STATISTICS IN TRANSITION new series, June 2023
Vol. 24, No. 3, pp. 95–116, https://doi.org/10.59170/stattrans-2023-038
Received – 11.09.2021; accepted – 17.08.2022

Efficient estimation of population mean in the presence of non-response and measurement error

Kuldeep Kumar Tiwari[1], **Vishwantra Sharma**[2]

ABSTRACT

In real-world surveys, non-response and measurement errors are common, therefore studying them together seems rational. Some population mean estimators are modified and studied in the presence of non-response and measurement errors. Bias and mean squared error expressions are derived under different cases. For all estimators, a theoretical comparison is made with the sample mean per unit estimator. The Monte-Carlo simulation is used to present a detailed picture of all estimators' performance.

Key words: non-response, measurement error, mean squared error, efficiency, mean estimation.

1. Introduction

The sampling technique is the most effective way to make population predictions by using a sample of units from the population. All of the survey's sampling strategies are theoretically based on some assumptions. One assumption that almost never holds true in real-world surveys is that all units in the sample will respond. Non-response could be due to a variety of factors, including the respondent's availability, discomfort with the questions/interviewer, or a lack of desire to contribute. However, the increase in error caused by non-response had a significant impact on the final results. In the presence of non-response in sample surveys, Hansen and Hurwitz (1946) proposed a method for estimating the population mean. They usually use it for mail surveys because they are less expensive. To use this method, first send a questionnaire to all of the units in the sample via mail. After that, select a sub-sample from the non-respondent units and conduct a direct or telephone interview with them. When contacted directly, he assumes that every unit in the non-respondent sub-sample responds. Hansen and Hurwitz (1946) defined the estimator of population mean in the presence of non-response as $\bar{y}_t^* = (\frac{n_1}{n})\bar{y}_{n_1} + (\frac{n_2}{n})\bar{y}_r$; where $\bar{y}_{n_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} y_i$, $\bar{y}_r = \frac{1}{r}\sum_{i=1}^{r} y_i$, n is sample size, n_1 is the number of respondent in the sample and n_2 is the number of non-respondent such that $n_1 + n_2 = n$. $r = \frac{n_2}{k}, k > 0$ is the size of sub-sample of non-respondent. Using this concept of handling non-response, many authors have proposed estimators for a variety of cases over decades. Some of them are Cochran (1977), Rao (1986), Okafor and Lee (2000), Kreuter et al. (2010), Khan et al. (2014), Luengo (2016), Khare and Sinha (2019), Sharma and Kumar (2020), Pandey et al. (2021), Sinha et al. (2022).

[1]Department of Mathematics, Chandigarh University, Mohali, Punjab, India.
E-mail: kuldeep.smvd@gmail.com. ORCID: https://orcid.org/0000-0001-5083-1206.
[2]Directorate of Census Operations, Jammu and Kashmir, India.
E-mail: vishwantrasharma07@gmail.com. ORCID: https://orcid.org/0000-0001-7387-0670.

Aside from non-response, measurement error is another error that affects the results of a real-life survey. We assume that all of the data that have been recorded and processed are accurate. However, in real-life surveys, this is purely hypothetical. Measurement error can be caused by a variety of factors, including interviewer bias, respondent bias, error in recording and processing the data, and so on. Some notable works on the estimation in the presence of measurement error are Cochran (1977), Fuller (1987), Shalabh (1997), Srivastava and Shalabh (2001), Gregoire and Salas (2009), Diane and Giordan (2012), Tiwari et al. 2022.

Since the presence of non-response and measurement error is expected to be in any survey, so it is desirable to study both of them at the same time. Very few works have been done so far on this. The contribution of researchers in this area is Kumar et al. (2015), Singh and Sharma (2015), Azeem and Hanif (2016), Kumar and Bhougal (2018), Kumar et al. (2018), Singh et al. (2018), Zahid et al. (2022), Tiwari et al. (2022).

So, here we carried out a study on the estimation of population mean in the presence of non-response and measurement error.

2. Notations

Let a finite population of size N be divided into two groups as respondent of size N_1 and non-respondent of size N_2. Let a sample of size n be taken from the population among which n_1 are respondent and n_2 are non-respondent. A sub-sample of size $r(=\frac{n_2}{k}), k>0$ is taken from n_2 non-respondents. At i^{th} unit of population, y_i and x_i be the observed values of study and auxiliary variables and $y_{t\,i}$, $x_{t\,i}$ be their true values, respectively.

The other notations are $\bar{x}_t^* = (\frac{n_1}{n})\bar{x}_{n_1} + (\frac{n_2}{n})\bar{y}_r$, $\bar{x}_{n_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i$, $\bar{x}_r = \frac{1}{r}\sum_{i=1}^{r} x_i$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

The population mean and variance for y are $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$, $S_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$. The population mean and variance for x are $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$, $S_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2$. Population variance for y and x for the group of non-respondent is
$S_{Y(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(Y_i-\bar{Y})^2$ and $S_{X(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(X_i-\bar{X})^2$ respectively.

Let $U_i = y_i - y_{t\,i}$ and $V_i = x_i - x_{t\,i}$ be the measurement error on the study and auxiliary variable respectively at i^{th} unit of the population. So, $\bar{y}^* = \bar{y}_t^* + \bar{U}^*$ and $\bar{x}^* = \bar{x}_t^* + \bar{V}^*$, where $\bar{U}^* = (\frac{n_1}{n})\bar{U}_{n_1} + (\frac{n_2}{n})\bar{U}_r$, $\bar{U}_{n_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} U_i$, $\bar{U}_r = \frac{1}{r}\sum_{i=1}^{r} U_i$ and $\bar{V}^* = (\frac{n_1}{n})\bar{V}_{n_1} + (\frac{n_2}{n})\bar{V}_r$, $\bar{V}_{n_1} = \frac{1}{n_1}\sum_{i=1}^{n_1} V_i$, $\bar{V}_r = \frac{1}{r}\sum_{i=1}^{r} V_i$.

Study variable y and auxiliary variable x are correlated with correlation coefficient ρ and ρ_2 is the correlation coefficient between y and x for the group of non-respondent. Since there is no relationship between measurement errors occurring on y and x, so U and V must be independent. Also, since there will be both under-reporting and over-reporting in measurement error, so we assume that mean of U and mean of V are zero. The population variance of measurement error assosiated with y is $S_U^2 = \frac{1}{N-1}\sum_{i=1}^{N}(U_i-\bar{U})^2$ and the population variance of measurement error assosiated with x is $S_V^2 = \frac{1}{N-1}\sum_{i=1}^{N}(V_i-\bar{V})^2$. Population variances of U and V for the group of non-respondent are $S_{U(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(U_i-\bar{U})^2$ and $S_{V(2)}^2 = \frac{1}{N_2-1}\sum_{i=1}^{N_2}(V_i-\bar{V})^2$ respectively. b_{yx} is sample regression coefficient of y on x.

Other notations used in the article are: $\theta = \frac{W_2(k-1)}{n}$, $W_2 = \frac{N_2}{N}$, $\lambda = \frac{1}{n} - \frac{1}{N}$, $R = \frac{\bar{Y}}{\bar{X}}$, $w_1 = \frac{n_1}{n}$, $w_2 = \frac{n_2}{n}$.

3. Brief review of literature

In this section, we discuss some estimators from the literature that will be further used.

Searls (1964) proposes an estimator $t_1 = k\bar{y}$ in simple random sampling, where k is a suitable constant. He shows that $MSE(t_1)$, that is mean square error of t_1 is less than the variance of $\bar{y}$, hence t_1 is preferable over usual estimator $\bar{y}$. Cochran (1940) defined the ratio estimator $t_2 = \bar{y}(\frac{\bar{X}}{\bar{x}})$. He further studied the ratio estimator in the presence of non-response when non-response occurs on both study variables and auxiliary variables in Cochran (1977). Rao (1986) studied the ratio estimator when there is non-response only on the study variable. Shalabh (1997) adapted the ratio estimator and presented a study on the ratio method of estimation in the presence of measurement error. Murthy (1964) proposes the product method of estimation by defining the estimator $t_3 = \bar{y}(\frac{\bar{x}}{\bar{X}})$. He shows that the product method of estimation is better to use when there is a high negative correlation between the study and the auxiliary variable. Khare and Srivastava (1993) studied ratio and product estimator in double sampling when there is non-response on both study and auxiliary variables. Cochran (1977) studied the usual regression estimator $t_4 = \bar{y} + b_{yx}(\bar{X} - \bar{x})$ and its properties when there is non-response on both study and auxiliary variables. Srivastava and Shalabh (2001) examined the regression estimator in the presence of measurement error. Okafor and Lee (2000) presented a study on ratio and regression estimator when there is non-response on both variables in the double sampling scheme. Srivastava (1967) generalise the ratio estimator by proposing $t_5 = \bar{y}(\frac{\bar{X}}{\bar{x}})^{\alpha}$. Rao (1991) proposed a difference estimator $t_6 = k_1\bar{y} + k_2(\bar{X} - \bar{x})$ in simple random sampling and shows that it works better than regression estimator. Bahl and Tuteja (1991) first time uses exponential function to estimate the population mean by defining ratio and product type estimator $t_7 = \bar{y}\exp(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}})$ and $t_8 = \bar{y}\exp(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}})$ respectively. Using ratio and product estimator, Singh and Espejo (2003) proposed an estimator as $t_9 = \bar{y}[a(\frac{\bar{X}}{\bar{x}}) + (1-a)(\frac{\bar{x}}{\bar{X}})]$ and show that its optimum mean square error (MSE) is the same as regression estimator. Kadilar and Cingi (2004) proposed an estimator using regression and ratio estimator as $t_{10} = [\bar{y} + b_{yx}(\bar{X} - \bar{x})](\frac{\bar{X}}{\bar{x}})$. Singh and Sharma (2015) studied the ratio and regression estimator in the presence of non-response and measurement error when non-response occurs on both study variable and auxiliary variable.

Now we will adapt the estimators t_1, t_2,..,t_{10}, and study it in the simultaneous presence of measurement error and non-response.

4. Adapted estimators

We have adapted estimators t_1, t_2,..,t_{10} to study them in the presence of non-response and measurement error. Here, we redefine them in two cases and will further investigate.

4.1. Case-1

When non-response occurs only on study variable then t_1, t_2,..,t_{10} can be redefined as

1 $t_{11} = k_1\bar{y}^*$, k_1 is constant.

2 $t_{12} = \bar{y}^*(\frac{\bar{X}}{\bar{x}})$

3 $t_{13} = \bar{y}^*(\frac{\bar{x}}{\bar{X}})$

4 $t_{14} = \bar{y}^* + b_1(\bar{X} - \bar{x})$, b_1 is constant.

5 $t_{15} = \bar{y}^*(\frac{\bar{X}}{\bar{x}})^{\alpha_1}$, α_1 is constant.

6 $t_{16} = k_{11}\bar{y}^* + k_{12}(\bar{X} - \bar{x})$, k_{11}, k_{12} are constants.

7 $t_{17} = \bar{y}^* \exp(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}})$

8 $t_{18} = \bar{y}^* \exp(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}})$

9 $t_{19} = \bar{y}^*[a_1(\frac{\bar{X}}{\bar{x}}) + (1-a_1)(\frac{\bar{x}}{\bar{X}})]$, a_1 is constant.

10 $t_{20} = [\bar{y}^* + d_1(\bar{X} - \bar{x})](\frac{\bar{X}}{\bar{x}})$, d_1 is constant.

with constants to be determined for minimum MSE.

4.2. Case-2

When non-response occurs on both study and auxiliary variable then t_1, t_2,..,t_{10} can be redefined as

1 $t_{21} = k_2\bar{y}^*$, k_2 is constant.

2 $t_{22} = \bar{y}^*(\frac{\bar{X}}{\bar{x}^*})$

3 $t_{23} = \bar{y}^*(\frac{\bar{x}^*}{\bar{X}})$

4 $t_{24} = \bar{y}^* + b_2(\bar{X} - \bar{x}^*)$, b_2 is constant.

5 $t_{25} = \bar{y}^*(\frac{\bar{X}}{\bar{x}^*})^{\alpha_2}$, α_2 is constant.

6 $t_{26} = k_{21}\bar{y}^* + k_{22}(\bar{X} - \bar{x}^*)$, k_{21} and k_{22} are constants.

7 $t_{27} = \bar{y}^* \exp(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*})$

8 $t_{28} = \bar{y}^* \exp(\frac{\bar{x}^*-\bar{X}}{\bar{x}^*+\bar{X}})$

9 $t_{29} = \bar{y}^*[a_2(\frac{\bar{X}}{\bar{x}^*}) + (1-a_2)(\frac{\bar{x}^*}{\bar{X}})]$, a_2 is constant.

10 $t_{30} = [\bar{y}^* + d_2(\bar{X} - \bar{x}^*)](\frac{\bar{X}}{\bar{x}^*})$, d_2 is constant.

with constants to be determined for minimum MSE.

5. Bias and Mean square error

We derive the bias and mean square error (MSE) of the estimators using the following terms.

$U^* = y_i^* - Y_i^*$ and $\omega_Y^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n (Y_i^* - \bar{Y})$, $\omega_U^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i^*$

Adding ω_Y^* and ω_U^* and dividing both side by $\sqrt{n}$, we have

$\frac{\omega_Y^* + \omega_U^*}{\sqrt{n}} = \frac{1}{n} \sum_{i=1}^n [(Y_i^* - \bar{Y}) + U_i^*]$ which is $\frac{\omega_Y^* + \omega_U^*}{\sqrt{n}} = \frac{1}{n} \sum_{i=1}^n y_i^* - \bar{Y}$

So,

$$\bar{y}^* = \bar{Y} + \xi_Y^*; \quad \text{where} \quad \xi_Y^* = \frac{\omega_Y^* + \omega_U^*}{\sqrt{n}}. \tag{1}$$

Similarly, for $\omega_X = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i - \bar{X})$ and $\omega_V = \frac{1}{\sqrt{n}} \sum_{i=1}^n V_i$, we have

$$\bar{x} = \bar{X} + \xi_X; \quad \text{where} \quad \xi_X = \frac{\omega_X + \omega_V}{\sqrt{n}}. \tag{2}$$

Again, for $V^* = x_i^* - X_i^*$ and $\omega_X^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^* - \bar{X})$, $\omega_V^* = \frac{1}{\sqrt{n}} \sum_{i=1}^n V_i^*$, we have

$$\bar{x}^* = \bar{X} + \xi_X^*; \quad \text{where} \quad \xi_X^* = \frac{\omega_X^* + \omega_V^*}{\sqrt{n}}. \tag{3}$$

Expected values of errors are

$$E(\xi_Y^{*2}) = A_{MSE} + A_{ME} = A, \tag{4}$$

where $A_{MSE} = \lambda S_Y^2 + \theta S_{Y(2)}^2$ and $A_{ME} = \lambda S_U^2 + \theta S_{U(2)}^2$.

$$E(\xi_X^2) = B_{MSE} + B_{ME} = B, \tag{5}$$

where $B_{MSE} = \lambda S_X^2$ and $A_{ME} = \lambda S_V^2$.

$$E(\xi_Y^* \xi_X) = \lambda \rho S_Y S_X = C, \tag{6}$$

$$E(\xi_X^{*2}) = D_{MSE} + D_{ME} = D, \tag{7}$$

where $D_{MSE} = \lambda S_X^2 + \theta S_{X(2)}^2$ and $D_{ME} = \lambda S_V^2 + \theta S_{V(2)}^2$.

$$E(\xi_Y^* \xi_X^*) = \lambda \rho S_Y S_X + \theta \rho_2 S_{Y(2)} S_{X(2)} = E, \tag{8}$$

and

$$E(\xi_Y^*) = E(\xi_X) = E(\xi_X^*) = E(\xi_U) = E(\xi_V) = 0. \tag{9}$$

Now, using these values we derive the bias and *MSE* for all the estimators.

5.1. Case-1

5.1.1 $t_{11} = k_1 \bar{y}^*$

Using equation (1), express t_{11} in terms of error as $t_{11} = k_1(\bar{Y} + \xi_Y^*)$, so

$$t_{11} - \bar{Y} = (k_1 - 1)\bar{Y} + k_1 \xi_Y^*. \tag{10}$$

Taking expectation on both sides of equation (10), we get

$$Bias(t_{11}) = (k_1 - 1)\bar{Y}. \tag{11}$$

Now squaring both sides of equation (10), we have

$$(t_{11} - \bar{Y})^2 = (k_1 - 1)^2 \bar{Y}^2 + k_1^2 \xi_Y^{*2} + 2k_1(k_1 - 1)\bar{Y}\xi_Y^*, \tag{12}$$

taking expectation to equation (12), we get

$$MSE(t_{11}) = (k_1 - 1)^2 \bar{Y}^2 + k_1^2 A. \tag{13}$$

Minimizing $MSE(t_{11})$ for k_1, we get the optimum value of k_1 as $k_1^o = \frac{\bar{Y}^2}{\bar{Y}^2 + A}$
Now, putting optimum value of k_1 in equation (13), we get minimum MSE of t_{11}.

$$MSE_{min}(t_{11}) = \frac{A\bar{Y}^2}{A + \bar{Y}^2}. \tag{14}$$

5.1.2 $t_{12} = \bar{y}^* (\frac{\bar{X}}{\bar{x}})$

Expanding t_{12} using equation (1) and (2), we have $t_{12} = (\bar{Y} + \xi_Y^*)\frac{\bar{X}}{(\bar{X} + \xi_X)}$ or $t_{12} = (\bar{Y} + \xi_Y^*)(1 + \frac{\xi_X}{\bar{X}})^{-1}$.

Assuming $|\xi| < 1$, expanding series in the right side and terminating the terms having ξ's degree greater than two, we have

$$t_{12} = (\bar{Y} + \xi_Y^*)\big(1 - \frac{\xi_X}{\bar{X}} + \frac{\xi_X^2}{\bar{X}^2}\big),$$

on simplifying, we get

$$t_{12} - \bar{Y} = \xi_Y^* - R\xi_X + \frac{R\xi_X^2}{\bar{X}} - \frac{\xi_Y^* \xi_X}{\bar{X}}, \tag{15}$$

where $R = \frac{\bar{Y}}{\bar{X}}$.
Taking expectation on both sides of equation (15), we get

$$Bias(t_{12}) = \frac{RB - C}{\bar{X}}. \tag{16}$$

Squaring equation (15) and terminating terms with ξ's degree greater than two and simplifying, we get

$$(t_{12}-\bar{Y})^2 = \xi_Y^{*2} - R^2\xi_X^2 - 2R\xi_Y^*\xi_X, \tag{17}$$

taking expectation to equation (17), we get

$$MSE(t_{12}) = A + R^2B - 2RC. \tag{18}$$

5.1.3 $t_{13} = \bar{y}^*(\frac{\bar{x}}{\bar{X}})$

Proceeding as in 5.1.2, we get the bias and MSE of t_{13} as

$$Bias(t_{13}) = \frac{C}{\bar{X}}, \tag{19}$$

and

$$MSE(t_{13}) = A + R^2B + 2RC. \tag{20}$$

5.1.4 $t_{14} = \bar{y}^* + b_1(\bar{X} - \bar{x})$

Using equation (1) and (2) expanding t_{14}, we get $t_{14} = \bar{Y} + \xi_Y^* - b_1\xi_X$
so we have

$$t_{14} - \bar{Y} = \xi_Y^* - b_1\xi_X. \tag{21}$$

On taking expectation to equation (21), we get

$$Bias(t_{14}) = 0. \tag{22}$$

Squaring both sides of equation (21) and taking expectation, we get

$$MSE(t_{14}) = A + b_1^2B - 2b_1C. \tag{23}$$

Minimizing $MSE(t_{14})$ for b_1, the optimum value of b_1 is $b_1^o = \frac{C}{B}$.
Using optimum value of b_1 in equation (23), we get

$$MSE_{min}(t_{14}) = A - \frac{C^2}{B}. \tag{24}$$

5.1.5 $t_{15} = \bar{y}^*(\frac{\bar{X}}{\bar{x}})^{\alpha_1}$

Proceeding as in 5.1.2, we get the bias and MSE of t_{15} as

$$Bias(t_{15}) = \frac{\alpha_1(\alpha_1+1)RB - 2\alpha_1C}{2\bar{X}}, \tag{25}$$

and

$$MSE(t_{15}) = A + \alpha_1^2R^2B - 2\alpha_1RC. \tag{26}$$

Minimizing $MSE(t_{15})$ for α_1, the optimum value of α_1 is $\alpha_1^o = \frac{C}{RB}$.

Putting optimum value of α_1 in equation (26), we get

$$MSE_{min}(t_{15}) = A - \frac{C^2}{B}. \tag{27}$$

5.1.6 $t_{16} = k_{11}\bar{y}^* + k_{12}(\bar{X} - \bar{x})$

Expressing t_{16} in terms of error using equation (1) and (2), we have $t_{16} = k_{11}(\bar{Y} + \xi_Y^*) + k_{12}(\bar{X} - \bar{X} - \xi_X)$. On simplifying, we get

$$t_{16} - \bar{Y} = (k_{11} - 1)\bar{Y} + k_{11}\xi_Y^* - k_{12}\xi_X, \tag{28}$$

taking expectation on both sides of equation (28), we get

$$Bias(t_{16}) = (k_{11} - 1)\bar{Y}. \tag{29}$$

Squaring both sides of equation (28), have

$$(t_{16} - \bar{Y})^2 = (k_{11} - 1)^2\bar{Y}^2 + k_{11}^2\xi_Y^{*2} + k_{12}^2\xi_X^2 + 2k_{11}(k_{11} - 1)\bar{Y}\xi_Y^* - 2k_{11}k_{12}\xi_Y^*\xi_X - 2k_{12}(k_{11} - 1)\bar{Y}\xi_X, \tag{30}$$

taking expectation on both sides of equation (30), we get

$$MSE(t_{16}) = (k_{11} - 1)^2\bar{Y}^2 + k_{11}^2 A + k_{12}^2 B - 2k_{11}k_{12}C. \tag{31}$$

Minimizing $MSE(t_{16})$ for k_{11} and k_{12}, we get the optimum values of k_{11} and k_{12} as $k_{11}^o = \frac{B\bar{Y}^2}{B\bar{Y}^2+AB-C^2}$ and $k_{12}^o = \frac{C\bar{Y}^2}{B\bar{Y}^2+AB-C^2}$.

Using optimum values of k_{11} and k_{12} in equation (31), we get minimum MSE.

$$MSE_{min}(t_{16}) = \frac{\bar{Y}^2(AB - C^2)}{B\bar{Y}^2 + AB - C^2}. \tag{32}$$

5.1.7 $t_{17} = \bar{y}^* \exp(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}})$

Expressing t_{17} in terms of error as $t_{17} = (\bar{Y} + \xi_Y^*)\exp\left(\frac{\bar{X}-\bar{X}-\xi_X}{\bar{X}+\bar{X}+\xi_X}\right)$. On simplifying, we get $t_{17} = (\bar{Y} + \xi_Y^*)\exp\left[-\frac{\xi_X}{2\bar{X}}\left(1 + \frac{\xi_X}{2\bar{X}}\right)^{-1}\right]$. Expand the series and ignore the terms having ξ's degree greater than two. After simplification, we get $t_{17} = (\bar{Y} + \xi_Y^*)(1 - \frac{\xi_X}{2\bar{X}} + \frac{3}{8}\frac{\xi_X^2}{\bar{X}^2})$. So, we have

$$t_{17} - \bar{Y} = \xi_Y^* - \frac{R\xi_X}{2} + \frac{3}{8}\frac{R\xi_X^2}{\bar{X}} - \frac{\xi_Y^*\xi_X}{2\bar{X}}. \tag{33}$$

Taking expectation on both sides of equation (33), we get

$$Bias(t_{17}) = \frac{3RB - 4C}{8\bar{X}}. \tag{34}$$

Squaring equation (33) on both sides and taking expectation, we get

$$MSE(t_{17}) = A + \frac{R^2B}{4} - RC. \tag{35}$$

5.1.8 $t_{18} = \bar{y}^* \exp(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}})$

Proceeding on the lines of 5.1.7, we get the bias and MSE of t_{18} as

$$Bias(t_{18}) = \frac{4C - RB}{8\bar{X}}, \tag{36}$$

$$MSE(t_{18}) = A + \frac{R^2B}{4} + RC. \tag{37}$$

5.1.9 $t_{19} = \bar{y}^*[a_1(\frac{\bar{X}}{\bar{x}}) + (1-a_1)(\frac{\bar{x}}{\bar{X}})]$

Express t_{19} in terms of error as $t_{19} = (\bar{Y} + \xi_Y^*)[a_1(\frac{\bar{X}}{\bar{X}+\xi_X}) + (1-a_1)\frac{(\bar{X}+\xi_X)}{\bar{X}}]$. After little simplification $t_{19} = (\bar{Y} + \xi_Y^*)[a_1(1+\frac{\xi_X}{\bar{X}})^{-1} + (1-a_1)(1+\frac{\xi_X}{\bar{X}})]$. Expand the series and ignore the terms having ξ's degree greater than two. On simplification, we get $t_{19} = (\bar{Y} + \xi_Y^*)[1 + (1-2a_1)\frac{\xi_X}{\bar{X}} + \frac{a_1\xi_X^2}{\bar{X}^2}]$. So, we have

$$t_{19} - \bar{Y} = \xi_Y^* + (1-2a_1)R\xi_X + \frac{a_1R\xi_X^2}{\bar{X}} + (1-2a_1)\frac{\xi_Y^*\xi_X}{\bar{X}}. \tag{38}$$

Taking expectation on both sides of equation (38), we get

$$Bias(t_{19}) = \frac{a_1RB + (1-2a_1)C}{\bar{X}}. \tag{39}$$

Squaring both sides of equation (38) and taking expectation, we get

$$MSE(t_{19}) = A + (1-2a_1)^2R^2B + 2R(1-2a_1)C. \tag{40}$$

Minimizing $MSE(t_{19})$ for a_1, we get the optimum value of a_1 as $a_1^o = \frac{1}{2} + \frac{C}{2RB}$. Putting optimum value of a_1 in equation (40), we get

$$MSE_{min}(t_{19}) = A - \frac{C^2}{B}. \tag{41}$$

5.1.10 $t_{20} = [\bar{y}^* + d_1(\bar{X} - \bar{x})](\frac{\bar{X}}{\bar{x}})$

Expressing t_{20} in terms of error and simplifying, we get

$$t_{20} - \bar{Y} = \xi_Y^* - (R+d_1)\xi_X + \frac{(R+d_1)\xi_X^2}{\bar{X}} - \frac{\xi_Y^*\xi_X}{\bar{X}}, \tag{42}$$

taking expectation on both sides of equation (42), we get

$$Bias(t_{20}) = \frac{(R+d_1)B - C}{\bar{X}}. \tag{43}$$

Squaring both sides of equation (42) and taking expectation, we get

$$MSE(t_{20}) = A + (R+d_1)^2 B - 2(R+d_1)C. \tag{44}$$

Minimizing $MSE(t_{20})$ for d_1, we get the optimum value of d_1 as $d_1^o = \frac{C}{B} - R$.
Putting optimum value of d_1 in equation (44), we get

$$MSE_{min}(t_{20}) = A - \frac{C^2}{B}. \tag{45}$$

The bias and MSEs in the next case can be obtained in similar steps used in Case-1. To save space, only the results are given.

5.2. Case-2

5.2.1 $t_{21} = k_2 \bar{y}^*$

$$Bias(t_{21}) = (k_2 - 1)\bar{Y}, \tag{46}$$

$$MSE(t_{21}) = (k_2 - 1)^2 \bar{Y}^2 + k_2^2 A. \tag{47}$$

Optimum value of k_2 is $k_2^o = \frac{\bar{Y}^2}{\bar{Y}^2 + A}$.

$$MSE_{min}(t_{21}) = \frac{A\bar{Y}^2}{A + \bar{Y}^2}. \tag{48}$$

5.2.2 $t_{22} = \bar{y}^* (\frac{\bar{X}}{\bar{x}^*})$

$$Bias(t_{22}) = \frac{RD - E}{\bar{X}}, \tag{49}$$

$$MSE(t_{22}) = A + R^2 D - 2RE. \tag{50}$$

5.2.3 $t_{23} = \bar{y}^* (\frac{\bar{x}^*}{\bar{X}})$

$$Bias(t_{23}) = \frac{E}{\bar{X}}, \tag{51}$$

$$MSE(t_{23}) = A + R^2 D + 2RE. \tag{52}$$

5.2.4 $t_{24} = \bar{y}^* + b_2(\bar{X} - \bar{x}^*)$

$$Bias(t_{24}) = 0, \tag{53}$$

$$MSE(t_{24}) = A + b_2^2 D - 2b_2 E. \tag{54}$$

Optimum value of b_2 is $b_2^o = \frac{E}{D}$.

$$MSE_{min}(t_{24}) = A - \frac{E^2}{D}. \tag{55}$$

5.2.5 $t_{25} = \bar{y}^* (\frac{\bar{X}}{\bar{x}^*})^{\alpha_2}$

$$Bias(t_{25}) = \frac{\alpha_2(\alpha_2+1)RD - 2\alpha_2 E}{2\bar{X}}, \tag{56}$$

$$MSE(t_{25}) = A + \alpha_2^2 R^2 D - 2\alpha_2 RE. \tag{57}$$

Optimum value of α_2 is $\alpha_2^o = \frac{E}{RD}$.

$$MSE_{min}(t_{25}) = A - \frac{E^2}{D}. \tag{58}$$

5.2.6 $t_{26} = k_{21}\bar{y}^* + k_{22}(\bar{X} - \bar{x}^*)$

$$Bias(t_{26}) = (k_{21} - 1)\bar{Y}, \tag{59}$$

$$MSE(t_{26}) = (k_{21} - 1)^2\bar{Y}^2 + k_{21}^2 A + k_{22}^2 D - 2k_{21}k_{22}E. \tag{60}$$

Optimum values of k_{21} and k_{22} are $k_{21}^o = \frac{D\bar{Y}^2}{D\bar{Y}^2+AD-E^2}$ and $k_{22}^o = \frac{E\bar{Y}^2}{D\bar{Y}^2+AD-E^2}$.

$$MSE_{min}(t_{26}) = \frac{\bar{Y}^2(AD - E^2)}{D\bar{Y}^2 + AD - E^2}. \tag{61}$$

5.2.7 $t_{27} = \bar{y}^* \exp(\frac{\bar{X}-\bar{x}^*}{\bar{X}+\bar{x}^*})$

$$Bias(t_{27}) = \frac{3RD - 4E}{8\bar{X}}, \tag{62}$$

$$MSE(t_{27}) = A + \frac{R^2 D}{4} - RE. \tag{63}$$

5.2.8 $t_{28} = \bar{y}^* \exp(\frac{\bar{x}^*-\bar{X}}{\bar{x}^*+\bar{X}})$

$$Bias(t_{28}) = \frac{4E - RD}{8\bar{X}}, \tag{64}$$

$$MSE(t_{28}) = A + \frac{R^2 D}{4} + RE. \tag{65}$$

5.2.9 $t_{29} = \bar{y}^*[a_2(\frac{\bar{X}}{\bar{x}^*}) + (1-a_2)(\frac{\bar{x}^*}{\bar{X}})]$

$$Bias(t_{29}) = \frac{a_1 RD + (1-2a_2)E}{\bar{X}}, \tag{66}$$

$$MSE(t_{29}) = A + (1-2a_2)^2 R^2 D + 2R(1-2a_2)E. \tag{67}$$

Optimum value of a_2 is $a_2^o = \frac{1}{2} + \frac{E}{2RD}$.

$$MSE_{min}(t_{29}) = A - \frac{E^2}{D}. \tag{68}$$

5.2.10 $t_{30} = [\bar{y}^* + d_2(\bar{X} - \bar{x}^*)](\frac{\bar{X}}{\bar{x}^*})$

$$Bias(t_{30}) = \frac{(R+b_2)D - E}{\bar{X}}, \tag{69}$$

$$MSE(t_{30}) = A + (R+b_2)^2 D - 2(R+b_2)E. \tag{70}$$

Optimum value of d_2 is $d_2^o = \frac{E}{D} - R$.

$$MSE_{min}(t_{30}) = A - \frac{E^2}{D}. \tag{71}$$

Note

The optimum MSEs of t_{i4}, t_{i5}, t_{i9} and t_{j0} are equal, where $i = 1,2$ and $j = 2,3$.

6. Efficiency comparison

In this section, we derive the conditions under which the estimators perform better than the usual estimator $\bar{y}^*$. As we know, an estimator t will be more efficient than $\bar{y}^*$ whenever the inequality $var(\bar{y}^*) - MSE(t) > 0$ is satisfied.

The variance of usual estimator $\bar{y}^*$ in the presence of non-response and measurement error is $var(\bar{y}^*) = \lambda S_Y^2 + \theta S_{Y(2)}^2 + \lambda S_U^2 + \theta S_{U(2)}^2$. That is

$$var(\bar{y}^*) = A. \tag{72}$$

6.1. Case-1

1. $MSE(t_{11}) < var(\bar{y}^*)$ *if* $\frac{A^2}{A+\bar{Y}^2} > 0$

2. $MSE(t_{12}) < var(\bar{y}^*)$ *if* $\frac{C}{B} > \frac{R}{2}$

3. $MSE(t_{13}) < var(\bar{y}^*)$ *if* $\frac{C}{B} < -\frac{R}{2}$
4. $MSE(t_{14}) < var(\bar{y}^*)$ *if* $\frac{C^2}{B} > 0$
5. $MSE(t_{15}) < var(\bar{y}^*)$ *if* $\frac{C^2}{B} > 0$
6. $MSE(t_{16}) < var(\bar{y}^*)$ *if* $\frac{A^2B-AC^2+\bar{Y}^2C^2}{B\bar{Y}^2+AB-C^2} > 0$
7. $MSE(t_{17}) < var(\bar{y}^*)$ *if* $\frac{C}{B} > \frac{R}{4}$
8. $MSE(t_{18}) < var(\bar{y}^*)$ *if* $\frac{C}{B} < -\frac{R}{4}$
9. $MSE(t_{19}) < var(\bar{y}^*)$ *if* $\frac{C^2}{B} > 0$
10. $MSE(t_{20}) < var(\bar{y}^*)$ *if* $\frac{C^2}{B} > 0$

6.2. Case-2

1. $MSE(t_{21}) < var(\bar{y}^*)$ *if* $\frac{A^2}{A+\bar{Y}^2} > 0$
2. $MSE(t_{22}) < var(\bar{y}^*)$ *if* $\frac{E}{D} > \frac{R}{2}$
3. $MSE(t_{23}) < var(\bar{y}^*)$ *if* $\frac{E}{D} < -\frac{R}{2}$
4. $MSE(t_{24}) < var(\bar{y}^*)$ *if* $\frac{E^2}{D} > 0$
5. $MSE(t_{25}) < var(\bar{y}^*)$ *if* $\frac{E^2}{D} > 0$
6. $MSE(t_{26}) < var(\bar{y}^*)$ *if* $\frac{A^2D-AE^2+\bar{Y}^2C^2}{D\bar{Y}^2+AD-E^2} > 0$
7. $MSE(t_{27}) < var(\bar{y}^*)$ *if* $\frac{E}{D} > \frac{R}{4}$
8. $MSE(t_{28}) < var(\bar{y}^*)$ *if* $\frac{E}{D} < -\frac{R}{4}$
9. $MSE(t_{29}) < var(\bar{y}^*)$ *if* $\frac{E^2}{D} > 0$
10. $MSE(t_{30}) < var(\bar{y}^*)$ *if* $\frac{E^2}{D} > 0$

7. Monte-Carlo Simulation

For validating the theoretical results in the previous sections, we perform a Monte-Carlo simulation study. We have used the following information to generate the data in R software: $N = 4000$, $n = 500$, $X = rnorm(N,4,7)$, $Y = 1+2X+\varepsilon$, $\varepsilon = rnorm(N,0,1)$, $U = rnorm(N,0,3)$, $V = rnorm(N,0,3)$. We have checked the performance of estimators for a different response rate. To get output more accurate, we have made 10000 replications to the process.

Percent relative efficiency (PRE) of an estimator t with respect to $\bar{y}^*$ is calculated by

$$PRE(t,\, \bar{y}^*) = \frac{var(\bar{y}^*)}{MSE(t)} \times 100. \tag{73}$$

To get PRE without measurement error, we use MSEs without terms of measurement error. That is, $var(\bar{y}^*) = \lambda S_Y^2 + \theta S_{Y(2)}^2$ and $A = A_{MSE} = \lambda S_Y^2 + \theta S_{Y(2)}^2$, $B = B_{MSE} = \lambda S_X^2$, $D = D_{MSE} = \lambda S_X^2 + \theta S_{X(2)}^2$ are used in expressions of MSEs of estimators.

We have compared the estimators using PREs in Table 1 and Table 2. From the definition of PRE in equation (73), higher PRE of an estimator means smaller MSE.

Table 1: PREs of estimators with respect to $\bar{y}^*$ in Case-1

N_1	N_2	Estimator	$PRE(.,\, \bar{y}^*)$ without measurement error					$PRE(.,\, \bar{y}^*)$ with measurement error				
			$1/k$					$1/k$				
			$1/2$	$1/3$	$1/4$	$1/5$	$1/10$	$1/2$	$1/3$	$1/4$	$1/5$	$1/10$
	500	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{11}	100.48	100.54	100.60	100.66	100.97	100.50	100.57	100.63	100.69	101.01
		t_{12}	699.90	420.18	318.36	265.68	175.09	267.41	225.51	200.39	183.64	145.61
		t_{13}	24.62	26.86	28.98	30.98	39.49	24.26	26.49	28.59	30.57	39.03
		t_{14}	773.62	442.70	329.80	272.86	177.20	337.22	266.94	228.79	204.83	154.31
		t_{15}	773.62	442.70	329.80	272.86	177.20	337.22	266.94	228.79	204.83	154.31
		t_{16}	774.11	443.25	330.41	273.53	178.17	337.72	267.51	229.42	205.53	155.33
		t_{17}	337.94	267.35	229.06	205.03	154.39	266.60	225.00	200.02	183.37	145.49
		t_{18}	44.34	47.26	49.88	52.26	61.42	44.47	47.39	50.01	52.39	61.54
		t_{19}	773.62	442.70	329.80	272.86	177.20	337.22	266.94	228.79	204.83	154.31
		t_{20}	773.62	442.70	329.80	272.86	177.20	337.22	266.94	228.79	204.83	154.31
	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{11}	100.54	100.66	100.79	100.91	101.52	100.57	100.69	100.82	100.95	101.59
		t_{12}	419.76	265.45	211.59	184.19	137.79	225.43	183.57	162.66	150.66	125.05
		t_{13}	26.87	30.99	34.67	37.98	50.51	26.49	30.58	34.24	37.53	50.03
		t_{14}	442.22	272.61	215.41	186.69	138.62	266.81	204.72	176.32	160.03	129.04
		t_{15}	442.22	272.61	215.41	186.69	138.62	266.81	204.72	176.32	160.03	129.04
		t_{16}	442.77	273.28	216.20	187.60	140.14	267.38	205.42	177.14	160.99	130.63
		t_{17}	267.21	204.92	176.44	160.12	129.08	224.92	183.29	162.47	149.98	124.99
		t_{18}	47.26	52.28	56.42	59.90	71.34	47.39	52.40	56.54	60.02	71.45
		t_{19}	442.22	272.61	215.41	186.69	138.62	266.81	204.72	176.32	160.03	129.04
		t_{20}	442.22	272.61	215.41	186.69	138.62	266.81	204.72	176.32	160.03	129.04
	1500	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{11}	100.60	100.79	100.97	101.15	102.06	100.63	100.82	101.01	101.20	102.16
		t_{12}	317.98	211.55	174.95	156.43	125.25	200.28	162.64	145.54	135.78	117.26
		t_{13}	28.99	34.68	39.52	43.69	58.13	28.60	34.24	39.06	43.21	57.66
		t_{14}	329.38	215.36	177.06	157.85	125.75	228.63	176.29	154.23	142.06	119.82
		t_{15}	329.38	215.36	177.06	157.85	125.75	228.63	176.29	154.23	142.06	119.82
		t_{16}	329.99	216.16	178.03	159.01	127.82	229.27	177.12	155.24	143.27	121.99
		t_{17}	228.89	176.41	154.30	142.11	119.84	199.92	162.45	145.42	135.69	117.23
		t_{18}	49.90	56.42	61.44	65.43	77.20	50.03	56.55	61.57	65.54	77.29
		t_{19}	329.38	215.36	177.06	157.85	125.75	228.63	176.29	154.23	142.06	119.82
		t_{20}	329.38	215.36	177.06	157.85	125.75	228.63	176.29	154.23	142.06	119.82

Table 2: PREs of estimators with respect to $\bar{y}^*$ in Case-2

N_1	N_2	Estimator	$PRE(., \bar{y}^*)$ without measurement error					$PRE(., \bar{y}^*)$ with measurement error				
			$1/k$					$1/k$				
			$1/2$	$1/3$	$1/4$	$1/5$	$1/10$	$1/2$	$1/3$	$1/4$	$1/5$	$1/10$
3500	500	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{21}	100.48	100.54	100.60	100.66	100.97	100.50	100.57	100.63	100.69	101.01
		t_{22}	4845.82	4845.35	4844.98	4844.68	4843.73	351.22	351.15	351.11	351.07	350.94
		t_{23}	22.23	22.23	22.23	22.23	22.23	21.89	21.89	21.89	21.89	21.89
		t_{24}	19691.95	19684.57	196778.67	19673.84	19658.76	509.55	509.47	509.41	509.36	509.19
		t_{25}	19691.95	19684.57	196778.67	19673.84	19658.76	509.55	509.47	509.41	509.36	509.19
		t_{26}	19692.44	19685.12	19679.28	19674.51	19659.73	510.06	510.05	510.05	510.06	510.21
		t_{27}	511.53	511.52	511.52	511.52	511.52	399.68	399.66	399.64	349.63	399.59
		t_{28}	41.08	41.08	41.08	41.08	41.08	41.20	41.20	41.20	41.20	41.20
		t_{29}	19691.95	19684.57	196778.67	19673.84	19658.76	509.55	509.47	509.41	509.36	509.19
		t_{30}	19691.95	19684.57	196778.67	19673.84	19658.76	509.55	509.47	509.41	509.36	509.19
3000	1000	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{21}	100.54	100.66	100.79	100.91	101.52	100.57	100.69	100.82	100.95	101.59
		t_{22}	4845.45	4845.65	4845.94	4845.29	4844.90	351.31	351.32	351.33	351.33	
		t_{23}	22.23	22.23	22.23	22.23	22.23	21.89	21.89	21.89	21.89	21.89
		t_{24}	19692.31	19686.50	19682.48	19679.54	19671.88	509.67	509.68	509.68	509.69	509.70
		t_{25}	19692.31	19686.50	19682.48	19679.54	19671.88	509.67	509.68	509.68	509.69	509.70
		t_{26}	19692.86	19687.17	19683.27	19680.45	19673.40	510.24	510.38	510.51	510.64	511.29
		t_{27}	511.53	511.53	511.53	511.53	511.51	349.71	349.70	349.70	349.70	349.70
		t_{28}	41.08	41.08	41.08	41.08	41.08	41.20	41.20	41.20	41.20	41.20
		t_{29}	19692.31	19686.50	19682.48	19679.54	19671.88	509.67	509.68	509.68	509.69	509.70
		t_{30}	19692.31	19686.50	19682.48	19679.54	19671.88	509.67	509.68	509.68	509.69	509.70
2500	1500	$\bar{y}^*$	100	100	100	100	100	100	100	100	100	100
		t_{21}	100.60	100.79	100.97	101.15	102.06	100.63	100.82	101.01	101.20	102.16
		t_{22}	4846.28	4846.21	4846.17	48461.14	4846.07	351.36	351.40	351.42	351.44	351.47
		t_{23}	22.23	22.23	22.23	22.23	22.23	21.89	21.89	21.89	21.89	21.90
		t_{24}	19693.88	19689.81	19687.26	19685.52	19681.43	509.74	509.78	509.80	509.82	509.86
		t_{25}	19693.88	19689.81	19687.26	19685.52	19681.43	509.74	509.78	509.80	509.82	509.86
		t_{26}	19694.49	19690.60	19688.24	19686.68	19683.50	510.37	510.61	510.82	511.03	512.03
		t_{27}	511.52	511.52	511.51	511.51	511.51	349.72	349.73	349.74	349.74	349.75
		t_{28}	41.08	41.08	41.08	41.08	41.08	41.20	41.20	41.20	41.20	41.20
		t_{29}	19693.88	19689.81	19687.26	19685.52	19681.43	509.74	509.78	509.80	509.82	509.86
		t_{30}	19693.88	19689.81	19687.26	19685.52	19681.43	509.74	509.78	509.80	509.82	509.86

From Table 1 and 2, it is concluded that:

1. Searls (1964) estimator t_{i2} has minute advantage over usual estimator $\bar{y}^*$.
2. PREs of estimators t_{i5}, t_{i9} and t_{j0} are equal to the PRE of regression estimator t_{i4}, as their optimum MSEs are the same.
3. Rao (1991) estimator t_{i6} perform slightly better than regression estimator.
4. In all the estimators, Rao (1991) estimator t_{i6} performs best in terms of having highest PREs.

Here $i = 1,2$ and $j = 2,3$.

8. Conclusion

We have considered ten estimators of population mean and studied them in the context of non-response and measurement error. We have obtained the expressions for bias and MSE for all the estimators in various cases. It is noted that optimum MSEs of Srivastava (1967) estimator t_{i5}, Singh and Espejo (2003) estimator t_{i9} and Kadilar and Cingi (2004) estimator t_{j0} are the same which is equal to the MSE of the regression estimator t_{i4} within the same sampling strategy i, $i = 1,2$; $j = 2,3$. This is also verified in the simulation study. It is also worth to mention that Rao (1991) difference estimator t_{i6} performs better than other estimators, although its efficiency over the regression estimator is very minute.

References

Azeem, M., Hanif, M., (2016). Joint influence of measurement error and non-response on estimation of population mean. *Commun Statist. Theory Methods*, 14(1), 12. DOI: 10.1080/03610926.2015.1026992.

Bahl, S., Tuteja, R. K., (1991). Ratio and product type exponential estimators. *Journal of Information and Optimization Sciences*, 12, pp. 159–164.

Cochran, W. G., (1940). The estimation of the yields of cereal experiments by sampling for the ratio gain to total produce. *J. Agric. Soc.*, 30, pp. 262-–275.

Cochran, W. G., (1977). Sampling techniques. *New York: Wiley.*

Diane, G., Giordan, M., (2012). Finite population variance estimation in presence of measurement errors. *Communications in Statistics - Theory and Methods*, 41(23), pp. 4302–4314. https://doi.org/10.1080/03610926.2011.573165.

Fuller, W. A., (1987), Measurement Error Models, *New York: Wiley*.

Gregoire, T. G., Salas, C., (2009). Ratio estimation with measurement error in the auxiliary variate. *Biometrics*, 65(2), 590—598. DOI:10.1111/j.1541-0420.2008.01110.x.

Hansen, M. H., Hurwitz, W. N., (1946). The problem of non-response in sample surveys. *J Amer Statist Assoc*, 41, pp. 517–529.

Kumar, S., Bhougal, S., Nataraja, N. S., (2015). Estimation of population mean in the presence of non-response and measurement error. *Revista Colombiana de Estadistica*, 38(1), pp. 145-–61. DOI:10.15446/rce.v38n1.48807.

Kumar, S., Bhougal, S., (2018). Study on Non Response and Measurement Error using Double Sampling Scheme. J. Stat. Appl. Pro. Lett., 5(1), pp. 43–52.

Kreuter, F., Olson, K., Wagner, J., et al., (2010). Using proxy measure and correlates of survey outcomes to adjust for non-response-examples from multiple surveys. *J Royal Statist Soc Ser A*, 173(3), pp. 1–21.

Khan, M., Shabbir, J., Hussain, Z., et al., (2014). A Class of Estimators for Finite Population Mean in Double Sampling under Nonresponse Using Fractional Raw Moments. *Journal of Applied Mathematics, Volume*. http://dx.doi.org/10.1155/2014/282065.

Kadilar, C., Cingi, H., (2004). Ratio estimators in simple random sampling. *Applied Mathematics and Computation*, 151(3), pp. 893—902. DOI 10.1080/03610926.2019.1682167.

Khare, B. B., Sinha, R. R., (2019). Estimation of product of two population means by multi-auxiliary characters under double sampling the non-respondent. *STATISTICS IN TRANSITION new series*, 20(3), pp. 81—95. DOI: 10.21307/stattrans-2019-025.

Khare, B. B., Srivastava, S., (1993). Estimation of population mean using auxiliary character in presence of non-response. *National Academy of Science and Letters, India*, 16(3), pp. 111–114.

Luengo, A. V. G., (2016). Ratio-cum-product estimation in presence of non-response in successive sampling. *JAMSI*, 12(1), pp. 55–83. https://doi.org/10.1515/jamsi-2016-0005.

Murthy, M. N., (1964). Product method of estimation. *Sankhya A*, 26, pp. 69—74.

Okafor, F. C., Lee, H., (2000). Double sampling for ratio and regression estimation with sub-sampling the non-respondents. *Survey Methodology*, 26, pp. 183—188.

Pandey, A. K., Usman, M., Singh, G. N., (2021). Optimality of ratio and regression type estimators using dual of auxiliary variable under non response, *Alexandria Engineer-*

ing Journal, 60(5), pp. 4461–4471. https://doi.org/10.1016/j.aej.2021.03.031.

Rao, P. S. R. S., (1986). Ratio estimation with sub-sampling the non-respondents. *Survey Methodology*, 12, pp. 217–230.

Rao, T. J., (1991). On certain methods of improving ratio and regression estimators, *Communications in Statistics-Theory and Methods*, 20(10), pp. 3325–3340.

Sharma, V., Kumar, S., (2020). Estimation of population mean using transformed auxiliary variable and non-response. *Revista Investigacio Operacional*, 41(3), pp. 438–444.

Sinha, R. R., Dhingra, H., Thakur, P., (2022). Estimation of Ratio of Two Means Using Regression-Cum-Exponential Estimators in the Presence of Non-response. *Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci.*, 92, pp. 57–64. https://doi.org/10.1007/s40010-020-00690-0.

Shalabh., (1997). Ratio method of estimation in the presence of measurement errors, *Indian Journal of Agricultural Statistics*, 50(2), pp. 150–155.

Srivastava, A. K., Shalabh., (2001). Effect of measurement errors on the regression method of estimation in survey sampling. *Journal of Statistical Research*, 35(2), pp. 35–44.

Singh, R. S., Sharma, P., (2015). Method of Estimation in the Presence of Non-response and Measurement Errors Simultaneously. *Journal of Modern Applied Statistical Methods*, 14(1), pp. 107-121. DOI: 10.22237/jmasm/1430453460.

Singh, P., Singh, R., Bouza, C. N., (2018). Effect of measurement error and non-response on estimation of population mean. *Revista Investigacion Operacional*, 39(1), pp. 108–120.

Searls, D. T., (1964). The Utilization of a Known Coefficient of Variation in the Estimation Procedure. Journal of the American Statistical Association, 59, pp. 1225–1226.

Srivastava, S. K., (1967). An estimator using auxiliary information in sample surveys, Calcutta Statistical Association Bulletin, 16, pp. 121-–132.

Singh, H. P., Espejo, M. R., (2003). On Linear Regression and Ratio-product Estimation of a Finite Population Mean. *The StatistiSingh, H. P. and Espejo, M. R.cian*, 52(1), pp. 59–67.

Tiwari, K. K., Bhougal, S., Kumar, S., Rather, K. U. I., (2022). Using Randomized Response to Estimate the Population Mean of a Sensitive Variable under the Influence of Measurement Error. *Journal of Statistical Theory and Practice*, 16(2), pp. 1-11. https://doi.org/10.1007/s42519-022-00251-1.

Tiwari, K. K., Bhougal, S., Kumar, S., Onyango, R., (2022). Assessing the Effect of Nonresponse and Measurement Error Using a Novel Class of Efficient Estimators. *Journal of Mathematics*, Article ID 4946265. DOI: 10.1155/2022/4946265.

Tiwari, K. K., Bhougal, S., Kumar, S., (2023). A General Class of Estimators in the Presence of Non-response and Measurement Error, Journal of Statistics Application & Probability Letters, 10(1), pp. 13-33. http://dx.doi.org/10.18576/jsapl/100102.

Zahid, E., Shabbir, J., Gupta, S., Onyango, R., Saeed, S., (2022). A generalized class of estimators for sensitive variable in the presence of measurement error and non-response. *PLoS ONE* 17(1), e0261561. https://doi.org/10.1371/journal. Pone.0261561.

Appendix

Here, we have to prove the equations (4), (5), (6), (7) and (8).
Using equation (1), we have

$$\xi_Y^* = \bar{y}^* - \bar{Y}.$$

Squaring and taking expectation on both sides, we have

$$E(\xi_Y^{*2}) = E(\bar{y}^* - \bar{Y})^2,$$

that is,

$$E(\xi_Y^{*2}) = V(\bar{y}^*). \tag{74}$$

here, V represents variance.

Since $\bar{y}^* = \bar{y}_t^* + \bar{U}^*$, so $V(\bar{y}^*) = V(\bar{y}_t^*) + V(\bar{U}^*) + Cov(\bar{y}_t^*, \bar{U}^*)$. As y and U are independent, so $Cov(\bar{y}_t^*, \bar{U}^*) = 0$. We have,

$$V(\bar{y}^*) = V(\bar{y}_t^*) + V(\bar{U}^*). \tag{75}$$

Now, we have to derive $V(\bar{y}_t^*)$.

$$\bar{y}_t^* = \left(\frac{n_1}{n}\right)\bar{y}_{n_1} + \left(\frac{n_2}{n}\right)\bar{y}_r,$$

so,

$$V(\bar{y}_t^*) = V_1[E_2(\bar{y}_t^*|n_1,n_2)] + E_1[V_2(\bar{y}_t^*|n_1,n_2)]. \tag{76}$$

Considering the first part of (76), we have

$$\begin{aligned}
V_1[E_2(\bar{y}_t^*|n_1,n_2)] &= V_1\left[E_2\left\{\left(\frac{n_1}{n}\bar{y}_{n_1} + \frac{n_2}{n}\bar{y}_r\right)|n_1,n_2\right\}\right] \\
&= V_1\left[\frac{n_1}{n}E_2(\bar{y}_{n_1})|n_1 + \frac{n_2}{n}E_2(\bar{y}_r)|n_2\right] \\
&= V_1\left[\frac{n_1}{n}\bar{y} + \frac{n_2}{n}\bar{y}\right] \\
&= V_1(\bar{y}) \\
&= \lambda S_Y^2. \qquad (77)
\end{aligned}$$

Considering the second part of equation (76), we have

$$\begin{aligned}
E_1[V_2(\bar{y}_t^*|n_1,n_2)] &= E_1\left[V_2\left\{\left(\frac{n_1}{n}\bar{y}_{n_1} + \frac{n_2}{n}\bar{y}_r\right)|n_1,n_2\right\}\right] \\
&= E_1\left[V_2\left\{\frac{n_1}{n}\bar{y}_{n_1}|n_1\right\} + V_2\left\{\frac{n_2}{n}\bar{y}_r|n_2\right\}\right] \\
&= E_1\left[\frac{n_2^2}{n^2}\left(\frac{1}{r} - \frac{1}{n_2}\right)S_r^2\right] \\
&= \frac{n_2^2}{n^2}\left(\frac{1}{r} - \frac{1}{n_2}\right)E_1(S_r^2) \\
&= \frac{n_2}{n^2}\left(\frac{n_2}{r} - 1\right)S_{Y(2)}^2 \\
&= \frac{W_2(k-1)}{n}S_{Y(2)}^2 \\
&= \theta S_{Y(2)}^2. \qquad (78)
\end{aligned}$$

Using equations (77), (78) in (76), we have

$$V(\bar{y}_t^*) = \lambda S_Y^2 + \theta S_{Y(2)}^2. \tag{79}$$

Similarly, we can derive

$$V(\bar{x}_t^*) = \lambda S_X^2 + \theta S_{X(2)}^2, \tag{80}$$

$$V(\bar{U}^*) = \lambda S_U^2 + \theta S_{U(2)}^2, \tag{81}$$

$$V(\bar{V}^*) = \lambda S_V^2 + \theta S_{V(2)}^2. \tag{82}$$

So, using equations (79), (81) in (75), we have

$$V(\bar{y}^*) = \lambda (S_Y^2 + S_U^2) + \theta (S_{Y(2)}^2 + S_{Y(2)}^2). \tag{83}$$

From equation (74) and (83), we have

$$E(\xi_Y^{*2}) = \lambda (S_Y^2 + S_U^2) + \theta (S_{Y(2)}^2 + S_{Y(2)}^2). \tag{84}$$

Which completes the proof of equation (4).

Similarly, we can show that

$$E(\xi_X^{*2}) = \lambda \{S_X^2 + S_V^2\} + \theta \{S_{X(2)}^2 + S_{V(2)}^2\}, \tag{85}$$

and

$$E(\xi_X^2) = \lambda \{S_X^2 + S_V^2\}. \tag{86}$$

Now, using equation (1) and (3), we have

$$\xi_Y^* \xi_X^* = (\bar{y}^* - \bar{Y})(\bar{x}^* - \bar{X}),$$

Taking expectation on both sides, we have

$$E(\xi_Y^* \xi_X^*) = Cov(\bar{y}^*, \bar{x}^*). \tag{87}$$

Now,

$$Cov(\bar{y}^*, \bar{x}^*) = E_1[Cov_2\{(\bar{y}^*, \bar{x}^*)|n_1, n_2\}] + Cov_1[E_2\{\bar{x}^*|n_1, n_2\}, E_2\{\bar{y}^*|n_1, n_2\}], \tag{88}$$

considering the second part,

$$\begin{aligned}
Cov_1[E_2\{\bar{x}^*|n_1, n_2\}, E_2\{\bar{y}^*|n_1, n_2\}] &= Cov_1[E_2\{(w_1\bar{x}_{n_1} + w_2\bar{x}_r)|n_1, n_2\}, \\
&\quad E_2\{(w_1\bar{y}_{n_1} + w_2\bar{y}_r)|n_1, n_2\}] \\
&= Cov_1[\{w_1\bar{x} + w_2\bar{x}\}, \{w_1\bar{y} + w_2\bar{y}\}] \\
&= Cov(\bar{x}, \bar{y}) \\
&= \lambda \rho S_Y S_X. \qquad (89)
\end{aligned}$$

Now, the first part of equation (88)

$$\begin{aligned}
E_1[Cov_2\{(\bar{y}^*,\bar{x}^*)|n_1,n_2\}] &= E_1[Cov_2\{(w_1\bar{x}_{n_1}+w_2\bar{x}_r)|n_1,n_2,(w_1\bar{y}_{n_1}+w_2\bar{y}_r)|n_1,n_2\}] \\
&= E_1[Cov_2\{(w_1\bar{x}_{n_1},w_1\bar{y}_{n_1})|n_1,n_2\}+Cov_2\{(w_1\bar{x}_{n_1},w_2\bar{y}_r)|n_1,n_2\} \\
&\quad +Cov_2\{(w_2\bar{x}_r,w_1\bar{y}_{n_1})|n_1,n_2\}+Cov_2\{(w_2\bar{x}_r,w_2\bar{y}_r)|n_1,n_2\}] \\
&= E_1[(w_2^2Cov_2\{\bar{x}_r,\bar{y}_r)|n_2\}] \\
&= w_2^2\left(\frac{1}{r}-\frac{1}{n_2}\right)E_1(S_{r_{YX(2)}}) \\
&= \frac{n_2}{n^2}\left(\frac{n_2}{r}-1\right)S_{YX(2)} \\
&= \theta S_{YX(2)} \\
&= \theta\rho_2 S_{Y(2)}S_{X(2)}. \qquad (90)
\end{aligned}$$

Using equations (89), (90) in (88), we have

$$Cov(\bar{y}^*,\bar{x}^*) = \lambda\rho S_Y S_X + \theta\rho_2 S_{Y(2)}S_{X(2)}. \qquad (91)$$

From equations (87) and (91), we have

$$E(\xi_Y^*\xi_X^*) = \lambda\rho S_Y S_X + \theta\rho_2 S_{Y(2)}S_{X(2)}. \qquad (92)$$

Similarly, we can derive

$$E(\xi_Y^*\xi_X) = \lambda\rho S_Y S_X. \qquad (93)$$