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Rotation schemes and Chebyshev polynomials

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Abstract

There is a continuing interplay between mathematics and survey methodology involving different branches of mathematics, not only probability. This interplay is quite obvious as regards the first of the two options: probability vs. non-probability sampling, as proposed and discussed in Kalton (2023). There, mathematics is represented by probability and mathematical statistics. However, sometimes connections between mathematics and survey methodology are less obvious, yet still crucial and intriguing. In this paper we refer to such an unexpected relation, namely between rotation sampling and Chebyshev polynomials. This connection, introduced in Kowalski and Wesołowski (2015), proved fundamental for the derivation of an explicit form of the recursion for the BLUE $\hat{\mu}_t$ of the mean on each occasion t in repeated surveys based on a cascade rotation scheme. This general result was obtained under two basic assumptions: ASSUMPTION I and ASSUMPTION II, expressed in terms of the Chebyshev polynomials. Moreover, in that paper, it was conjectured that these two assumptions are always satisfied, so the derived form of recursion is universally valid. In this paper, we partially confirm this conjecture by showing that ASSUMPTION I is satisfied for rotation patterns with a single gap of an arbitrary size.

1. Introduction

Existence of connections between survey methodology and mathematics is a trivial statement. The most natural ones are triggered by probability sampling, the first option in dychotomy between probability and non-probability sampling proposed in the review, Kalton (2023), in this issue of SiT. Of course, it involves probability theory and mathematical statistics on the mathematical side. The title (and the content) of the popular monograph "Model Assisted Sampling Survey" by Särndal, Swensson and Wretman (1992) is the best reference to appreciate this connection. Some other areas of mathematics are also typically involved in this interplay; as (convex) optimization theory in optimal allocation problems, or graph theory in modelling dependence structure in adaptive sampling. The second part of Kalton's dychotomy may open new doors for involvement of mathematics in survey methodology. But even within survey methodology based on probability sampling, unexpected and useful connections between the two areas happen. A good example is a connection between rotation sampling and Chebyshev polynomials, which we are going to explore in this paper.

Rotation of the sample is a standard method used in repeated surveys. It allows not only catching the dynamics of the population under study and lower the burden of surveys for respondents, but also can be used to improve estimation of parameters at the given occasion

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by proper treatment of observations from the past occasions. Typical examples are the Labour Force Survey in the EU with the rotation pattern 110011 (also referred to by 2-2-2), i.e. a unit (group of units) is in a sample for two consecutive occasions, leaves the survey for next two occassions, then enters the sample for two more consecutive occasions and then leaves the survey for good, or the Current Population Survey in the US with the rotation pattern 1111000000001111 (i.e. 4-8-4). Such methodology was proposed in the seminal paper Patterson (1950), who postulated the recurrence form for the best linear unbiased estimators (BLUEs) of the mean on each occasion. Patterson considered a model with exponentially time-dependent correlations for each unit of the population and independence between units. He assumed that the rotation pattern is such that any unit leaving the sample cannot return to the survey. In such setting it was proved that for any occasion t the BLUE $\hat{\mu}_t$ (based on all past observations) of the current mean μ_t satisfies the linear one-step recursion of the form

$$\hat{\mu}_t = a_1(t)\hat{\mu}_{t-1} + \underline{r}_0^T(t)\underline{X}_t + \underline{r}_1^T(t)\underline{X}_{t-1}, \tag{1}$$

where $\underline{X}_i$ is the vector of observations at time $i = t, t-1$ and the recursion coefficients, i.e. the number $a_1(t)$ and the vectors $\underline{r}_0(t)$, $\underline{r}_1(t)$ were identified in terms of the correlation coefficient ρ.

The assumption that a unit leaving the sample never returns to the survey was crucial for derivation of (1). Therefore, it was expected that the first order recursion for the optimal BLUE's would no longer hold for more general rotation patterns which do not satisfy Patterson's condition. A postulated form of the recursion would be of the form

$$\hat{\mu}_t = a_1(t)\hat{\mu}_{t-1} + \ldots + a_p(t)\hat{\mu}_{t-p} + \underline{r}_0^T(t)\underline{X}_t + \underline{r}_1^T(t)\underline{X}_{t-1} + \ldots + \underline{r}_p^T(t)\underline{X}_{t-p}, \tag{2}$$

where p is a natural number and $a_1(t), \ldots, a_p(t)$, $\underline{r}_0(t), \ldots, \underline{r}_p(t)$ are numeric and vector coefficients. However, such extension posed major difficulties, see, e.g. Yansaneh and Fuller (1998). Therefore, for years researchers have been mostly focused on sub-optimal estimators. Already Hansen, Hurwitz, Nisselson and Steinberg (1955) proposed an alternative sub-optimal K-composite estimator, where the optimality was sought under additional assumption of one-step recursion, that is, under assumption that $p = 1$ in (2). This approach was further developed in Rao and Graham (1964), Gurney and Daly (1965), Cantwell (1990), Cantwell and Caldwell (1998), Ciepiela, Gniado, Wesołowski and Wojtyś (2012). Another approach, based on the so-called regression composite extimator has been proposed and studied in Bell (2001), Fuller and Rao (2001), Singh, Kennedy and Wu (2001), Kowalczyk and Juszczak (2018). Different rotation patterns and comparisons of efficiencies of different methods are presented in McLaren and Steel (2000), and Steel and McLaren (2002,2008). For a relatively new review see Karna and Nath (2015). Polish experiences with rotation sampling are described in a review by Kordos (2012). An alternative methodology, which we do not consider here, is based on time series theory, with random means on subsequent occasions while here we assume that they are constants depending on t. An overview of the time series approach to rotation sampling is given e.g. in Binder and Hidiroglou (1988).

The first result going beyond Patterson's scheme of a rotation pattern without gaps, i.e. of the form 11...11, was obtained in Kowalski (2009), where it was proved that for rotation patterns with arbitrary number of singleton gaps, i.e. of the form 1...101...101...1, the recursion (2) holds with $p = 2$ and all coefficients were identified. Moreover, it was observed in that paper that the coefficients stabilize quickly as t grows, which suggested an approach to the general case by recursion with coefficients not depending on t, equivalently for the stationary situation, i.e. the case when $t \to \infty$. Then, the recursion assumes the form

$$\hat{\mu}_t = a_1\hat{\mu}_{t-1} + \ldots + a_p\hat{\mu}_{t-p} + \underline{r}_0^T\underline{X}_t + \underline{r}_1^T\underline{X}_{t-1} + \ldots + \underline{r}_p^T\underline{X}_{t-p}, \tag{3}$$

Under such a setting a general solution for arbitrary rotation pattern was obtained in Kowalski and Wesołowski (2015) (referred to by KW in the sequel). According to the main result in KW the recursion depth, p, is the size of the maximal gap in the rotation pattern increased by 1 (therefore it was 1 in the Patterson model, 2 in for rotation patterns with gaps of size 1) and 3 in the LFS rotation pattern 110011 (the last one settled in Wesołowski (2010)). The form of the coefficients in (3), as given in KW, is explicit, and rather unexpectedly, involves the Chebyshev polynomials of the first kind defined by

$$T_k(x) = \cos(k \arccos x), \quad k = 0, 1, \ldots$$

For a thorough review of Chebyshev polynomials readers are encouraged to consult Paszkowski (1975). It has to be emphasized that the solution, valid for any cascade rotation pattern, was obtained in KW under two specific assumptions: ASSUMPTION I concerning roots of a special polynomial Q_p of degree p expressed through Chebyshev polynomials and ASSUMPTION II concerning full rank of certain matrix S being a function of these roots. However, in numerous simulations both these ASSUMPTIONs were always satisfied. Therefore, it was conjectured, see p. 101 of KW, that both ASSUMPTIONs are always satisfied and the solution obtained is universally valid. The goal of the present paper is to show that the conjecture holds true, at least partially. Actually, it will be shown that ASSUMPTION I holds true for rotation patterns with a single gap of arbitrary size. The rest of the paper is organized as follows. In Section 2, we present the general setting of the rotation scheme in mathematical language and adjust ASSUMPTIONs I and II to rotation patterns with a single gap of arbitrary size. In Section 3, we give a short introduction to Chebyshev polynomials emphasizing tools needed to analyze roots of the polynomial Q_p. In Section 4, we prove the main result which says that ASSUMPTION I is satisified for rotation patterns with single gap of arbitrary size. Section 5 is devoted to a representation of Q_p as an affine perturbation of a Chebyshev polynomial of a proper degree, which is the main tool for the proof.

2. General setting and rotation patterns with a single gap of arbitrary size

Consider a doubly-infinite matrix of random variables (X_{ij}), $i, j \in \mathbb{Z}$ such that for any $j \in \mathbb{Z}$

$$\mathbb{E}X_{i,j} = \mu_j, \quad \text{for all } i \in \mathbb{Z},$$

and, without loss of generality we assume that $\mathbb{V}\mathrm{ar}(X_{i,j}) = 1$ for all $i, j \in \mathbb{Z}$. The correlation structure is described by

$$\mathbb{C}\mathrm{orr}(X_{i,j}, X_{k,l}) = I(k = i)\rho^{|j-l|}, \tag{4}$$

where $0 < |\rho| < 1$.

For natural number $N \geq 2$ consider a sequence $\underline{X}_j = (X_{j,j}, \dots, X_{j+N-1,j})$, $j \in \mathbb{Z}$, of N-variate random vectors. Note that from (4) it follows that the covariance matrix $\mathbf{C} = \mathbb{C}\mathrm{ov}\,(\underline{X}_j, \underline{X}_{j+1})$, of dimensions $N \times N$, has all entries equal zero except the ones just above the diagonal, which are all equal ρ. Moreover, (4) yields

$$\mathbb{C}\mathrm{ov}(\underline{X}_j, \underline{X}_k) = \mathbf{C}^{|k-j|}$$

and note that $\mathbf{C}^j$ is a matrix with all entries equal zero except the jth over diagonal with all entries equal ρ^j when $j \leq N-1$ and it is a zero matrix when $j > N-1$.

A rotation pattern is any vector $(\varepsilon_1, \dots, \varepsilon_N)$ with 0-1 entries such that $\varepsilon_1 = \varepsilon_N = 1$. Let $M = \{j \in \{1, \dots, N\} : \varepsilon_j = 0\}$. Then $N = n + m$, where $m = \#M$ is the number of zeros among the entries and n is the number of ones (note that $n \geq 2$). Each zero in rotation pattern results in a "hole" in the sample and the largest set of subsequent zeros determines a gap in the rotation pattern. Let $p-1$ denote the dimension of the largest gap in the rotation pattern.

We modify vectors $\underline{X}_j$ into

$$\underline{Y}_j = (X_{j+k-1,j}, k \in \{1, \dots, N\} \setminus M), \quad j \in \mathbb{Z}.$$

For a given $t \in \mathbb{Z}$ let $\hat{\mu}_t$ denote the BLUE of μ_t based on $\underline{Y}_s$, $s \leq t$.

We study the recurrence formula for the BLUE estimators of the following form

$$\hat{\mu}_t = \tilde{a}_1 \hat{\mu}_{t-1} + \ldots + \tilde{a}_s \hat{\mu}_{t-s} + \underline{\tilde{r}}_0^T \underline{Y}_t + \underline{\tilde{r}}_1^T \underline{Y}_{t-1} + \ldots + \underline{\tilde{r}}_s^T \underline{Y}_{t-s},$$

for any $t \in \mathbb{Z}$, where s, $\tilde{a}_1, \dots, \tilde{a}_s \in \mathbb{R}$ and $\underline{\tilde{r}}_0, \underline{\tilde{r}}_1, \dots, \underline{\tilde{r}}_s \in \mathbb{R}^m$ are unknown. The goal is to find s and to identify remaining parameters in terms of p, ρ and N.

Alternatively, $\hat{\mu}_t$ can be defined as optimal unbiased linear estimator $\sum_{s \leq t} \underline{w}_s^T \underline{X}_s$, with additional constraints

$$w_{s,j}(1 - \varepsilon_j) = 0, \quad j = 1, \dots, N, \quad s \leq t, \tag{5}$$

imposed by the gaps in the rotation pattern. Therefore, the above recursion can be written in the form

$$\hat{\mu}_t = a_1 \hat{\mu}_{t-1} + \ldots a_s \hat{\mu}_{t-s} + \underline{r}_0^T \underline{X}_t + \underline{r}_1^T \underline{X}_{t-1} + \ldots + \underline{r}_s^T \underline{X}_{t-s}, \tag{6}$$

for any $t \in Z$, where $a_1, \dots, a_s \in \mathbb{R}$ and $\underline{r}_0, \underline{r}_1, \dots, \underline{r}_s \in \mathbb{R}^N$.

Note that (5) forces respective entries of vectors $\underline{r}_j \in \mathbb{R}^N$, $j = 0, \dots, s$, to be equal zero.

The problem is to prove that the recurrence (6) holds for $s = p$ and to determine scalar parameters a_i, $i = 1,\dots,p$ and vector parameters $\underline{r}_j \in \mathbb{R}^N$, $j = 0,1,\dots,p$. As it has been already mentioned, under two basic assumptions there exist formulas which completely answer this question. The first of these assumptions is concerned with localization of roots of certain polynomial and the second deals with unique solvability of certain linear system of equations. There is a strong numerical evidence that these assumptions may be universally satisfied. However, no proof of this fact has been available until now. It has been theoretically confirmed only for $m = 0,1$ and any $n \geq 2$ and for the rotation pattern 110011 (here $m = 2$). In this paper we will show that the first assumption (ASSUMPTION I below) is satisfied for all rotation patterns with a single gap of arbitrary size m. We do not know how to prove that the second assumption (ASSUMPTION II below) is satisfied in this case.

From now on we consider only rotation patterns with a single gap of arbitrary size $m \in \{0,1,\dots\}$. In the remaining part of this section we will present ASSUMPTIONs I and II for such rotation patterns only. A reader interested in the general case is encouraged to look into KW.

Recall that the Chebyshev polynomials of the first kind (T_n) are defined through a three step recurrence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,\dots \tag{7}$$

and $T_0(x) = 1$, $T_1(x) = x$, that is $T_n(\cos t) = \cos(nt)$, $n = 0,1,\dots$.

Consider a polynomial Q_p of degree p defined by

$$Q_p(x) = 1-\rho^2 + (N-1)(1+\rho^2-2\rho x) - (1+\rho^2-2\rho x)^2 \ \mathrm{tr}\left(\mathbf{T}_m(x)\mathbf{R}_m^{-1}(\rho)\right), \tag{8}$$

where $\mathbf{T}_m$ is an $m \times m$ symmetric Toeplitz matrix of the Chebyshev polynomials of the form

$$\mathbf{T}_m = \begin{bmatrix} T_0 & T_1 & T_2 & \dots & T_{m-2} & T_{m-1} \\ T_1 & T_0 & T_1 & \dots & T_{m-3} & T_{m-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ T_{m-2} & T_{m-3} & T_{m-4} & \dots & T_0 & T_1 \\ T_{m-1} & T_{m-2} & T_{m-3} & \dots & T_1 & T_0 \end{bmatrix} \tag{9}$$

and $\mathbf{R}_m$ is an $m \times m$ invertible constant three-diagonal matrix

$$\mathbf{R}_m = \begin{bmatrix} 1+\rho^2 & -\rho & 0 & \dots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \dots & -\rho & 1+\rho^2 \end{bmatrix} \tag{10}$$

ASSUMPTION I: **Roots of Q_p are distinct and do not belong to $[-1,1]$.**

As mentioned above, our goal is to show that ASSUMPTION I is satisfied. It is done in the remaining three sections below. In Section 3, we present some basic facts on the Chebyshev polynomials of the first and second kind we need in the sequel. The proof, given in Section 4, to large extent is based on a representation of Q_p derived in Section 5.

But before we analyze ASSUMPTION I we will introduce also ASSUMPTION II, which is conjectured to be also satisfied, but we do not know, how to prove it.

Note that Q_p is a polynomial of pth degree. If its roots $x_1,\ldots,x_p$ are simple and are outside of the interval $[-1,1]$ (which will be proved in the sequel), then there exist unique $d_1,\ldots,d_p$, which can be complex, such that $|d_i|<1$ and $\frac{1}{2}\left(d_i+d_i^{-1}\right)=x_i$, $i=1,\ldots,p$.

For such numbers $d_1,\ldots,d_p$ define a $p^2\times p^2$ matrix

$$\mathbf{S}=\mathbf{S}(d_1,\ldots,d_p)=\begin{bmatrix} \widetilde{\mathbf{G}}(d_1) & \widetilde{\mathbf{G}}(d_2) & \cdots & \widetilde{\mathbf{G}}(d_p) \\ \mathbf{G}(d_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{G}(d_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{G}(d_p) \end{bmatrix}$$

where $\widetilde{\mathbf{G}}(d_i)$ are $p\times p$ matrices

$$\widetilde{\mathbf{G}}(d)=\tfrac{1}{1-\rho^2}\begin{bmatrix} (N-1)(1-d\rho)+1-\rho^2 & (1-d\rho)\underline{1}_h^T \\ & \\ (1-d\rho)\underline{1}_{p-1} & \widetilde{\mathbf{H}}_{p-1} \end{bmatrix}$$

with $\widetilde{\mathbf{H}}_{p-1}(d)$ being a $(p-1)\times(p-1)$ upper bi-diagonal matrix

$$\widetilde{\mathbf{H}}_{p-1}(d)=\begin{bmatrix} 1 & -d\rho & & \\ & \ddots & \ddots & \\ & & \ddots & -d\rho \\ & & & 1 \end{bmatrix}.$$

and $\mathbf{G}(d_i)$ are $(p-1)\times p$ matrices

$$\mathbf{G}(d)=\tfrac{1}{1-\rho^2}\left[(1-d\rho)(d-\rho)\underline{1}_h,\ d\,\mathbf{H}_{p-1}\right],$$

with $\mathbf{H}_{p-1}=\mathbf{H}_{p-1}(d)$ being a $(p-1)\times(p-1)$ tri-diagonal matrix

$$\mathbf{H}_{p-1}(d)=\begin{bmatrix} 1+\rho^2 & -d\rho & & \\ -\rho/d & \ddots & \ddots & \\ & \ddots & \ddots & -d\rho \\ & & -\rho/d & 1+\rho^2 \end{bmatrix}.$$

Here is the second main assumption:

ASSUMPTION II: $\mathbf{det}\,\mathbf{S}(\mathbf{d_1},\ldots,\mathbf{d_p}) \neq \mathbf{0}$.

Unfortunately, we are unable to prove that it is satisfied in the setting of a single gap of arbitrary size m. As mentioned above, only the cases of $m = 0,1,2$ have been settled until now.

When ASSUMPTION I and ASSUMPTION II are satisfied, then Theorem 3.1 proved in KW says that $p = m+1$ and gives explicit formulas for a_i, $i = 1,\ldots,p$, and $\underline{r}_i$, $i = 0,1,\ldots,p$, in terms of the $d_1,\ldots,d_p$ determined through roots of Q_p and the solution $\underline{c}$ of the linear equation $\mathbf{S}\underline{c} = (1,0,...,0)^T$. For details consult KW.

3. Chebyshev polynomials

The Chebyshev polynomials of the second kind $(U_n)_{n\geq 0}$ are defined through the same three step recurrence as $(T_n)_{n\geq 0}$, that is

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad n = 1,2,\ldots \tag{11}$$

but the boundary conditions are slightly different: $U_0(x) = 1$ and $U_1(x) = 2x$, that is $U_n(\cos t) = \frac{\sin((n+1)t)}{\sin t}$, if only $\sin t \neq 0$.

We will also use two important identities connecting two forms of the Chebyshev polynomials for any $n = 1,2,\ldots$ (in the formulas below we denote $U_{-1} \equiv 0$):

$$T_n' = nU_{n-1}, \tag{12}$$

and

$$T_n^2(x) + (1-x^2)U_{n-1}^2(x) = 1. \tag{13}$$

Moreover, two representations of the Chebyshev polynomials given in Lemma 3.1 (cf. Paszkowski, 1975) below will be very useful.

Lemma 3.1. *For any $x \neq 0$ and $n = 0,1,\ldots$*

$$T_n(\tfrac{1}{2}(x+x^{-1})) = \tfrac{1}{2}\left(x^n + x^{-n}\right) \tag{14}$$

and for $x \neq 0,\pm 1$ we have

$$U_n(\tfrac{1}{2}(x+x^{-1})) = \tfrac{x^{n+1}-x^{-(n+1)}}{x-x^{-1}}. \tag{15}$$

It is known (cf. Paszkowski, 1975) that

$$U_n(x) = \det \mathbf{V}_n(x), \quad n = 1,2,\ldots, \tag{16}$$

where $\mathbf{V}_n(x)$ is an $n\times n$ tridiagonal matrix defined by

$$\mathbf{V}_n(x) = \begin{bmatrix} 2\,x & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2\,x & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2\,x & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & 2\,x & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2\,x \end{bmatrix}. \tag{17}$$

We see that $\mathbf{V}_n(x)$ is non-singular for any $x \geq 1$. In this case the explicit form of the inverse of $\mathbf{V}_n(x)$ is known. Let $\mathbf{A}$ denote the inverse of $\mathbf{V}_n(x)$. Then $\mathbf{A} = [a_{i,j}]_{i,j\in\{1,\ldots,n\}}$ is a symmetric matrix such that

$$a_{i,j} = \tfrac{1}{U_n(x)}\, U_{i-1}(x)U_{n-j}(x), \quad 1 \leq i \leq j \leq n. \tag{18}$$

We will apply the following useful formulae (cf. Paszkowski, 1975)

$$\sum_{j=1}^{n} T_j(x)\, U_{n-j}(x) = \tfrac{n}{2}\, U_n(x), \tag{19}$$

$$2\,(x-y) \sum_{j=0}^{n}{}' T_j(x)\, U_{n-j}(y) = T_{n+1}(x) - T_{n+1}(y), \tag{20}$$

under notation

$$\sum_{j=s}^{n}{}' b_j = \tfrac{1}{2} b_s + b_{s+1} + \ldots + b_n, \quad n > l.$$

4. Roots of Q_p

In this section we show that ASSUMPTION 1 is satisfied for rotation patterns with a single gap of an arbitrary size. In the proof we strongly rely on properties of the Chebyshev polynomials and the representation of Q_p in terms of an affine additive perturbation of the Chebyshev polynomial of the first kind derived in Section 5.

Theorem 4.1. *For any $p \geq 1$ the polynomial Q_p defined by* (8), (9) *and* (10) *has exactly one (when p is odd) or exactly two (when p is even) real roots (i.e. the remaining roots are complex). These roots are outside of interval $[-1,1]$. All roots of Q_p are simple.*

Proof. Note that due to Prop. 5.2

$$(\det \mathbf{R}_m)\, Q_p(x) = \det \mathbf{R}_m\, (n-2)(1+\rho^2-2\rho x) + 2 - 2\rho^{m+1} T_{m+1}(x). \tag{21}$$

Therefore, the roots of Q_p are identical to the roots of polynomial $\widetilde{Q}_p$ defined by

$$\widetilde{Q}_p(x) = a + bx + T_{m+1}(x),$$

where

$$a = -\tfrac{(\det \mathbf{R}_m)(n-2)(1+\rho^2)+2}{2\rho^{m+1}} = -rU_m(r)(n-2) - \rho^{-m-1} \qquad \text{and} \qquad b = \tfrac{(\det \mathbf{R}_m)(n-2)}{\rho^m} = U_m(r)(n-2).$$

Assume that z_0 is a multiple root of $\widetilde{Q}_p$. That is

$$a + bz_0 + T_{m+1}(z_0) = 0. \tag{22}$$

Moreover, z_0 is necessarily a root of derivative of $\widetilde{Q}_p$. Thus from (12) we get

$$b + (m+1)U_m(z_0) = 0. \tag{23}$$

Combining (22) and (23) through (13) we obtain

$$(a+bz_0)^2 + (1-z_0^2)\left(\tfrac{b}{m+1}\right)^2 = 1.$$

That is, z_0 is a solution of the quadratic equation

$$b^2\left(1-\frac{1}{(m+1)^2}\right)x^2+2abx+a^2+\left(\frac{b}{m+1}\right)^2-1=0. \tag{24}$$

whose discriminant is

$$\Delta=4b^2\left[\frac{a^2}{(m+1)^2}-\left(\frac{b^2}{(m+1)^2}-1\right)\left(1-\frac{1}{(m+1)^2}\right)\right].$$

If $b \leq m+1$ clearly $\Delta > 0$. For $b > m+1$ note that

$$a^2=\frac{[(\det \mathbf{R}_m)(n-2)(1+\rho^2)+2]^2}{4\rho^{2m+2}} > \frac{[(\det \mathbf{R}_m)(n-2)(1+\rho^2)]^2}{4\rho^{2m+2}} = b^2\left(\frac{1+\rho^2}{2\rho}\right)^2 > b^2.$$

Therefore, $\Delta > 0$ also in this case. Thus, the quadratic equation (24) has only real solutions. Consequently, $\widetilde{Q}_p$ does not have multiple complex roots.

Note that $\widetilde{Q}_p$ can be written as

$$\widetilde{Q}_p(x)=-\frac{1}{\rho^{m+1}}\left[1+\tfrac{1}{2}(n-2)(1+\rho^2-2\rho x)\det \mathbf{R}_m\right]+T_{m+1}(x).$$

Clearly, the expression in brackets is greater or equal 1 for $x \in [-1,1]$. Since the Chebyshev polynomials T_n, $n=1,2,\ldots$, on $[-1,1]$ assume values in $[-1,1]$ it follows that on $[-1,1]$ the polynomial $\widetilde{Q}_p$ is either strictly positive (when $\rho^{m+1} < 0$) or strictly negative (when $\rho^{m+1} > 0$).

It is well known that

- if m is an even number then: U_m is strictly decreasing on $(-\infty,-1)$, strictly increasing on $(1,\infty)$ and $U_m(\pm 1)=m+1$;
- if m is an odd number then: U_m is strictly increasing on $(-\infty,-1)$ and on $(1,\infty)$ and $U_m(\pm 1) = \pm(m+1)$.

Consequently, only the following four cases are possible:

1. If m is even and $\rho > 0$ then $a < -1$ and $b \geq 0$. Thus, $\widetilde{Q}_p$ (of odd degree) has exactly one real root $x_1 > 1$. Note that it is simple. The reason for that is that the derivative of $\widetilde{Q}_p$ which equals $b+(m+1)U_m(x)$ is bounded from below by $b+(m+1)^2$ on $(1,\infty)$. Therefore, $\widetilde{Q}_p$ cannot have a multiple real root > 1.
2. If m is even and $\rho < 0$ then $a > 1$ and $b \geq 0$. Thus, $\widetilde{Q}_p$ (of odd degree) has exactly one real root $x_1 < -1$. Similarly, as above $\widetilde{Q}'_p(x) = b+(m+1)U_m(x) > b+(m+1)^2$ on $(-\infty,-1)$, and thus $\widetilde{Q}_p$ does not have a multiple root < -1.
3. If m is odd and $\rho > 0$ then $a < -1$ and $b \geq 0$. Thus, $\widetilde{Q}_p$ (of even degree) has exactly two real roots: $x_1 < -1$ and $x_2 > 1$. Similarly as above $\widetilde{Q}'_p(x) = b+(m+1)U_m(x) > b+(m+1)^2 > 0$ for $x > 1$, and thus the root x_2 is simple. Note also that the quadratic polynomial (24) is strictly positive on negative half line, that is $\widetilde{Q}_p$ cannot have negative multiple roots, in particular, the root x_1 is not multiple.
4. If m is odd and $\rho < 0$ then $a < -1$ and $b \leq 0$. Thus, $\widetilde{Q}_p$ (of even degree) has exactly two real roots: $x_1 < -1$ and $x_2 > 1$. This time the derivative, $\widetilde{Q}'_p(x) = b+(m+1)U_m(x) < b-(m+1)^2 < 0$ for $x < -1$, and thus the root x_1 is simple. Similarly as above, to check that x_2 is simple, we refer to (24) having the left-hand side strictly positive for $x > 0$, which means that there are no multiple positive roots.

$\square$

Remark 4.1. *Note that from* (21) *for* $n=2$ *we get* $(\det \mathbf{R}_m)\, Q_p(x) = 2 - 2\rho^{m+1} T_{m+1}(x)$. *Consequently, to find roots of* Q_p *it suffices to look for solutions of the equation*

$$T_{m+1}(x) = \rho^{-m-1}.$$

For $x = \frac{1}{2}(d + 1/d)$ *we obtain*

$$d^{m+1} + d^{-m-1} = \frac{2}{\rho^{m+1}}$$

and thus for $z = d^{m+1}$ *we get a quadratic equation*

$$z^2 - 2\frac{z}{\rho^{m+1}} + 1 = 0$$

with two real solutions

$$z = \frac{1 \pm \sqrt{1-\rho^{2(m+1)}}}{\rho^{m+1}}.$$

Therefore

$$d_j = d_{\pm} \exp\left[i\frac{2j\pi}{m+1}\right], \quad j = 0, 1, \ldots, m,$$

where

$$d_{\pm} = \frac{\sqrt[m+1]{1 \pm \sqrt{1-\rho^{2(m+1)}}}}{|\rho|}.$$

Note that $0 < d_- < 1 < d_+$.

5. Q_p through additive first degree perturbation of T_p

In this section we derive a convenient representation of Q_p in terms of T_p with changed terms of degree zero and one. It is preceded by a simple expression for determinant of $\mathbf{R}_m$ involving the second order Chebyshev polynomial U_m.

Lemma 5.1. *Let* $r = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$. *For any* $m = 0, 1, \ldots$

$$\det \mathbf{R}_m = \rho^m\, U_m(r), \quad m = 0, 1, 2, \ldots. \tag{25}$$

Proof. Notice that $\mathbf{R}_m = \rho \mathbf{V}_m(r)$, so by (16) we have

$$\det \mathbf{R}_m = \rho^m \det \mathbf{V}_m(r) = \rho^m\, U_m(r).$$

The proof is complete. □

Proposition 5.2.

$$(1+\rho^2-2\rho x)^2 \operatorname{tr}\left(\mathbf{T}_m(x)\, \mathbf{R}_m^{-1}\right) = (m+1)(1+\rho^2-2\rho x) + 1 - \rho^2 - 2\frac{1-\rho^{m+1} T_{m+1}(x)}{\det \mathbf{R}_m}. \tag{26}$$

Proof. Denote $r = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$. Then, $1+\rho^2-2\rho x = -2\rho(x-r)$.

From Lemma 3.1 it follows that (26) is equivalent to

$$4\rho^2(x-r)^2 \operatorname{tr}\left(\mathbf{T}_m(x)\, \mathbf{R}_m^{-1}\right) = \frac{2\rho}{U_m(r)}(T_{m+1}(x) - T_{m+1}(r)) - 2\rho(m+1)(x-r). \tag{27}$$

We see that $\mathbf{R}_m^{-1} = \frac{1}{\rho}\,\mathbf{A}$, where $\mathbf{A}$ is a symmetric matrix with entries defined by (18). Note that the symmetric Toeplitz structure of the matrix $\mathbf{T}_m$ and the fact that $\mathbf{A}$ is symmetric imply

$$\rho\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = 2\sum_{k=0}^{m-1}{}'\,T_k(x)\sum_{i=1}^{m-k} a_{i,i+k}.$$

We interchange two sums in the above equation. Then we have

$$\rho\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = 2\sum_{i=1}^{m}\sum_{k=0}^{m-i}{}'\,T_k(x)\,a_{i,i+k}$$

From (18) we get

$$(\rho\,U_m(r))\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = 2\sum_{i=1}^{m} U_{i-1}(r)\sum_{k=0}^{m-i}{}'\,T_k(x)\,U_{m-i-k}(r). \tag{28}$$

From (20) it follows that

$$2\,(x-r)\sum_{k=0}^{m-i}{}'\,T_k(x)\,U_{m-i-k}(r) = T_{m-i+1}(x) - T_{m-i+1}(r). \tag{29}$$

This together with (28) gives

$$(x-r)(\rho\,U_m(r))\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = \sum_{i=1}^{m} U_{i-1}(r)(T_{m-i+1}(x) - T_{m-i+1}(r)),$$

which can be rewritten as follows

$$(x-r)(\rho\,U_m(r))\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = S_1 - S_2, \tag{30}$$

where

$$S_1 = \sum_{i=1}^{m} U_{i-1}(r)\,T_{m-i+1}(x), \quad S_2 = \sum_{i=1}^{m} U_{i-1}(r)\,T_{m-i+1}(r). \tag{31}$$

From (19) we have

$$S_2 = \sum_{j=1}^{m} T_j(r)\,U_{m-j}(r) = \tfrac{m}{2}\,U_m(r).$$

Note that (20) implies

$$2(x-r)S_1 = 2(x-r)\sum_{j=1}^{m} T_j(x)\,U_{m-j}(x) = (T_{m+1}(x) - T_{m+1}(r)) - (x-r)\,U_m(r).$$

This together with (30) gives

$$4(x-r)^2(\rho\,U_m(r))\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = 4(x-r)(S_1 - S_2).$$

Finally, we obtain

$$4(x-r)^2(\rho\,U_m(r))\ \mathrm{tr}\left(\mathbf{T}_m(x)\,\mathbf{R}_m^{-1}\right) = 2\,(T_{m+1}(x) - T_{m+1}(r)) - 2\,(m+1)\,(x-r)\,U_m(r). \tag{32}$$

From this (27) follows immediately. The proof of (26) is now complete. □

6. Conclusions

This paper shows, through a particular example, why sampling survey methodology needs mathematics and vice versa, how it can be a source of intriguing purely mathematical problems. We were concerned with a connection between rotation sampling design and the Chebyshev polynomials, which was used in KW to give a complete description of the recursion for BLUEs of means on every occasion. The recursion depth was identified through the largest gap in the rotation pattern and the recursion coefficients in terms of the Chebyshev polynomials depending on correlations for a single unit. According to the standard Patterson model, these correlations are assumed to be exponential in time and the same for every unit, with independence between units. The general form of the recurssion was derived in KW under ASSUMPTIONS I and II and expressed in terms of the Chebyshev polynomials. There is a strong numerical evidence that both the assumptions are not needed for the recursion to hold true. In this paper, using intrinsic properties of the Chebyshev polynomials of the first and the second kind, we proved that, at least for rotation designs with one arbitrary large gap, ASSUMPTION I is always satisfied. However, the problem if ASSUMPTION II is also satisfied, even in such a simplified rotation pattern, remains a challenging mathematical question.

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