https://doi.org/10.26485/0459-6854/2018/68.1/7 PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2018 Vol. LXVIII

Recherches sur les déformations no. 1

pp. 97–110

Marek Aleksiejczyk

THE NUMERICAL RANGES OF COMPLEX MATRICES

Summary

The aim of this paper is to recall the most important properties of the sets $W(A) := \{< Ax, x >;\ ||x|| = 1\}$ called the numerical ranges and some their generalizations. We refer their convexity (or being star-shaped), their forms for special matrices. Some examples are also given.

Keywords and phrases: complex matrix, the numerical range of matrix, Rayleigh quotient, k-numerical range of a matrix

1. The numerical range (classical) of a matrix

Let $M_n(\mathbf{C})$ denote the algebra of all $n \times n$ complex matrices, $< \cdot, \cdot >$ denote the standard inner product on $\mathbf{C}^n$, $|| \cdot ||$ - Euclidean norm, S-unit sphere in this norm and A^* - adjoint (conjugate transpose) of a matrix A.

In this work we present basic properties of some subsets of the complex plain called the numerical ranges of complex matrices in $M_n(\mathbf{C})$. Some properties of numerical ranges are applied in the proofs of some linear algebra theorems [5], numerical methods ([9], [23]), and even so seemingly remote knowledge areas as quantum physics [20]. The numerical range is also defined for bounded operators acting on an infinite-dimensional Hilbert spaces, Banach spaces etc. - we restrict our consideration to finite-dimensional case.

Let us start from introducing the main definition.

Definition 1.1. The numerical range of $A \in M_n(\mathbf{C})$ is the set

$$W(A) := \{< Ax, x >;\ ||x|| = 1\}.$$

The number

$$r(A) := \sup\{|z|;\ z \in W(A)\}$$

is called the numerical radius of the matrix A.

The numerical range is called also *the field of values*. The numerical range of a matrix A is equal to the set of all possible values of the *Rayleigh quotient*:

$$\frac{< Ax, x >}{< x, x >}; \quad x \neq 0.$$

The numerical range is also written in equivalent way:

$$W(A) = \{x^* Ax; \ x^* x = 1\}.$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbf{C}^n$. Then the matrix U with columns $\{e_1, \ldots, e_n\}$ is unitary and

$$U^* AU = \begin{bmatrix} < Ae_1, e_1 >, & < Ae_2, e_1 >, & \ldots, & < Ae_n, e_1 > \\ < Ae_1, e_2 >, & < Ae_2, e_2 >, & \ldots, & < Ae_n, e_2 > \\ \cdots & \cdots & \cdots & \cdots \\ < Ae_1, e_n >, & < Ae_2, e_n >, & \ldots, & < Ae_n, e_n > \end{bmatrix}.$$

Now let $\{e_1, \ldots, e_n\}$ varies on all possible orthonormal basis of $\mathbf{C}^n$. The numerical range is equal to the set of all complex numbers on the arbitrary (fixed) entry on the main diagonal of the matrix A (fixing an element out of diagonal we obtain the set $W_0(A)$ (it will be defined in the further part of this work).

Theorem 1.1 (General properties of the numerical range)**.** *Let $A, B \in M_n(\mathbf{C})$ and $\alpha, \beta \in \mathbf{C}$. Then*

(1) *$W(A)$ is a compact subset of a complex plain,*
(2) *$W(A)$ includes eigenvalues of A,*
(3) *$W(UAU^*) = W(A)$ whenever U is unitary,*
(4) *$W(A^*) = \overline{W(A)}$,*
(5) *$W(A^T) = W(A)$,*
(6) *$W(\alpha I + \beta A) = \alpha + \beta W(A)$,*
(7) *$W(A + B) \subset W(A) + W(B)$,*
(8) *$W(\frac{A+A^*}{2}) = \mathrm{Re}(W(A))$,*
(9) *$r(A)$ is a norm and*

$$\frac{1}{2}||A|| \leq r(A) \leq ||A||.$$

The proofs of above facts may be found for example in [12], [10]. In view of definition of the numerical range and properties of inner product we obtain:

Corollary 1.1. *Let $A \in M_n(\mathbf{C}), \alpha, \beta, \lambda \in \mathbf{C}$. Then*

(1) *$W(A) = \{\lambda\}$ if and only if $A = \lambda I$,*
(2) *$W(A) \subset \mathbf{R}$ if and only if $A = A^*$,*
(3) *$W(A)$ is a line segment if and only if $A = \alpha I + \beta H$ for some $\alpha, \beta \in \mathbf{C}$, $H = H^*$,*
(4) *If $A \in M_n(\mathbf{R})$, then $W(A)$ is symmetric with respect to the real line.*

Now we will determine the numerical ranges of two 2×2 matrices:

Example 1.1. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$
Then
$$W(A) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix};\ x\overline{x} + y\overline{y} = 1 \right\} = \{x\overline{x};\ x\overline{x} + y\overline{y} = 1\} = [0,1].$$

There is a little more difficult to obtain:

Example 1.2. Let
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Then
$$W(A) = \left\{ z;\ |z| \leq \frac{1}{2} \right\}.$$

Let us now recall the most important theorem concerning numerical ranges, which is recognized as the beginning of the theory of the numerical ranges.

Theorem 1.2 (Hausdorff-Toeplitz (1918/1919))**.** *Let $A \in M_n(\mathbf{C})$. Then $W(A)$ is a convex subset of the complex plain.*

Original proofs were published in [24] i [11], later interesting proofs may be found in [8], [15], [21]. Hausdorff-Toeplitz theorem assumes a special form in case of normal matrices ($A^*A = AA^*$).

Theorem 1.3. *Let $A \in M_n(\mathbf{C})$ be normal. Then the numerical range of A is a convex hull of its spectrum.*

It is result of fact that normal matrix is unitary similar to diagonal matrix and definition of convex hull of of the finite set. The numerical radius is a norm, but not a matrix norm, it means that it does not satisfy the condition $r(AB) \leq r(A)r(B)$, for example if
$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$
we have $r(A) = r(B) = 1/2$, but $r(AB) = 1$. However, in 1960 Berger proved the following *power inequality*:

Theorem 1.4. *Let $A \in M_n(\mathbf{C})$. Then*
$$r(A^n) \leq (r(A))^n$$
for any natural number n

(for a proof see [19] i [6]). Next result is due to F. D. Murnaghan in 1932 ([17]). We recall it in a form proposed by R. A. Horn and C. R. Johnson in [12]:

Theorem 1.5. *Let $A \in M_2(\mathbf{C})$ and let $A_0 = A - (\frac{1}{2}\mathrm{tr}A)I$. Then the numerical range of A is an elliptical disc (convex hull of corresponding ellipse) with eigenvalues of A as a foci, major axis of the length $\sqrt{\mathrm{tr}A_0^*A_0 + 2|\det A_0|}$ and minor axis of the length $\sqrt{\mathrm{tr}A_0^*A_0 - 2|\det A_0|}$*

The above theorem assumes special form in case of A is upper triangular form (in view of theorem of Schur [22] every matrix is unitary similar to upper triangular one).

Corollary 1.2. *Let*

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

Then the numerical range of A is an elliptical disc with a and c as a foci and minor axis of the length $|b|$.

Let us consider the diameter of the numerical range. It is easy to notice that for a normal matrix $A \in M_n(\mathbf{C})$ the diameter of its numerical range

$$\operatorname{diam} W(A) := \max\{|z_1 - z_2|;\ z_1, z_2 \in W(A)\}$$

and in some cases its "minimal width"

$$\operatorname{mw} W(A) := \min_{\theta \in \mathbf{C}, |\theta|=1} \max\{\mathrm{Re}(\theta(z_1 - z_2));\ z_1, z_2 \in W(A)\}$$

are equal to maximal and minimal distance, respectively between its eigenvalues, so corresponding eigenvectors are orthogonal. N. K. Tsing w 1983 in [25] generalized this result. At first he proved the following lemma:

Lemma 1.1. *Let $A \in M_n(\mathbf{C})$. For any pair of vectors $x, y \in S$ there exists vectors $u, v \in S$ such,that $< u, v >= 0$ and*

$$< Ax, x > - < Ay, y >= 1 - \sqrt{| < x, y > |^2}(< Au, u > - < Av, v >).$$

He proved two results also. First of them concerns the diameter of the numerical range.

Theorem 1.6. *Let $A \in M_n(\mathbf{C})$ be a non-scalar matrix (scalar matrix is a complex multiplicity of identity matrix) and let $x, y \in S$ be such that*

$$| < Ax, x > - < Ay, y > | = \operatorname{diam} W(A).$$

Then $< x, y >= 0$.

Before the next theorem we shall introduce the notion of the supporting line.

Definition 1.2. Let $X \subset \mathbf{C}$ be a compact set. Supporting line of a set X in the complex plane is a line that contains a point of X, but does not separate any two points of X. In other words, X lies completely in one of the two closed half-planes defined by X and has at least one point on supporting line.

Now we can recall second result of Tsing:

Theorem 1.7. *Let $A \in M_n(\mathbf{C})$ and let l_1, l_2 be two parallel supporting lines of $W(A)$ such that their distance is equal to* $\mathrm{mw}(A)$. *If* $\mathrm{mw}(A) > 0$ *and $x, y \in S$ are such that points $< Ax, x >$ and $< Ay, y >$ belong to the lines l_1 and l_2, respectively, then $< x, y >= 0$.*

Now we will take care of the numerical determination of the numerical ranges of matrices. As it follows from results presented above, an analytic determination of the numerical range of a matrix is possible in very special cases (for example 2×2 matrices). Even for $n \times n$ normal matrices (where numerical range is simply a convex hull of a spectrum) for $n > 4$ is possible using numerical methods. Moreover, in general case, the only way to present the numerical range of a matrix is "plotting" it. Below we present some algorithms for computing the numerical ranges of matrices. The simplest possible algorithm of numerical determination of the numerical range of the matrix of order n is following: we perform N times the following operations:

1. we generate $2n$ real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$
2. we generate vector $v = x + iy$, where $x = [x_1, \ldots, x_n],\ y = [y_1, \ldots, y_n]$
3. we normalize vector v
4. we compute $< Av, v >$ and mark this point (usually on the computer screen)

In practice, such algorithm is effective for small values of n, for largest values it needs many millions operations. Much more interesting algorithm was proposed in [16] by M. Marcus i C. Pesce - of the basis proved quote above authors theorem:

Theorem 1.8. *Let $A \in M_n(\mathbf{C})$. Then the numerical range of A is a sum of numerical ranges of matrices*

$$A_{xy} = \begin{bmatrix} < Ax, x > & < Ay, x > \\ < Ax, y > & < Ay, y > \end{bmatrix},$$

where $(x, y) \in \mathbf{C}^n \times \mathbf{C}^n$ varies on all pairs of orthonormal vectors.

They proved also properties of the numerical range of 2×2 matrices. Probably the best algorithm was proposed by C. R. Johnson in [13]. Its idea is following: for some number of numbers $0 = \varphi_1 \leq \varphi_2 \leq \ldots \leq \varphi_n = 2\pi$ we perform the following operations:

1. we multiply matrix A by number of modulus one and argument φ_i and obtain matrix A_i
2. we determine Hermitian part $A_i^{'} = (A_i + A_i^*)/2$ of matrix A_i
3. we compute the largest eigenvalue of the obtained matrix and corresponding eigenvector x
4. we determine $< A_i^{'}x, x > / < x, x >$ and mark it.

Next we join obtained points (we approximate the boundary of the boundary of the numerical range by polygon).

2. Selected generalizations of the classical numerical range

In this section we discuss some concepts of generalizations of the classical numerical range.

2.1. k-numerical range of a matrix

Definition 2.1. Let $A \in M_n(\mathbf{C})$ and let $1 \leq k \leq n$. The set

$$W_k(A) := \left\{ \sum_{i=1}^{k} < Ax_i, x_i >;\ < x_i, x_j >= \delta_{ij} \right\},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ in other cases, is called the k-numerical range of a matrix A.

Notice that

Corollary 2.1. *Let $A \in M_n(\mathbf{C})$. Then*
(1) $W_1(A) = W(A)$,
(2) $W_n(A) = \{\mathtt{tr} A\}$.

It is natural the question about convexity of the k-numerical range. This problem was proposed by Halmos and in the case of normal matrices was positively solved by R. C. Thompson w 1963. In the same year C. A. Berger proved the convexity of the k-numerical range in general case. Further generalization of the numerical range is the c-numerical range.

2.2. c-numerical range

Definition 2.2. Let $A \in M_n(\mathbf{C})$ and let $c = (c_1, \ldots, c_n)^t \in \mathbf{C}^n$. The set

$$W_c(A) := \left\{ \sum_{i=1}^{n} c_i < Ax_i, x_i >;\ < x_i, x_j >= \delta_{ij} \right\},$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ in other cases, is called the c-numerical range of a matrix A.

There is also natural to ask about convexity of this generalization. In this case situation is more complicated: R. Westwick proved in [27] that $W_c(A)$ is convex if $c \in \mathbf{R}^n$ and in general case ($c \in \mathbf{C}^n$) it not must be convex if $n > 2$. The c-numerical ranges were intensively studied by numerous authors - for example by N. Bebiano and J. da Oliveira (1986-1995). It is easy to notice that c-numerical range can be written as

$$W_c(A) = \{\mathrm{tr}(CU^*AU);\ U \text{ is unitary}\},$$

where $C = \mathrm{diag}(c_1, \ldots, c_n)$. This is why we easily obtain the next generalization of the numerical range proposed by M. Goldberg i E. G. Straus in 1977.

2.3. C-numerical range of a matrix

Definition 2.3. Let $A, C \in M_n(\mathbf{C})$. The set

$$W_C(A) = \{tr(CU^*AU);\ U \in \mathcal{U}\},$$

where $\mathcal{U}$ is the set of unitary matrices, is called the C-numerical range of a matrix A.

In the following lemma we collect the most important properties of the C-numerical range.

Lemma 2.1. *Let $A, B, C \in M_n(\mathbf{C})$ and let $\alpha, \beta \in \mathbf{C}$. Then*
(1) $W_C(U^*AU) = W_C(A)$ *for any unitary matrix* U,
(2) $W_C(A + B) \subset W_C(A) + W_C(B)$,
(3) $W_C(\alpha A + \beta I) = \alpha W_C(A) + \beta \mathrm{tr} C$.

Moreover it is easy to notice that $W_C(A) = W_A(C)$.

Let us recall some results concerning convexity of the C-numerical range. This result of Westwick was referred earlier in another way.

Theorem 2.1. *Let $A, C \in M_n(\mathbf{C})$ and let C be selfadjoint. Then $W_C(A)$ is a convex set.*

The next result is due to Tsing ([26]):

Theorem 2.2. *Let $A, C \in M_n(\mathbf{C})$ and let C be of rank one. Then $W_C(A)$ is a convex set.*

Hence 2×2 matrix A can be transform by translation into 2×2 matrix B of rank one without changing the shape of the q-numerical range, we obtain

Corollary 2.2. *Let $A, C \in M_2(\mathbf{C})$. Then $W_C(A)$ is a convex set.*

Let us recall by [14] two results concerning the C-numerical range in special cases. The first theorem characterizes matrices for which C-numerical range is a point.

Theorem 2.3. *Let $A, C \in M_n(\mathbf{C})$. Then the following are equivalent:*
(i) *C numerical range of A consist of one point,*
(ii) *at least on of matrices A and C is scalar.*

Next theorem characterizes matrices for which C-numerical range is a line segment.

Theorem 2.4. *Let $A, C \in M_n(\mathbf{C})$. Then the following are equivalent:*
(i) *C-numerical range of A is a line segment,*
(ii) *classical numerical ranges of A and C are line segments.*

In general case the C-numerical range may not be convex, even for normal matrices, but it has weaker property ([7]).

Theorem 2.5. *Let $C \in M_n(\mathbf{C})$. Then $W_C(A)$ is star-shaped with respect to the point*

$$\frac{\mathrm{tr}A\,\mathrm{tr}C}{n}$$

for any $A \in M_n(\mathbf{C})$.

2.4. q-numerical range of a matrix

Now we turn our attention on the next generalization of the classical numerical range.

Definition 2.4. Let $A \in M_n(\mathbf{C})$ and let $q \in \mathbf{C},\ |q| \leq 1$. The q-numerical range of matrix A is the set

$$W_q(A) := \{< Ax, y >;\ ||x|| = ||y|| = 1,\ < x, y >= q\}$$

Similarly as in classical case we define the q-numerical radius:

$$r_q(A) := \sup\{|z|;\ z \in W_q(A)\}.$$

The notion of the q-numerical range was introduced by N. K. Tsing in 1984 ([26]). In the same paper the convexity of the q-numerical range was proved. Let us recall basic properties of the q-numerical range.

Lemma 2.2. *Let $A \in M_n(\mathbf{C})$. Then*
(1) *$W_q(U^*AU)$ whenever U is unitary,*
(2) *$W_q(\alpha A + \beta I) = \alpha W_q(A) + \beta q$,*
(3) *$W_{\mu q}(A) = \mu W_q(A)$ for $\mu \in \mathbf{C}$ satisfying $|\mu| = 1$.*

Properties presented above are natural generalizations of corresponding properties of the classical numerical range, but not all of them carries out for the general case. For example the boundary of the q-numerical range is a smooth curve if $|q| < 1$. Some results on the diameter of the generalized numerical range can be found in [1].

It is easy to notice that

$$W_1(A) = W(A).$$

The convexity of the q numerical range assumes a special form in case of $q = 0$.

Theorem 2.6. *Let $A \in M_n(\mathbf{C})$. Then $W_0(A)$ is a circular disc centered in point $z = 0$ and radius*

$$r_0(A) = \{\inf ||A - \lambda I||;\ \lambda \in \mathbf{C}\}.$$

It is worth to recall result of E. Asplund i V. Ptak ([4]):

$$\inf_{\lambda \in \mathbf{C}} \sup_{x \in S} ||(A - \lambda I)x|| = \sup_{x \in S} \inf_{\lambda \in \mathbf{C}} ||(A - \lambda I)x||.$$

In proofs of results concerning the q-numerical range the following fact is used very often:

Proposition 2.1. *Let x, y be unit vectors and let $< x, y >= q$. Then*

$$y = \overline{q}x + pv,$$

where v is orthonormal to x and $p = \sqrt{1 - |q|^2}$.

We obtain immediately

Corollary 2.3. *Let x, y be unit vectors with $< x, y >= q$ and let $A \in M_n(\mathbf{C})$. Then*

$$< Ax, y >= q < Ax, x > +p < Ax, v >,$$

where v and p are like above.

Notice that q-numerical range is also special case of the C-numerical range: $W_q(A) = W_{C_q}(A)$, where

$$C_q = \begin{bmatrix} q & \sqrt{1-|q|^2} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is trivial to notice:

Corollary 2.4. *Let $A \in M_n(\mathbf{C})$ and let $|q| \leq 1$. Then*

$$q\sigma(A) \subset W_q(A).$$

Another interesting result concerns the continuity of the q-numerical range as a function of $q \in \mathbf{C}$, as of $A \in M_n(\mathbf{C})$ in Hausdorff metric (q-numerical ranges are compact sets)

$$\operatorname{dist}(F, G) = \max\left\{\max_{\lambda \in F} \min_{\mu \in G} |\lambda - \mu|, \max_{\lambda \in G} \min_{\mu \in F} |\lambda - \mu|\right\}.$$

Theorem 2.7. *Let $A, B \in M_n(\mathbf{C})$, let $q_1, q_2 \in \mathbf{C}$ be complex numbers of modulus not greater than 1 and let $M := \max\{||A||, ||B||\}$. Then*

$$\operatorname{dist}(W_{q_1}(A), W_{q_2}(B)) \leq M\sqrt{|q_1 - q_2|^2 + 2|q_1 - q_2|} + ||A - B||.$$

Moreover, if we take $|q_1| = \sin\phi_1, |q_2| = \sin\phi_2$, where $\phi_1, \phi_2 \in [0, \pi/2]$, then

$$\operatorname{dist}(W_{q_1}(A), W_{q_2}(B)) \leq M\mu|q_1 - q_2| + ||A - B||,$$

where

$$\mu = \begin{cases} \sqrt{2(1 + \cos(\phi_1 - \phi_2))}/(\cos\phi_1 + \cos\phi_2) & \text{if } |q_1| \neq |q_2| \\ 1 & \text{in another cases} \end{cases}$$

Now we turn our attention on q-numerical radius as a norm. We easily obtain

Theorem 2.8. *q-numerical radius is a seminorm for all $|q| \leq 1$. It is a norm if and only if $q \neq 0$.*

(0-numerical radius of any scalar matrix is of course equal to zero). The following fact is a simple consequence of definitions of matrix norm and q-numerical radius.

Proposition 2.2. *Let $A \in M_n(\mathbf{C})$. Then*

$$\max_{|q|\leq 1} r_q(A) = ||A||.$$

Let us now consider the q-numerical range of 2×2 matrices. Hence the 0-numerical range is always circular disc (Theorem 2.6) and 1-numerical range of 2×2 matrices are elliptical discs (Theorem 1.5). Then it is natural a question "Is the q-numerical range of 2×2 matrix an elliptical disc?". This problem has a positive answer due to H. Nakazato ([18]).

Theorem 2.9. *Let*

$$A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},$$

where $0 \leq b \leq a$ and let $0 \leq q \leq 1$. Then

$$W_q(A) = \left\{ x + iy;\ \frac{x^2}{A^2} + \frac{y^2}{B^2} \leq 1 \right\},$$

where

$$A = \frac{(a+b) + \sqrt{1-q^2}(a-b)}{2}, \quad B = \frac{(a-b) + \sqrt{1-q^2}(a+b)}{2}$$

Some results on square matrices, their subclass of magic matrices and their numerical ranges see also [3] and [2]. Finally we shall demonstrate differences between difficulties of computation of the classical numerical range and the general q-numerical range. Determination of the classical numerical range of an orthogonal projection is as simple as in Example 1.1. Let us determine q-numerical range of an orthogonal projection P ($P = P^2 = P^*$).

Example 2.1. Let $P \in M_n(\mathbf{C})$ be an orthogonal projection (non-scalar) and let $0 \leq q \leq 1$. Then

$$W_q(P) = \left\{ x + iy :\ \frac{(x - q/2)^2}{1/4} + \frac{y^2}{(1-q^2)/4} \leq 1 \right\}.$$

Indeed, without loss of generality we may assume that $P = I_k \oplus 0_{n-k}$. Notice that the set

$$\left\{ x + iy :\ \frac{(x - q/2)^2}{1/4} + \frac{y^2}{(1-q^2)/4} \leq 1 \right\}$$

may be written in a form

$$\{z \in \mathbf{C} :\ |z| + |z - q| \leq 1\}.$$

Every vector $u = [u_1, \ldots, u_k, u_{k+1}, \ldots, u_n] \in \mathbf{C}^n$ may be written in a form

$$u = \underbrace{[u_1, \ldots, u_k, 0, \ldots, 0]}_{u_1} + \underbrace{[0, \ldots, 0, u_{k+1}, \ldots, u_n]}_{u_2}.$$

Let $x, y \in S$ be an arbitrary vectors such that $< x, y >= q$ and let x_1, y_1, x_2, y_2 be of above form. Then

$$
\begin{aligned}
&| < Px, y > | + |q - < Px, y > | = | < Px, y > | + | < Ix, y > - < Px, y > | \\
&= | < Px, y > | + |(I - P)x, y| = | < x_1, y_1 > | + | < x_2, y_2 > | \leq 1.
\end{aligned}
$$

We have

$$
W_q(P) \subset \left\{ x + iy : \ \frac{(x - q/2)^2}{1/4} + \frac{y^2}{(1 - q^2)/4} \leq 1 \right\}.
$$

We must exhibit, that for any point z of the boundary of the q-numerical range of P there exists unit vectors x, y such that $< x, y >= q$ and $< Px, y >= z$. Notice that

$$
\left\{ x + iy : \ \frac{(x - q/2)^2}{1/4} + \frac{y^2}{(1 - q^2)/4} = 1 \right\} = \left\{ \frac{\cos t + q}{2} + i \frac{\sqrt{1 - q^2} \sin t}{2} \ : \ t \in \mathbf{R} \right\}.
$$

For any fixed $t \in \mathbf{R}$ it is enough to point out vectors $x(t), y(t) \in S$ such that $< x(t), y(t) >= q$ and

$$
< Px(t), y(t) >= \frac{\cos t + q}{2} + i \frac{\sqrt{1 - q^2} \sin t}{2}.
$$

Let

$$
\begin{aligned}
u(t) &= \sqrt{\frac{q + \cos t}{2} + i \frac{\sqrt{1 - q^2} \sin t}{2}}, \\
v(t) &= \sqrt{\frac{q - \cos t}{2} - i \frac{\sqrt{1 - q^2} \sin t}{2}}
\end{aligned}
$$

and let

$$
\begin{aligned}
x(t) &= [u(t), 0, \ldots, 0, v(t)]^t, \\
y(t) &= [\overline{u(t)}, 0, \ldots, 0, \overline{v(t)}]^t.
\end{aligned}
$$

Then

$$
\begin{aligned}
&||x(t)||^2 = |u(t)|^2 + |v(t)|^2 = |u^2(t)| + |v^2(t)| = \left| \frac{q + \cos t}{2} + i \frac{\sqrt{1 - q^2} \sin t}{2} \right| \\
&+ \left| \frac{q - \cos t}{2} - i \frac{\sqrt{1 - q^2} \sin t}{2} \right| = \sqrt{\frac{q^2 + 2q \cos t + \cos^2 t + (1 - q^2) \sin^2 t}{4}} \\
&+ \sqrt{\frac{q^2 - 2q \cos t + \cos^2 t + (1 - q^2) \sin^2 t}{4}} = \frac{1 + q \cos t}{2} + \frac{1 - q \cos t}{2} = 1.
\end{aligned}
$$

Similarly we exhibit that $||y(t)|| = 1$. Then $x(t), y(t) \in S$, and moreover

$$
< x(t), y(t) >= \frac{q + \cos t}{2} + i \frac{\sqrt{1 - q^2} \sin t}{2} + \frac{q - \cos t}{2} - i \frac{\sqrt{1 - q^2} \sin t}{2} = q
$$

and

$$< Px(t), y(t) >= u^2(t) = \frac{q + \cos t}{2} + i\frac{\sqrt{1-q^2}\sin t}{2}.$$

Finally we obtain

$$W_q(P) = \left\{ x + iy : \frac{(x - q/2)^2}{1/4} + \frac{y^2}{(1-q^2)/4} \leq 1 \right\} \quad \text{for} \quad q = 1, \ W_q(P) = [0, 1].$$

It is easy to notice that in above example we have $P^n = P$ for any natural numbers n, but $r_q(P) = (1+q)/2$. So for an orthogonal projection we have $r_q(P^n) > (r_q(P))^n$ for $q < 1$ and $n \geq 2$. We obtain the following corollary

Corollary 2.5. *"Power inequality" is not valid in case $|q| < 1$.*

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Department of Complex Analysis
Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn
Sloneczna 54, PL-10-710 Olsztyn
Poland
E-mail: marek.aleksiejczyk@gmail.com

Presented by Jacek Hejduk at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on June 27, 2018.

OBRAZY NUMERYCZNE MACIERZY ZESPOLONYCH

S t r e s z c z e n i e

W pracy tej omówione są najważniejsze własności zbiorów

$$W(A) := \{< Ax, x >; ||x|| = 1\},$$

które nazywane są obrazami numerycznymi. Podane są również ich uogólnienia. Rozważana jest ich wypukłość, gwiaździstość oraz ich geometria dla wybranych macierzy. Przedstawiono również szereg przykładów macierzy i ich obrazów.

Słowa kluczowe: macierz zespolona, k-numeryczny obraz macierzy