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ON $F(p,n)$-FIBONACCI QUATERNIONS

Summary

In this paper we introduce and study a special one-parameter generalization of Fibonacci quaternions. We investigate their properties and we give generalizations of some classical results for Fibonacci quaternions.

Keywords and phrases: Fibonacci numbers, Lucas numbers, quaternions, recurrence relations

1. Introduction

Let F_n be the nth Fibonacci number defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = F_1 = 1$. The nth Lucas number L_n is defined recursively by $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$ with the initial terms $L_0 = 2$, $L_1 = 1$.

Apart Fibonacci and Lucas numbers there are known numbers defined recursively by the second order linear recurrence relations. It is necessary to mention Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, Jacobsthal-Lucas numbers, for details see [1]. These numbers are also named as numbers of the Fibonacci type and they have applications in distinct areas of mathematics. In this paper we use Fibonacci numbers in the theory of quaternions.

Let $\mathbb{H}$ be the set of quaternions q of the form

$$q = a + bi + cj + dk \tag{1}$$

where $a, b, c, d \in \mathbb{R}$ and

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2}$$

Note that (2) implies $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

Quaternions were introduced by Hamilton in 1843 as an extension of the complex numbers.

Analogously as for complex numbers the addition, the substraction and the multiplication of quaternions were defined.

Let $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ and $q_2 = a_2 + b_2 i + c_2 j + d_2 k$ be two quaternions. Then the addition and the subtraction of them is defined as follows

$q_1 \pm q_2 = (a_1 \pm a_2) + (b_1 \pm b_2)i + (c_1 \pm c_2)j + (d_1 \pm d_2)k.$

The quaternion multiplication also is defined analogously as the complex numbers multiplication using the rule (2).

For details of the quaternion theory see [17].

In 1963 Horadam [6] introduced the nth Fibonacci and Lucas quaternions as follows

$G_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$

and

$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$

respectively.

Distinct properties of the Fibonacci and Lucas quaternions can be found for example in [5], [6], [9]. In [7] Horadam indicated the possibility of introducing Pell quaternions and generalized Pell quaternions. Interesting results of Pell quaternions, Pell-Lucas quaternions obtained recently can be found in [3], [15]. Also Jacobsthal quaternions and Jacobsthal-Lucas quaternions were introduced and studied, see [14].

There are also many papers containing some generalization of Fibonacci quaternions, see [4], [8], [13], [16]. Some types of these quaternions relate to distinct generalizations of numbers of the Fibonacci type. In [12] Polatli, Kizilates and Kesim studied the split k-Fibonacci and k-Lucas quaternions. Catarino in [2] derived the generating function and some identities for the modified Pell and the modified k-Pell quaternions. Also Kilic [10] considered split k-Jacobsthal and k-Jacobsthal-Lucas quaternions and presented their properties.

Motivated by their investigations and results in this paper we introduce and study $F(p, n)$-Fibonacci quaternions which generalize the Fibonacci quaternions.

2. $F(p, n)$- Fibonacci numbers

Besides the usual Fibonacci and Lucas numbers many kinds of generalizations of these numbers have been presented in the literature, see their list in [1]. In [11] Kwaśnik and I. Włoch introduced the generalized Fibonacci numbers $F(p, n)$ and the generalized Lucas numbers $L(p, n)$ defined as follows

$$\begin{aligned} &F(p,n) = n+1, \text{ for } n = 0, 1, \ldots, p-1, \\ &F(p,n) = F(p,n-1) + F(p,n-p), \text{ for } n \geq p, \end{aligned} \tag{3}$$

$$\begin{aligned} &L(p,n) = n+1, \text{ for } n = 0, 1, \ldots, 2p-1, \\ &L(p,n) = L(p,n-1) + L(p,n-p), \text{ for } n \geq 2p, \end{aligned} \tag{4}$$

where $p \geq 2$, $n \geq 0$.
Note that for $n \geq 0$ we have that $F(2,n) = F_{n+1}$ and for $n \geq 2$ $L(2,n) = L_n$.

The numbers $F(p,n)$ and $L(p,n)$ were introduced and studied as a generalization of the classical Fibonacci and Lucas numbers in the context of graph theory. They were used for determining the total number of p-independent sets (i.e. subsets of vertices which induce the empty subgraph) in special classes of graphs.

The following Table presents the initial words of the generalized Fibonacci numbers and the generalized Lucas numbers for special case of n and p.

Table 1. The values of $F(p,n)$, $L(p,n)$, F_n and L_n.

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55	89
$F(2,n)$	1	2	3	5	8	13	21	34	55	89	144
$F(3,n)$	1	2	3	4	6	9	13	19	28	41	60
$F(4,n)$	1	2	3	4	5	7	10	14	19	26	36
$F(5,n)$	1	2	3	4	5	6	8	11	15	20	26
L_n	2	1	3	4	7	11	18	29	47	76	123
$L(2,n)$	1	2	3	4	7	11	18	29	47	76	123
$L(3,n)$	1	2	3	4	5	6	10	15	21	31	46
$L(4,n)$	1	2	3	4	5	6	7	8	13	19	26

In [18] some combinatorial properties of the generalized Fibonacci numbers and the generalized Lucas numbers were obtain by A. Włoch. We recall some of them.

Theorem 2.1 ([18]). *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\sum_{l=0}^{n-p} F(p,l) = F(p,n) - p. \tag{5}$$

Theorem 2.2 ([18]). *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp-1) + 1 = F(p,np). \tag{6}$$

Theorem 2.3 ([18]). *Let $p \geq 2$, $n \geq 2p-2$ be integers. Then*

$$F(p,n) = \sum_{l=0}^{p-1} F(p,n-(p-1)-l). \tag{7}$$

Theorem 2.4 ([18]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p,pl) = L(p,np+1) - (p+2). \tag{8}$$

Theorem 2.5 ([18]). *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$L(p,n) = pF(p,n-(2p-1)) + F(p,n-p). \tag{9}$$

3. $F(p,n)$-Fibonacci quaternions

The nth $F(p,n)$-Fibonacci quaternion FQ_n^p and the nth $L(p,n)$-Lucas quaternion LQ_n^p are defined as

$$FQ_n^p = F(p,n) + iF(p,n+1) + jF(p,n+2) + kF(p,n+3), \tag{10}$$

$$LQ_n^p = L(p,n) + iL(p,n+1) + jL(p,n+2) + kL(p,n+3), \tag{11}$$

respectively.

Theorem 3.1. *Let $p \geq 2$ be integer. Then for $n \geq p+1$*

$$\begin{aligned}\sum_{l=0}^{n-p} FQ_l^p = FQ_n^p - p - i\left(p + F(p,0)\right) - j\left(p + F(p,0) + F(p,1)\right) + \\ - k\left(p + F(p,0) + F(p,1) + F(p,2)\right).\end{aligned} \tag{12}$$

Proof. Using (5) and (10) we have

$$\begin{aligned}
&\sum_{l=0}^{n-p} FQ_l^p = FQ_0^p + FQ_1^p + \ldots + FQ_{n-p}^p = \\
&= F(p,0) + iF(p,1) + jF(p,2) + kF(p,3) + \\
&+ F(p,1) + iF(p,2) + jF(p,3) + kF(p,4) + \ldots + \\
&+ F(p,n-p) + iF(p,n-p+1) + jF(p,n-p+2) + kF(p,n-p+3) = \\
&= F(p,0) + F(p,1) + \ldots + F(p,n-p) + \\
&+ i\left(F(p,1) + \ldots + F(p,n-p+1) + F(p,0) - F(p,0)\right) + \\
&+ j\left(F(p,2) + \ldots + F(p,n-p+2) + F(p,0) + F(p,1) - F(p,0) - F(p,1)\right) + \\
&+ k\left(F(p,3) + \ldots + F(p,n-p+3) + F(p,0) + F(p,1) + F(p,2) + \right.\\
&\left. - F(p,0) - F(p,1) - F(p,2)\right) = \\
&= F(p,n) - p + i\left(F(p,n+1) - p - F(p,0)\right) + \\
&+ j\left(F(p,n+2) - p - F(p,0) - F(p,1)\right) + \\
&+ k((F(p,n+3) - p - F(p,0) - F(p,1) - F(p,2)) = \\
&= FQ_n^p - p - i\left(p + F(p,0)\right) - j\left(p + F(p,0) + F(p,1)\right) + \\
&- k\left(p + F(p,0) + F(p,1) + F(p,2)\right),
\end{aligned}$$

which ends the proof. □

Remark. If $p = 2$ then we obtain the known equality for the Fibonacci quaternions

G_n (see [5])

$$\sum_{l=0}^{n-2} G_l = G_n - (2+3i+5j+8k) = G_n - G_1. \tag{13}$$

Lemma 3.2. *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} F(p,lp) = F(p,np+1) - F(p,1), \tag{14}$$

$$\sum_{l=1}^{n} F(p,lp+1) = F(p,np+2) - F(p,2), \tag{15}$$

$$\sum_{l=1}^{n} F(p,lp+2) = F(p,np+3) - F(p,3). \tag{16}$$

Proof. Using the second equation from (3) we have

$$F(p,n-1) = F(p,n) - F(p,n-p), \text{ for n} \geq \text{p}.$$

For integers p, $2p$, $\ldots, np$ we obtain

$$\begin{aligned}
&F(p,p) = F(p,p+1) - F(p,1)\\
&F(p,2p) = F(p,2p+1) - F(p,p+1)\\
&F(p,3p) = F(p,3p+1) - F(p,2p+1)\\
&\vdots\\
&F(p,np) = F(p,np+1) - F(p,(n-1)p+1).
\end{aligned}$$

Adding these equalities we obtain (14).

In the same way one can easily prove (15) and (16). $\square$

Theorem 3.3. *Let $p \geq 2$, $n \geq p$ be integers. Then*

$$\sum_{l=1}^{n} FQ^p_{lp-1} = FQ^p_{np} - \left(F(p,0) + iF(p,1) + jF(p,2) + kF(p,3)\right). \tag{17}$$

Proof. Using (10) we have

$$\begin{aligned}
&\sum_{l=1}^{n} FQ^p_{lp-1} = FQ^p_{p-1} + FQ^p_{2p-1} + \ldots + FQ^p_{np-1} =\\
&= F(p,p-1) + iF(p,p) + jF(p,p+1) + kF(p,p+2) +\\
&+ F(p,2p-1) + iF(p,2p) + jF(p,2p+1) + kF(p,2p+2) + \ldots +\\
&+ F(p,np-1) + iF(p,np) + jF(p,np+1) + kF(p,np+2) =\\
&= F(p,p-1) + F(p,2p-1) + \ldots + F(p,np-1) +\\
&+ i\left(F(p,p) + F(p,2p) + \ldots + F(p,np)\right) +\\
&+ j\left(F(p,p+1) + F(p,2p+1) + \ldots + F(p,np+1)\right) +\\
&+ k\left(F(p,p+2) + F(p,2p+2) + \ldots + F(p,np+2)\right).
\end{aligned}$$

Writing (6) as $\sum_{l=1}^{n} F(p, lp-1) = F(p, np)-1 = F(p, np)-F(p, 0)$ and using (14)–(16) we obtain (17). $\square$

Remark. If $p = 2$ then we obtain the known equality for the Fibonacci quaternions G_n (see [5])

$$\sum_{l=1}^{n} G_{2l-1} = G_{2n} - (1 + 2i + 3j + 5k) = G_{2n} - G_0. \tag{18}$$

Theorem 3.4. *Let $p \geq 2$, $n \geq 2p - 2$ be integers. Then*

$$FQ_n^p = \sum_{l=0}^{p-1} FQ_{n-(p-1)-l}^p. \tag{19}$$

Proof. Using (7) and (10) we have

$$\begin{aligned}
&\sum_{l=0}^{p-1} FQ_{n-(p-1)-l}^p = FQ_{n-(p-1)}^p + FQ_{n-(p-1)-1}^p + \ldots + FQ_{n-(p-1)-(p-1)}^p = \\
&= F(p, n-(p-1)) + iF(p, n-(p-1)+1)+ \\
&+ jF(p, n-(p-1)+2) + kF(p, n-(p-1)+3)+ \\
&+ F(p, n-(p-1)-1) + iF(p, n-(p-1))+ \\
&+ jF(p, n-(p-1)+1) + kF(p, n-(p-1)+2) + \ldots + \\
&+ F(p, n-(p-1)-(p-1)) + iF(p, n-(p-1)-(p-1)+1)+ \\
&+ jF(p, n-(p-1)-(p-1)+2)+ \\
&+ kF(p, n-(p-1)-(p-1)+3) = \\
&= F(p, n) + iF(p, n+1) + jF(p, n+2) + kF(p, n+3) = FQ_n^p,
\end{aligned}$$

which ends the proof. $\square$

Remark. If $p = 2$ and $n \geq 2$ then we obtain the basic equality for the Fibonacci quaternions G_n

$$G_n = G_{n-1} + G_{n-2}. \tag{20}$$

Lemma 3.5. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} L(p, pl+1) = L(p, np+2) - L(p, p+2), \tag{21}$$

$$\sum_{l=2}^{n} L(p, pl+2) = L(p, np+3) - L(p, p+3), \tag{22}$$

$$\sum_{l=2}^{n} L(p, pl+3) = L(p, np+4) - L(p, p+4). \tag{23}$$

Proof. Using the second equation from (4) we have

$$L(p,n-1) = L(p,n) - L(p,n-p), \text{ for n} \geq \text{2p}.$$

For integers $2p+1,\ 3p+1,\ \ldots, np+1$ we obtain

$$\begin{aligned}
&L(p,2p+1) = L(p,2p+2) - L(p,p+2)\\
&L(p,3p+1) = L(p,3p+2) - L(p,2p+2)\\
&L(p,4p+1) = L(p,4p+2) - L(p,3p+2)\\
&\vdots\\
&L(p,np+1) = L(p,np+2) - L(p,(n-1)p+2).
\end{aligned}$$

Adding these equalities we obtain (21).

In the same way one can easily prove (22) and (23). □

Theorem 3.6. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$\sum_{l=2}^{n} LQ^p_{pl} = LQ^p_{np+1} - LQ^p_{p+1}. \tag{24}$$

Proof. Using (11) we have

$$\begin{aligned}
&\sum_{l=2}^{n} LQ^p_{pl} = LQ^p_{2p} + LQ^p_{3l} + \ldots + LQ^p_{nl} =\\
&= L(p,2p) + iL(p,2p+1) + jL(p,2p+2) + kL(p,2p+3)+\\
&+ L(p,3p) + iL(p,3p+1) + jL(p,3p+2) + kL(p,3p+3) + \ldots +\\
&+ L(p,np) + iL(p,np+1) + jL(p,np+2) + kL(p,np+3)+\\
&= L(p,2p) + L(p,3p) + \ldots + L(p,np)+\\
&+ i\left(L(p,2p+1) + L(p,3p+1) + \ldots + L(p,np+1)\right)+\\
&+ j\left(L(p,2p+2) + L(p,3p+2) + \ldots + L(p,np+2)\right)+\\
&+ k\left(L(p,2p+3) + L(p,3p+3) + \ldots + L(p,np+3)\right).
\end{aligned}$$

Writing (8) as $\sum\limits_{l=2}^{n} L(p,pl) = L(p,np+1) - L(p,p+1)$ and using (21)–(23) we obtain (24). □

Theorem 3.7. *Let $p \geq 2$, $n \geq 2p$ be integers. Then*

$$LQ^p_n = p \cdot FQ^p_{n-(2p-1)} + FQ^p_{n-p}. \tag{25}$$

Proof. Using (10) we have

$$\begin{aligned}
FQ^p_{n-(2p-1)} &= F(p,n-(2p-1)) + iF(p,n-(2p-1)+1)+\\
&+ jF(p,n-(2p-1)+2) + kF(p,n-(2p-1)+3)
\end{aligned}$$

and

$$FQ^{p}_{n-p} = F(p, n-p) + iF(p, n-p+1) + \\ + jF(p, n-p+2) + kF(p, n-p+3),$$

consequently

$$\begin{aligned} & p \cdot FQ^{p}_{n-(2p-1)} + FQ^{p}_{n-p} = \\ & = p \cdot F(p, n-(2p-1)) + F(p, n-p) + \\ & + i\left(p \cdot F(p, (n+1)-(2p-1)) + F(p, (n+1)-p)\right) + \\ & + j\left(p \cdot F(p, (n+2)-(2p-1)) + F(p, (n+2)-p)\right) + \\ & + k\left(p \cdot F(p, (n+3)-(2p-1)) + F(p, (n+3)-p)\right). \end{aligned}$$

Using (9) we have

$$\begin{aligned} & p \cdot FQ^{p}_{n-(2p-1)} + FQ^{p}_{n-p} = \\ & = L(p, n) + iL(p, n+1) + jL(p, n+2) + kL(p, n+3), \end{aligned}$$

which ends the proof. □

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KWATERNIONY $F(p,n)$-FIBONACCIEGO

S t r e s z c z e n i e

W pracy wprowadzamy i badamy jednoparametrowe uogólnienie kwaternionów Fibonacciego. Podajemy własności kwaternionów $F(p,n)$-Fibonacciego, a także uogólnienia klasycznych wyników dla kwaternionów Fibonacciego.

Słowa kluczowe: liczby Fibonacciego, liczby Lucasa, kwaterniony, zależności rekurencyjne