 0); $t_{k_2} \rightarrow t_{lr}$

(iii). for (ζ_2, ψ_2, θ, Λ, η, λ)=(0, 1, 0, 1, 1, 0); $t_{k_2} \rightarrow t_r$

(iv). for (ζ_4, ψ_4, Λ, Δ, η, λ)=(1, 1, 0, 0, η, λ); $t_{k_4} \rightarrow t_{a_i},\ i = 5,6,...,13$

Several other estimators can be generated for different values of scalars. Some members of the proposed classes of estimators are given in Table 1.

Theorem 3.1. *The bias and MSE of the suggested classes of estimators* $t_{k_i},\ i = 1,2,3,4$ *are given as*

$$Bias(t_{k_i}) = \bar{Y}\left[\ \zeta_i I_i + \psi_i J_i - 1\ \right] \tag{3.5}$$

$$MSE(t_{k_i}) = \bar{Y}^2\left[1 + \zeta_i^2 F_i + \psi_i^2 G_i + 2\zeta_i\psi_i H_i - 2\zeta_i I_i - 2\psi_i J_i\right] \tag{3.6}$$

Proof. Refer to Appendix *B* for the outline of the derivation and definitions of parametric functions ζ_i, ψ_i, F_i, G_i, H_i, I_i and J_i. □

Corollary 3.1. *The minimum MSE of the suggested classes of estimators* $t_{k_i},\ i = 1,2,3,4$ *are given as*

$$minMSE(t_{k_i}) = \bar{Y}^2\left[1 - \frac{(F_i J_i^2 + G_i I_i^2 - 2H_i I_i J_i)}{(F_i G_i - H_i^2)}\right] \tag{3.7}$$

Proof. Refer to Appendix *B*. □

Table 1: Some members of the existing and proposed classes of estimators

Value of		Class of estimators				
η	λ	Singh *et al.* (2008) estimators t_{s_j}	Zaman and Kadilar (2019) estimators t_{zk_j}	Yadav and Zaman (2020) estimators t_{yz_j}	Bhushan and Gupta (2020) estimators t_{bg_j}	Proposed estimators $t_{k_{i(j)}},\ i = 1,2,3,4$
1	$\beta_2(\phi)$	t_{s_1}	t_{zk_1}	t_{yz_1}	t_{bg_1}	$t_{k_{i(1)}}$
1	C_ϕ	t_{s_2}	t_{zk_2}	t_{yz_2}	t_{bg_2}	$t_{k_{i(2)}}$
$\beta_2(\phi)$	C_ϕ	t_{s_3}	t_{zk_3}	t_{yz_3}	t_{bg_3}	$t_{k_{i(3)}}$
C_ϕ	$\beta_2(\phi)$	t_{s_4}	t_{zk_4}	t_{yz_4}	t_{bg_4}	$t_{k_{i(4)}}$
1	ρ	t_{s_5}	t_{zk_5}	t_{yz_5}	t_{bg_5}	$t_{k_{i(5)}}$
C_ϕ	ρ	t_{s_6}	t_{zk_6}	t_{yz_6}	t_{bg_6}	$t_{k_{i(6)}}$
ρ	C_ϕ	t_{s_7}	t_{zk_7}	t_{yz_7}	t_{bg_7}	$t_{k_{i(7)}}$
$\beta_2(\phi)$	ρ	t_{s_8}	t_{zk_8}	t_{yz_8}	t_{bg_8}	$t_{k_{i(8)}}$
ρ	$\beta_2(\phi)$	t_{s_9}	t_{zk_9}	t_{yz_9}	t_{bg_9}	$t_{k_{i(9)}}$
S_ϕ	$\beta_2(\phi)$	$t_{s_{10}}$	$t_{zk_{10}}$	$t_{yz_{10}}$	$t_{bg_{10}}$	$t_{k_{i(10)}}$

4. Efficiency Conditions

On comparing the minimum MSE of the proposed classes of estimators t_{k_i} $i = 1,2,3,4$ given in (3.7) with the minimum MSE of the of existing estimators given in (A.1), (A.2), (A.3), (A.4), (A.5), (A.6), (A.7), (A.9) and (A.10), we get the following conditions.

$$MSE(t_m) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma C_y^2 \tag{4.1}$$

$$MSE(t_r) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma[C_y^2 + C_\phi^2 - 2\rho C_yC_\phi] \tag{4.2}$$

$$MSE(t) \geq MSE(t_{k_i});\ t = t_{lr}, t_J, t_i, t_{a_i}, i = 1,2,3,4, t_{ss}, t_{z_i},\ i = 1,2,...,5 \text{ and } t_{yz}$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - [\gamma C_y^2(1-\rho^2)] \tag{4.3}$$

$$MSE(t_{sr}) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma\left[C_y^2 + \frac{C_\phi^2}{4} - \rho C_yC_\phi\right] \tag{4.4}$$

$$MSE(t_{sp}) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma\left[C_y^2 + \frac{C_\phi^2}{4} + \rho C_yC_\phi\right] \tag{4.5}$$

$$MSE(t_s) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma\left[\upsilon^2C_\phi^2 + C_y^2(1-\rho^2)\right] \tag{4.6}$$

$$MSE(t_{a_i}) \geq MSE(t_{k_i}),\ i = 1,2,3,4$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma\left[C_y^2 + \upsilon^2C_\phi^2 - 2\upsilon\rho C_yC_\phi\right] \tag{4.7}$$

$$MSE(t_{zc}) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq 1 - \gamma\left[\lambda^2C_\phi^2 + C_y^2 - 2\lambda\rho C_yC_\phi\right] \tag{4.8}$$

$$MSE(t_{bg}) \geq MSE(t_{k_i})$$

$$\frac{(F_iJ_i^2 + G_iI_i^2 - 2H_iI_iJ_i)}{(F_iG_i - H_i^2)} \geq \frac{(AG^2 + BD^2 - 2DFG)}{(4AB - F^2)} \tag{4.9}$$

Under the conditions of (4.1) to (4.9), the proposed classes of estimators $t_{k_i},\ i = 1,2,3,4$ dominate the usual mean estimator, classical ratio, product and regression estimator, Jhajj *et al.* (2006) estimator, Singh *et al.* (2007) estimator, Singh *et al.* (2008) estimator, Abd-Elfattah *et al.* (2010) estimators, Solanki and Singh (2013) estimator, Zaman and Kadilar

(2019) estimator, Zaman (2019a, 2020) estimators, Yadav and Zaman (2020) estimators and Bhushan and Gupta (2020) estimator. Further, these conditions are supported with an empirical study using three different populations.

5. Empirical Study

To have clear idea about the properties of the proposed estimators, we consider an empirical study over two real populations. The description of the populations is given below:
Population 1. (Source: Sukhatme and Sukhatme (1970), pp. 256)
y: Number of villages in the circles, ϕ: A circle consisting of more than five villages, N=89, n=23, $\bar{Y}$=3.36, P=0.124, C_y=0.601, C_ϕ=2.678, ρ=0.766 and $\beta_2(\phi)$= 3.492.
Population 2. (Source: Sukhatme and Sukhatme (1970), pp. 256)
y: Area (in acres) under the wheat crop within the circles, ϕ: A circle consisting of more than five villages, N=89, n=23, $\bar{Y}$=1102, P=0.124, C_y=0.65, C_ϕ=2.678, ρ=0.624 and $\beta_2(\phi)$=3.492.
Population 3. (Source: Zaman *et al.* (2014))
y: The number of teachers, ϕ: The number of teachers is more than 60, N=111, n=30, $\bar{Y}$=29.279, P=0.117, C_y=0.872, C_ϕ=2.758, ρ=0.797 and $\beta_2(\phi)$=3.898.
For the above populations, we have calculated the percent relative efficiency (PRE) of various estimators T with respect to the usual mean estimator using the following expression.

$$PRE = \frac{MSE(T)}{MSE(t_m)} \times 100 \tag{5.1}$$

The results of the numerical study for the above populations are disclosed in Table 2. It has been seen from Table 2 that the members $t_{k_{i(j)}}$, i=1,2,3,4; j=1,2,...,10, of the suggested classes of estimators t_{k_i} are superior than:

(i). the usual mean estimator t_m, classical ratio estimator t_r and regression estimator t_{lr} envisaged by Naik and Gupta (1996), Jhajj (2006) estimator t_J, Abd-Elfattah *et al.* (2010) estimators t_{a_i}, $i = 1,2,3,9$, ratio type estimators t_{a_i}, $i = 10,11,12,13,14$ defined on the lines of Kadilar and Cingi (2006), ratio and product exponential estimators t_{sr}, t_{sp} suggested by Singh *et al.* (2007), Solanki and Singh (2013) estimator t_{ss}, Zaman (2019a) estimators t_{z_i}, $i = 1,2,3,4$ and Zaman (2020) estimator t_z.

(ii). the members t_{s_j}, j=1 to 10; of the class of estimators t_s suggested by Singh *et al.* (2008).

(iii). the members t_{zk_j}, j=1 to 10; of the class of estimators t_{zk} introduced by Zaman and Kadilar (2019).

(iv). the members t_{yz_j}, j=1 to 10; of the class of estimators t_{yz} introduced by Yadav and Zaman (2020).

(v). the members t_{bg_j}, j=1 to 10; of the class of estimators t_{bg} investigated by Bhushan and Gupta (2020).

On comparing the findings of Table 2, we have seen that the *PRE* of the members $t_{k_{i(j)}}$, i=1,2,3,4; j=1,2,...,10, of the suggested classes of estimators dominate the estimators discussed in the earlier section. Moreover, we have also observed that the estimator t_{k_1} consisting of the information $(\beta_2(\phi),\ \rho)$ is the most efficient among the suggested classes of estimators in each population.

6. Conclusion

In this paper, we have introduced various novel classes of estimators by combining difference estimator with Srivastava, Walsh and Log type estimators and Srivastava type estimator with Walsh type estimator for estimating the population mean of study variable utilizing the information on an auxiliary attribute and compared them with the relevant contemporary estimators till date. It is important to consider various classes of estimators in a single study so that their relative efficiencies can be compared to get a better understanding regarding the performance of such estimators with the existing estimators. The bias and *MSE* of these estimators are derived up to the first order of approximation. The efficiency conditions have been obtained under which the proposed estimators dominate various estimators available till date. These efficiency conditions are further verified by an empirical study using three real populations. The empirical results show the dominance of the proposed classes of estimators over the usual mean estimator, classical ratio, regression and difference estimators, Srivastava (1967) type estimator, Walsh (1970) type estimator, Jhajj *et al.* (2006) estimator, Singh *et al.* (2007) estimator, Singh *et al.* (2008) estimator, Abd-Elfattah *et al.* (2010) estimator, Solanki and Singh (2013) estimator, Log type estimator envisaged on the lines of Bhushan and Gupta (2016), Zaman and Kadilar (2019) estimator, Yadav and Zaman (2020) estimators, Zaman (2019a, 2020) estimators and Bhushan and Gupta (2020) estimator. The empirical results also show that the estimator t_{k_1} based on the information $(\beta_2(\phi),\ \rho)$ is found to be the most efficient among the proposed classes of estimators in each population. Thus, the proposed classes of estimators are enthusiastically recommended for the estimation of population mean when information is available in the form of auxiliary attribute.

Moreover, the proposed classes of estimators can also be developed in stratified sampling based on auxiliary attribute and it is the authors research work in forthcoming studies.

Acknowledgments

The authors are thankful to the reviewers and the editor for their valuable comments and suggestions that led to improving the article.

Table 2: PRE of different estimators with respect to t_m

Estimators	Populations 1	2	3	Estimators	Populations 1	2	3
t_r	7.100	7.792	16.772	t_{bg_9}	144.303	116.249	146.009
t	241.987	163.766	274.129	$t_{bg_{10}}$	158.350	120.130	156.516
t_{sr}	39.207	37.415	102.029	$\mathbf{t_{k_{1(1)}}}$	**242.987**	**165.043**	**275.675**
t_{sp}	10.664	12.796	16.606	$\mathbf{t_{k_{1(2)}}}$	**242.959**	**165.030**	**275.590**
t_{s_1}	229.046	158.582	267.891	$\mathbf{t_{k_{1(3)}}}$	**243.084**	**165.125**	**275.596**
t_{s_2}	221.175	155.311	262.220	$\mathbf{t_{k_{1(4)}}}$	**242.986**	**165.053**	**275.574**
t_{s_3}	125.330	69.105	106.443	$\mathbf{t_{k_{1(5)}}}$	**243.057**	**165.218**	**275.406**
t_{s_4}	177.630	135.387	236.259	$\mathbf{t_{k_{1(6)}}}$	**244.550**	**166.421**	**276.697**
t_{s_5}	125.208	92.841	189.138	$\mathbf{t_{k_{1(7)}}}$	**242.964**	**165.033**	**275.610**
t_{s_6}	44.861	37.616	83.637	$\mathbf{t_{k_{1(8)}}}$	**245.630**	**167.153**	**278.057**
t_{s_7}	229.074	160.253	266.318	$\mathbf{t_{k_{1(9)}}}$	**242.995**	**165.050**	**275.695**
t_{s_8}	33.302	28.908	59.173	$\mathbf{t_{k_{1(10)}}}$	**243.018**	**165.058**	**275.751**
t_{s_9}	234.100	161.655	270.086	$\mathbf{t_{k_{2(1)}}}$	**243.377**	**165.243**	**276.367**
$t_{s_{10}}$	240.418	163.150	273.439	$\mathbf{t_{k_{2(2)}}}$	**243.437**	**165.273**	**276.528**
t_{a_5}	114.539	110.475	116.065	$\mathbf{t_{k_{2(3)}}}$	**243.598**	**165.334**	**277.041**
t_{a_6}	135.724	124.115	123.244	$\mathbf{t_{k_{2(4)}}}$	**243.497**	**165.296**	**276.642**
t_{a_7}	215.966	143.873	205.746	$\mathbf{t_{k_{2(5)}}}$	**243.965**	**165.524**	**277.833**
t_{a_8}	142.278	127.965	148.582	$\mathbf{t_{k_{2(6)}}}$	**243.603**	**165.155**	**277.953**
t_{a_9}	230.244	162.838	192.843	$\mathbf{t_{k_{2(7)}}}$	**243.412**	**165.254**	**276.480**
$t_{a_{10}}$	133.081	79.247	264.617	$\mathbf{t_{k_{2(8)}}}$	**243.386**	**164.980**	**277.779**
$t_{a_{11}}$	126.667	115.075	118.228	$\mathbf{t_{k_{2(9)}}}$	**243.356**	**165.226**	**276.332**
$t_{a_{12}}$	39.332	30.183	203.993	$\mathbf{t_{k_{2(10)}}}$	**243.313**	**165.211**	**276.245**
$t_{a_{13}}$	110.994	106.513	112.654	$\mathbf{t_{k_{3(1)}}}$	**242.950**	**165.064**	**275.642**
$t_{a_{14}}$	104.650	103.453	104.980	$\mathbf{t_{k_{3(2)}}}$	**242.896**	**165.020**	**275.527**
t_{zk_1}	112.5124	109.071	107.681	$\mathbf{t_{k_{3(3)}}}$	**242.713**	**164.883**	**275.150**
t_{zk_2}	116.458	111.789	110.919	$\mathbf{t_{k_{3(4)}}}$	**242.829**	**164.952**	**275.425**
t_{zk_3}	161.007	138.049	144.370	$\mathbf{t_{k_{3(5)}}}$	**242.558**	**164.748**	**274.924**
t_{zk_4}	134.946	123.647	121.670	$\mathbf{t_{k_{3(6)}}}$	**242.449**	**164.692**	**274.709**
t_{zk_5}	161.081	144.836	139.218	$\mathbf{t_{k_{3(7)}}}$	**242.920**	**165.013**	**275.564**
t_{zk_6}	237.106	161.551	207.797	$\mathbf{t_{k_{3(8)}}}$	**242.433**	**164.688**	**274.663**
t_{zk_7}	112.497	107.393	108.668	$\mathbf{t_{k_{3(9)}}}$	**242.970**	**165.038**	**275.671**
t_{zk_8}	241.118	151.750	241.650	$\mathbf{t_{k_{3(10)}}}$	**243.009**	**165.052**	**275.743**
t_{zk_9}	109.517	105.678	106.104	$\mathbf{t_{k_{4(1)}}}$	**242.123**	**163.990**	**275.007**
$t_{zk_{10}}$	104.069	103.027	102.453	$\mathbf{t_{k_{4(2)}}}$	**242.104**	**164.003**	**274.944**
t_{bg_1}	138.1014	111.875	142.173	$\mathbf{t_{k_{4(3)}}}$	**242.000**	**164.136**	**274.500**
t_{bg_2}	131.138	108.767	135.331	$\mathbf{t_{k_{4(4)}}}$	**242.043**	**164.061**	**274.765**
t_{bg_3}	93.983	89.283	102.100	$\mathbf{t_{k_{4(5)}}}$	**241.999**	**164.176**	**274.550**
t_{bg_4}	109.671	98.167	119.360	$\mathbf{t_{k_{4(6)}}}$	**242.044**	**164.462**	**274.173**
t_{bg_5}	93.951	85.960	104.900	$\mathbf{t_{k_{4(7)}}}$	**242.122**	**163.982**	**274.987**
t_{bg_6}	74.142	75.212	87.121	$\mathbf{t_{k_{4(8)}}}$	**242.110**	**164.559**	**274.129**
t_{bg_7}	138.130	113.963	139.952	$\mathbf{t_{k_{4(9)}}}$	**242.138**	**163.974**	**275.040**
t_{bg_8}	71.899	74.593	85.370	$\mathbf{t_{k_{4(10)}}}$	**242.169**	**163.961**	**275.121**

where $t = t_{lr},\ t_i,\ t_{a_i},\ i = 1,2,3,4,\ t_J,\ t_{ss},\ t_{z_i},\ i = 1,2,...,5$ and $t_{yz_i},\ i = 1,2,...,10$

References

Abd-Elfattah, A.M., El-Sherpieny, E.A., Mohamed, S.M., Abdou, O.F., (2010). Improvement in estimating the population mean in simple random sampling using information on auxiliary attribute. *Applied Mathematics and Computation*, 215, pp. 4198–4202.

Bahl, S., Tuteja, R.K., (1991). Ratio and product type exponential estimator. *Information and Optimization Sciences*, 12, pp. 159–163.

Bhushan, S., Gupta, R., (2019). Some log-type classes of estimators using auxiliary attribute. *Advances in Computational Sciences and Technology*, 12(2), pp. 99–108.

Bhushan, S., Gupta, R., (2020). An improved log-type family of estimators using attribute. *Journal of Statistics and Management Systems*, 23(3), pp. 593–602.

Bhushan, S., Kumar, A., (2020). Log type estimators of population mean under ranked set sampling. *Predictive Analytics using Statistics and Big Data: Concepts and Modeling*, 28, pp. 47–74.

Bhushan, S., Kumar, A., (2022). On optimal classes of estimators under ranked set sampling. *Communications in Statistics - Theory and Methods*, 51(8), pp. 2610–2639.

Bhushan, S., Gupta, R., Singh, S., Kumar, A., (2020a). A modified class of log-type estimators for population mean using auxiliary information on variables. *International Journal of Applied Engineering Research*, 15(6), pp. 612–627.

Bhushan, S., Gupta, R., Singh, S., Kumar, A., (2020b). Some improved classes of estimators using auxiliary information. *International Journal for Research in Applied Science & Engineering Technology*, 8(VI), pp. 1088–1098.

Bhushan, S., Gupta, R., Singh, S., Kumar, A., (2020c). A new efficient log-type class of estimators using auxiliary variable. *International Journal of Statistics and Systems*, 15(1), pp. 19–28.

Bhushan, S., Gupta, R., Singh, S., Kumar, A., (2020d). Some new improved classes of estimators using multiple auxiliary information. *Global Journal of Pure and Applied Mathematics*, 16(3), pp. 515–528.

Bhushan, S., Gupta, R., Singh, S., Kumar, A., (2020e). Some log-type classes of estimators using multiple auxiliary information. *International Journal of Scientific Engineering and Research*, 8(6), pp. 12–17.

Bhushan, S., Kumar, A., Singh, S., (2021). Some efficient classes of estimators under stratified sampling. *Communications in Statistics - Theory and Methods*, pp. 1–30. DOI:https://doi.org/10.1080/03610926.2021.1939052.

Grover, L. K., Kaur, P., (2011). An improved exponential estimator of finite population mean in simple random sampling using an auxiliary attribute. *Applied Mathematics and Computation*, 218(7), pp. 3093–3099.

Hansen, M. H. Hurwitz, W. N., Madow, W. G., (1953). *Sample Survey Methods and Theory*. John Wiley and Sons, New York, U.S.A.

Jhajj, H. S., Sharma, M. K., Grover, L. K., (2006). A family of estimators of population mean using information on auxiliary attribute. *Pakistan Journal of Statistics*, 22, pp. 43–50.

Kadilar, C., Cingi, H., (2006). Improvement in estimating the population mean in simple random sampling. *Applied Mathematics Letters*, 19, pp. 75–79.

Koyuncu, N., (2012). Efficient estimators of population mean using auxiliary attributes. *Applied Mathematics and Computation*, 218(22), pp. 10900–10905.

Naik, V. D., Gupta, P.C., (1996). A note on estimation of mean with known population proportion of an auxiliary character. *Journal of Indian Society of Agricultural Statistics*, 48(2), pp. 151–158.

Ozel, K. G., (2016). A new exponential type estimator for the population mean in simple random sampling. *Journal of Modern Applied Statistical Methods*, 15(2), pp. 207–214.

Solanki, R. S., Singh, H. P., (2013). Improved estimation of population mean using population proportion of an auxiliary character. *Chilean Journal of Statistics*, 4(1), pp. 3–17.

Singh, R., Chouhan, P., Sawan, N., Smarandache, F., (2007). Ratio-product type exponential estimator for estimating finite population mean using information on auxiliary attribute. *Renaissance High Press, USA*. pp. 18–32.

Singh, R., Chauhan, P., Sawan, N., Smarandache, F., (2008). Ratio estimators in simple random sampling using information on auxiliary attribute. *Pakistan Journal of Statistics and Operation Research*, IV(1), pp. 47–53.

Singh, H. P., Solanki, R. S., (2012). Improved estimation of population mean in simple random sampling using information on auxiliary attribute. *Applied Mathematics and Computation*, 218(15), pp. 7798–7812.

Srivastava, S. K., (1967). An estimator using auxiliary information. *Calcutta Statistical Association Bulletin*, 16, pp. 121–132.

Sukhatme, P. V., Sukhatme, B. V., (1970). *Sampling Theory of Surveys with Applications*. Iowa State University Press, Ames, U.S.A.

Walsh, J. E., (1970). Generalization of ratio estimator for population total. *Sankhya, A*, 32, pp. 99–106.

Yadav, S. K., Zaman, T., (2020). A general exponential family of estimators for population mean using auxiliary attribute. *Journal of Science and Arts*, 20(1), pp. 25–34.

Zaman, T., Saglam, V., Sagir, M., Yucesoy, E., Zobu, M., (2014). Investigation of some estimators via Taylor series approach and an application. *American Journal of Theoretical and Applied Statistics*, 3(5), pp. 141–147.

Zaman, T., Kadilar, C., (2019). Novel family of exponential estimators using information of auxiliary attribute. *Journal of Statistics and Management Systems*, 22(8), pp. 1499–1509.

Zaman, T., Toksoy, E., (2019). Improvement in estimating the population mean in simple random sampling using information on two auxiliary attributes and numerical application in agricultural engineering. *Fresenius Environmental Bulletin*, 28(6), pp. 4584–4590.

Zaman, T., (2019a). Improved estimators using coefficient of skewness of auxiliary attribute. *Journal of Reliability and Statistical Studies*, pp. 177–186.

Zaman, T., (2019b). Efficient estimators of population mean using auxiliary attribute in stratified random sampling. *Advances and Applications in Statistics*, 56(2), pp. 153–171.

Zaman, T., (2020). Generalized exponential estimators for the finite population mean. *Statistics in Transition - new series*, 21(1), pp. 159–168.

Zaman, T., Kadilar, C., (2020). On estimating the population mean using auxiliary character in stratified random sampling. *Journal of Statistics and Management Systems*, 23(8), pp. 1415–1426.

Zaman, T., (2021). An efficient exponential estimator of the mean under stratified random sampling. *Mathematical Population Studies*, 28(2), pp. 104–121.

Zaman, T., Kadilar, C., (2021a). Exponential ratio and product type estimators of the mean in stratified two-phase sampling. *AIMS Mathematics*, 6(5), pp. 4265–4279.

Zaman, T., Kadilar, C., (2021b). New class of exponential estimators for finite population mean in two-phase sampling. *Communications in Statistics-Theory and Methods*, 50(4), pp. 874–889.

Appendix A

To the first degree of approximation, the *MSE* of the estimators reviewed in Section 2 are respectively given below.

$$MSE(t_m) = \gamma\bar{Y}^2 C_y^2 \tag{A.1}$$

$$MSE(t_r) = \gamma\bar{Y}^2 \left[C_y^2 + C_\phi^2 - 2\rho C_y C_\phi\right] \tag{A.2}$$

$$minMSE(t) = \bar{Y}^2\gamma C_y^2(1-\rho^2);\ t = t_{lr},\ t_i,\ t_{a_i},\ i = 1,2,3,4,\ t_J,\ t_{ss},\ t_{z_i},\ i = 1,2,...,5, t_{yz} \tag{A.3}$$

$$MSE(t_{sr}) = \gamma\bar{Y}^2 \left[C_y^2 + \frac{C_\phi^2}{4} - \rho C_y C_\phi\right] \tag{A.4}$$

$$MSE(t_{sp}) = \gamma\bar{Y}^2 \left[C_y^2 + \frac{C_\phi^2}{4} + \rho C_y C_\phi\right] \tag{A.5}$$

$$MSE(t_s) = \gamma\bar{Y}^2 \left[\upsilon^2 C_\phi^2 + C_y^2(1-\rho^2)\right] \tag{A.6}$$

$$MSE(t_{a_i}) = \gamma\bar{Y}^2 \left[C_y^2 + \upsilon^2 C_\phi^2 - 2\upsilon\rho C_y C_\phi\right],\ i = 5,6,...,13 \tag{A.7}$$

$$MSE(t_{ss}) = \gamma\bar{Y}^2 \left[C_y^2 + \frac{\alpha C_\phi^2}{4}\left(\alpha - 4\rho\frac{C_y}{C_\phi}\right)\right] \tag{A.8}$$

$$MSE(t_{zk}) = \gamma\bar{Y}^2 \left[\lambda^2 C_\phi^2 + C_y^2 - 2\lambda\rho C_y C_\phi\right] \tag{A.9}$$

$$minMSE(t_{bg}) = \bar{Y}^2 \left[1 - \frac{(AG^2 + BD^2 - 2DFG)}{(4AB - F^2)}\right] \tag{A.10}$$

The optimum values of scalars involved in the estimators t_{lr}, t_i, $i=1,2,3,4$, t_{ss}, t_{z_i}, $i=1,2,...,5$, t_{yz} and t_{bg} are tabulated below:

$$\beta_{\phi\,(opt)} = \rho\frac{C_y}{C_\phi} \tag{A.11}$$

$$\theta_{(opt)} = -\rho\frac{C_y}{C_\phi} = \Lambda_{(opt)} = \Delta_{(opt)} = \gimel_{(opt)} \tag{A.12}$$

$$\alpha_{(opt)} = 2\rho\frac{C_y}{C_\phi} \tag{A.13}$$

$$\omega_{(opt)} = \frac{(\rho C_y - C_\phi\zeta_i)}{C_\phi(\zeta_1-\zeta_i)},\ i=1,2,3,4 \tag{A.14}$$

$$k_{1(opt)} = 1-\rho\frac{C_y}{\upsilon C_\phi} \tag{A.15}$$

$$k_{2(opt)} = \rho\frac{C_y}{\upsilon C_\phi} \tag{A.16}$$

$$\theta_{(opt)} = \upsilon-\rho\frac{C_y}{C_\phi} \tag{A.17}$$

$$w_{1(opt)} = \frac{(GF-2BD)}{(4AB-F^2)} \tag{A.18}$$

$$w_{2(opt)} = \bar{Y}\frac{(DF-2GA)}{(4AB-F^2)} \tag{A.19}$$

where $\upsilon=\eta P/(\eta P+\lambda)$, $\zeta_1=P/(P+\beta_1(\phi))$, $\zeta_2=\beta_2(\phi)P/(\beta_2(\phi)P+\beta_1(\phi))$, $\zeta_3=C_\phi P/(C_\phi P+\beta_1(\phi))$, $\zeta_4=\beta_1(\phi)P/(\beta_1(\phi)P+C_\phi)$, $A=1+\gamma(C_y^2+\alpha^2\upsilon^2C_\phi^2+4\alpha\upsilon\rho C_yC_\phi-\alpha\upsilon C_\phi^2)$, $B=1+\gamma(C_\phi^2+\alpha^2\upsilon^2C_\phi^2-\alpha\upsilon^2C_\phi^2+4\alpha\upsilon C_\phi^2)$, $D=\gamma(\alpha\upsilon^2C_\phi^2-2\alpha\upsilon\rho C_yC_\phi)-2$, $G=\gamma(\alpha\upsilon^2C_\phi^2-2\alpha\upsilon C_\phi^2)-2$, $F=2+2\gamma(2\alpha\upsilon C_\phi^2+2\alpha\upsilon\rho C_yC_\phi+\rho C_yC_\phi-\alpha\upsilon^2C_\phi^2+\alpha^2\upsilon^2C_\phi^2)$.

Appendix B

Consider the first estimator

$$t_{k_1} = \zeta_1\{\bar{y}+\theta(p^*-P^*)\}+\psi_1\bar{y}\left\{1+\log\left(\frac{p^*}{P^*}\right)\right\}^{\gimel} \tag{B.20}$$

Now, using the notations defined in the earlier section, we express this estimator in terms of $e's$ as

$$t_{k_1}-\bar{Y} = \bar{Y}\left[\zeta_1\left(1+e_0+\frac{\theta}{R}\eta e_1\right)+\psi_1\left(1+e_0+\gimel\upsilon e_1-\gimel\upsilon^2e_1^2+\frac{\gimel^2}{2}\upsilon^2e_1^2+\gimel\upsilon e_0e_1\right)\right] \tag{B.21}$$

Taking expectation both sides of Equation (B.21), we get the bias of the estimator t_{k_1} up to the first order of approximation as

$$Bias(t_{k_1}) = \bar{Y}\left[\ \zeta_1 I_1 + \psi_1 J_1 - 1\ \right] \tag{B.22}$$

Now, squaring and taking expectation both sides of Equation (B.21), we get the *MSE* of the estimator up to the first order of approximation as

$$MSE(t_{k_1}) = \bar{Y}^2\left[1 + \zeta_1^2 F_1 + \psi_1^2 G_1 + 2\zeta_1\psi_1 H_1 - 2\zeta_1 I_1 - 2\psi_1 J_1\right] \tag{B.23}$$

The *MSE* of the estimator t_{k_1} is minimized by

$$\zeta_{1(opt)} = \frac{(G_1 I_1 - H_1 J_1)}{(F_1 G_1 - H_1^2)} \quad \text{and} \quad \psi_{1(opt)} = \frac{(F_1 J_1 - H_1 I_1)}{(F_1 G_1 - H_1^2)} \tag{B.24}$$

The minimum *MSE* at $\zeta_{1(opt)}$ and $\psi_{1(opt)}$ is given by

$$minMSE(t_{k_1}) = \bar{Y}^2\left[1 - \frac{(F_1 J_1^2 + G_1 I_1^2 - 2H_1 I_1 J_1)}{(F_1 G_1 - H_1^2)}\right] \tag{B.25}$$

Similarly, the *MSE* of the other estimators can be obtained. In general, we can write

$$MSE(t_{k_i}) = \bar{Y}^2\left[1 + \zeta_i^2 F_i + \psi_i^2 G_i + 2\zeta_i\psi_i H_i - 2\zeta_i I_i - 2\psi_i J_i\right] \tag{B.26}$$

The $MSE(t_{k_i})$ is minimized for

$$\zeta_{i(opt)} = \frac{(G_i I_i - H_i J_i)}{(F_i G_i - H_i^2)} \quad \text{and} \quad \psi_{i(opt)} = \frac{(F_i J_i - H_i I_i)}{(F_i G_i - H_i^2)} \tag{B.27}$$

The minimum *MSE* at the optimum values of the above scalars is given as

$$minMSE(t_{k_i}) = \bar{Y}^2\left[1 - \frac{(F_i J_i^2 + G_i I_i^2 - 2H_i I_i J_i)}{(F_i G_i - H_i^2)}\right] \tag{B.28}$$

where

$$F_1 = \left[1 + \gamma\left\{C_y^2 + \left(\frac{\theta}{R}\right)^2 \eta^2 C_\phi^2 + 2\left(\frac{\theta}{R}\right)\eta\rho C_y C_\phi\right\}\right]$$

$$G_1 = \left[1 + \gamma\left\{C_y^2 + (2\gimel^2 - 2\gimel)\upsilon^2 C_\phi^2 + 4\gimel\upsilon\rho C_y C_\phi\right\}\right]$$

$$H_1 = \left[1 + \gamma\left\{C_y^2 + \left(\frac{\gimel^2}{2}\upsilon^2 - \gimel\upsilon^2 + \frac{\theta\gimel}{R}\eta\upsilon\right)C_\phi^2 + \left(\frac{\theta}{R}\eta + 2\gimel\upsilon\right)\rho C_y C_\phi\right\}\right]$$

$$I_1 = 1$$

$$J_1 = \left[1 + \gamma\left\{\left(\frac{\gimel^2}{2} - \gimel\right)\upsilon^2 C_\phi^2 + \gimel\upsilon\rho C_y C_\phi\right\}\right]$$

$$F_2 = \left[1+\gamma\left\{C_y^2+\left(\frac{\theta}{R}\right)^2\eta^2C_\phi^2+2\left(\frac{\theta}{R}\right)\eta\rho C_yC_\phi\right\}\right]$$
$$G_2 = [1+\gamma\{C_y^2+(2\Lambda^2\upsilon^2+\Lambda\upsilon^2)C_\phi^2-4\Lambda\upsilon\rho C_yC_\phi\}]$$
$$H_2 = \left[1+\gamma\left\{C_y^2+\left(\frac{\Lambda(\Lambda+1)}{2}\upsilon^2-\frac{\theta\Lambda}{R}\eta\upsilon\right)C_\phi^2+\left(\frac{\theta}{R}\eta-2\Lambda\upsilon\right)\rho C_yC_\phi\right\}\right]$$
$$I_2 = 1$$
$$J_2 = \left[1+\gamma\left\{\frac{\Lambda(\Lambda+1)}{2}\upsilon^2C_\phi^2-\Lambda\upsilon\rho C_yC_\phi\right\}\right]$$
$$F_3 = \left[1+\gamma\left\{C_y^2+\left(\frac{\theta}{R}\right)^2\eta^2C_\phi^2+2\left(\frac{\theta}{R}\right)\eta\rho C_yC_\phi\right\}\right]$$
$$G_3 = [1+\gamma\{C_y^2+3\Delta^2\upsilon^2C_\phi^2-4\Delta\upsilon\rho C_yC_\phi\}]$$
$$H_3 = \left[1+\gamma\left\{C_y^2+\left(\Delta^2\upsilon^2-\frac{\theta\Delta}{R}\eta\upsilon\right)C_\phi^2+\left(\frac{\theta}{R}\eta-2\Delta\upsilon\right)\rho C_yC_\phi\right\}\right]$$
$$I_3 = 1$$
$$J_3 = [1-\gamma\{\Delta\upsilon\rho C_yC_\phi-\Delta^2\upsilon^2C_\phi^2\}]$$
$$F_4 = [1+\gamma\{C_y^2+(2\Lambda^2\upsilon^2+\Lambda\upsilon^2)C_\phi^2-4\Lambda\upsilon\rho C_yC_\phi\}]$$
$$G_4 = [1+\gamma\{C_y^2+3\Delta^2\upsilon^2C_\phi^2-4\Delta\upsilon\rho C_yC_\phi\}]$$
$$H_4 = \left[1+\gamma\left\{C_y^2+\left(\Delta^2\upsilon^2+\Lambda\Delta\upsilon^2+\frac{\Lambda(\Lambda+1)}{2}\upsilon^2\right)C_\phi^2-2\upsilon(\Lambda+\Delta)\rho C_yC_\phi\right\}\right]$$
$$I_4 = \left[1+\gamma\left\{\frac{\Lambda(\Lambda+1)}{2}\upsilon^2C_\phi^2-\Lambda\upsilon\rho C_yC_\phi\right\}\right]$$
$$J_4 = [1+\gamma\{\Delta^2\upsilon^2C_\phi^2-\Delta\upsilon\rho C_yC_\phi\}]$$