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Dedicated to the memory of
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INTEGRAL THEOREMS FOR MONOGENIC FUNCTIONS IN AN INFINITE-DIMENSIONAL SPACE WITH A COMMUTATIVE MULTIPLICATION

Summary

We consider monogenic functions taking values in a topological vector space being an expansion of a certain infinite-dimensional commutative Banach algebra associated with the three-dimensional Laplace equation. We establish also integral theorems for monogenic functions taking values in the mentioned algebra and the mentioned topological vector space.

Keywords and phrases: Laplace equation, spatial potentials, harmonic algebra, topological vector space, differentiable in the sense of Gâteaux function, monogenic function, Cauchy-Riemann conditions

1. Introduction

A commutative algebra $\mathbb{A}$ with unit is called harmonic (see [1, 2, 3, 4]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors e_1, e_2, e_3 satisfying the relations

$$e_1^2 + e_2^2 + e_3^2 = 0, \qquad e_k^2 \neq 0,\ k = 1, 2, 3\,.$$

Such a triad e_1, e_2, e_3 is also called harmonic.

In the papers [1, 2, 3, 4, 5, 6, 7, 8] harmonic algebras are used for constructions of spatial *harmonic* functions, i.e. doubly continuously differentiable functions $u(x, y, z)$

satisfying the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = 0\,. \tag{1}$$

I. Mel'nichenko [3, 4] found all three-dimensional harmonic algebras and developed a method for finding all harmonic bases in these algebras. But it is impossible to obtain all solutions of equation (1) in the form of components of differentiable in the sense of Gâteaux functions taking values in finite-dimensional commutative algebras (see, e.g., [4, p. 43]).

In the papers [4, 6] spherical functions are obtained as the first components of decompositions of corresponding analytic functions with respect to the basis of an infinite-dimensional commutative Banach algebra $\mathbb{F}$. To obtain all solutions of equation (1) in the form of components of differentiable in the sense of Gâteaux functions, in the papers [7] we included corresponding algebras in topological vector spaces.

In the paper [8] we constructed spatial harmonic functions in the form of principal extensions of analytic functions of a complex variable into a complexification $\mathbb{F}_{\mathbb{C}}$ of the algebra $\mathbb{F}$. We considered special extensions of differentiable in the sense of Gâteaux functions with values in a topological vector space $\widetilde{\mathbb{F}}_{\mathbb{C}}$ being an expansion of the algebra $\mathbb{F}_{\mathbb{C}}$. Moreover, we considered also relations between the mentioned extensions and spatial potentials, in particular, axial-symmetric potentials.

For monogenic functions given in an infinite-dimensional algebra or a topological vector space associated with axial-symmetric potentials, analogues of classical integral theorems of complex analysis was proved in the paper [9].

In the present paper, using ideas of the paper [9], we prove integral theorems for monogenic functions taking values in an infinite-dimensional algebra $\mathbb{F}_{\mathbb{C}}$ and a topological vector space $\widetilde{\mathbb{F}}_{\mathbb{C}}$.

2. An infinite-dimensional algebra $\mathbb{F}_{\mathbb{C}}$

Consider an infinite-dimensional commutative associative Banach algebra over the field of real numbers $\mathbb{R}$, namely:

$$\mathbb{F} := \Big\{a = \sum_{k=1}^{\infty} a_k e_k : a_k \in \mathbb{R}, \sum_{k=1}^{\infty} |a_k| < \infty\Big\}$$

with the norm $\|a\|_{\mathbb{F}} := \sum\limits_{k=1}^{\infty} |a_k|$ and the basis $\{e_k\}_{k=1}^{\infty}$, where the multiplication table for the basis elements is of the following form:

$$e_n e_1 = e_n, \qquad e_{2n+1}e_{2n} = \frac{1}{2}\, e_{4n} \quad \forall\, n \geq 1\,,$$

$$e_{2n+1}e_{2m} = \frac{1}{2}\Big(e_{2n+2m} - (-1)^m e_{2n-2m}\Big) \quad \forall\, n > m \geq 1\,,$$

$$e_{2n+1}e_{2m} = \frac{1}{2}\left(e_{2n+2m} + (-1)^n e_{2m-2n}\right) \quad \forall\, m > n \geq 1\,,$$

$$e_{2n+1}e_{2m+1} = \frac{1}{2}\left(e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\right) \quad \forall\, n \geq m \geq 1\,,$$

$$e_{2n}e_{2m} = \frac{1}{2}\left(-e_{2n+2m+1} + (-1)^m e_{2n-2m+1}\right) \quad \forall\, n \geq m \geq 1\,.$$

This algebra was proposed in the paper [4] (see also [7]). Inasmuch as $e_1^2+e_2^2+e_3^2 = 0$, the algebra $\mathbb{F}$ is harmonic and the vectors e_1, e_2, e_3 form a harmonic triad.

Now, consider a complexification $\mathbb{F}_{\mathbb{C}} := \mathbb{F} \oplus i\mathbb{F} \equiv \{a + ib : a, b \in \mathbb{F}\}$ of the algebra $\mathbb{F}$ such that the norm in $\mathbb{F}_{\mathbb{C}}$ is given as $\|c\| := \sum\limits_{k=1}^{\infty} |c_k|$, where $c = \sum\limits_{k=1}^{\infty} c_k e_k$, $c_k \in \mathbb{C}$, and $\mathbb{C}$ is the field of complex numbers.

Note that the algebra $\mathbb{F}_{\mathbb{C}}$ is isomorphic to the algebra $\mathbf{F}_{\mathbb{C}}$ of absolutely convergent trigonometric Fourier series

$$c(\theta) = c_0 + \sum_{k=1}^{\infty}\left(a_k\, i^k \cos k\theta + b_k\, i^k \sin k\theta\right)$$

with $c_0, a_k, b_k \in \mathbb{C}$ and the norm $\|c\|_{\mathbf{F}_{\mathbb{C}}} := |c_0| + \sum\limits_{k=1}^{\infty}\left(|a_k| + |b_k|\right)$. In this case, we have the isomorphism $e_{2k-1} \leftrightarrow i^{k-1}\cos{(k-1)\theta}$, $e_{2k} \leftrightarrow i^k \sin k\theta$ between elements of bases.

3. Monogenic and analytic functions taking values in the algebra $\mathbb{F}_{\mathbb{C}}$

Below, we shall consider functions given in subsets of the linear manifold $E_4 := \{\xi = xe_1 + s\, ie_1 + ye_2 + ze_3 \,:\, x, s, y, z \in \mathbb{R}\}$ containing the complex plane $\mathbb{C}$. With a set $Q \subset \mathbb{R}^4$ we associate the set $Q_\xi := \{\xi = xe_1 + s\, ie_1 + ye_2 + ze_3 : (x, s, y, z) \in Q\}$ in E_4. In what follows, $\xi = xe_1 + s\, ie_1 + ye_2 + ze_3$.

A function $\Psi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ is called *analytic* in a domain Q_ξ if in a certain neighborhood of each point $\xi_0 \in Q_\xi$ it can be represented in the form of the sum of convergent power series

$$\Psi(\xi) = \sum_{k=1}^{\infty} c_k(\xi - \xi_0)^k, \quad c_k \in \mathbb{F}_{\mathbb{C}}. \tag{2}$$

A continuous function $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ is called *monogenic* in a domain $Q_\xi \subset E_4$ if Φ is differentiable in the sense of Gâteaux in every point of Q_ξ, i. e., if for every $\xi \in Q_\xi$ there exists an element $\Phi'(\xi) \in \mathbb{F}_{\mathbb{C}}$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\xi + \varepsilon h) - \Phi(\xi)\right) \varepsilon^{-1} = h\Phi'(\xi) \qquad \forall\, h \in E_4\,. \tag{3}$$

It is obvious that an analytic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ is monogenic in the domain Q_ξ and its derivative $\Phi'(\xi)$ is also monogenic in Q_ξ.

Below, we establish sufficient conditions for a monogenic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ to be analytic in a domain $Q_\xi \subset E_4$.

Let us emphasize that in the case where a monogenic function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ has the continuous Gâteaux derivatives Φ', Φ'', it satisfies the identity $\Delta_3\Phi(\xi) \equiv 0$ because

$$\Delta_3\Phi(\xi) \equiv \Phi''(\xi)\ (e_1^2 + e_2^2 + e_3^2) \equiv 0\,.$$

Thus, for every component $U_k : Q \to \mathbb{C}$ of the decomposition

$$\Phi(\xi) = \sum_{k=1}^{\infty} U_k(x,s,y,z)\ e_k \tag{4}$$

of such a function Φ, the functions $\mathrm{Re}\, U_k(x,s,y,z)$, $\mathrm{Im}\, U_k(x,s,y,z)$ are spatial harmonic functions for every fixed s.

We say that the functions $U_k : Q \to \mathbb{C}$ of the decomposition (4) are $\mathbb{R}$-*differentiable* in Q if for all points $(x,s,y,z) \in Q$ the following relations are true:

$$U_k(x+\Delta x, s+\Delta s, y+\Delta y, z+\Delta z) - U_k(x,s,y,z) =$$
$$= \frac{\partial U_k}{\partial x}\Delta x + \frac{\partial U_k}{\partial s}\Delta s + \frac{\partial U_k}{\partial y}\Delta y + \frac{\partial U_k}{\partial z}\Delta z + o(\|\Delta\xi\|)\,,$$
$$\Delta\xi := e_1\Delta x + ie_1\Delta s + e_2\Delta y + e_3\Delta z \to 0.$$

The following theorem can be proved similarly to Theorem 4.1 [6].

Theorem 1. *Let a function $\Phi : Q_\xi \to \mathbb{F}_\mathbb{C}$ be continuous in a domain $Q_\xi \subset E_4$ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *be $\mathbb{R}$-differentiable in Q. In order the function Φ be monogenic in the domain Q_ξ, it is necessary and sufficient that the conditions*

$$\frac{\partial\Phi}{\partial s} = \frac{\partial\Phi}{\partial x}\, i\,, \qquad \frac{\partial\Phi}{\partial y} = \frac{\partial\Phi}{\partial x}\, e_2\,, \qquad \frac{\partial\Phi}{\partial z} = \frac{\partial\Phi}{\partial x}\, e_3 \tag{5}$$

be satisfied in Q_ξ and the following relations be fulfilled in Q:

$$\sum_{k=1}^{\infty} \left|\frac{\partial U_k(x,s,y,r)}{\partial x}\right| < \infty, \tag{6}$$

$$\lim_{\varepsilon\to 0+0} \sum_{k=1}^{\infty} \Bigg| U_k(x+\varepsilon h_1, s+\varepsilon h_2, y+\varepsilon h_3, r+\varepsilon h_4) - U_k(x,s,y,r) -$$
$$- \frac{\partial U_k(x,s,y,r)}{\partial x}\ \varepsilon h_1 - \frac{\partial U_k(x,s,y,r)}{\partial s}\ \varepsilon h_2 - \frac{\partial U_k(x,s,y,r)}{\partial y}\ \varepsilon h_3 -$$
$$- \frac{\partial U_k(x,s,y,r)}{\partial r}\ \varepsilon h_4 \Bigg| \varepsilon^{-1} = 0 \qquad \forall\, h_1, h_2, h_3, h_4 \in \mathbb{R}\,. \tag{7}$$

Note that the first of conditions (5) means that every function U_k from the equality (4) is holomorphic with respect to the variable $x+is$ for each fixed pair (y,z).

4. Integral theorems for monogenic functions taking values in the algebra $\mathbb{F}_{\mathbb{C}}$

In the paper [10] for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. The convexity of the domain in the mentioned results from [10] is withdrawn by E. K. Blum [11].

Below we establish similar results for monogenic functions $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ given only in a domain Q_ξ of the linear manifold E_4 instead of domain of whole algebra. Let us note that *a priori* the differentiability of the function Φ in the sense of Gâteaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Let us also note that in the paper [9] similar results were established for monogenic functions in an other infinite-dimensional algebra associated with axial-symmetric potentials.

In the case where Γ is a Jordan rectifiable curve in $\mathbb{R}^4$ we shall say that Γ_ξ is also a Jordan rectifiable curve. For a continuous function $\Phi : \Gamma_\xi \to \mathbb{F}_{\mathbb{C}}$ of the form (4), where $(x,s,y,r) \in \Gamma$ and $U_k : \Gamma \to \mathbb{C}$, we define an integral along the curve Γ_ξ with $d\xi := e_1\,dx + ie_1\,ds + e_2\,dy + e_3\,dz$ by the equality

$$\int\limits_{\Gamma_\xi} \Phi(\xi)d\xi := \sum_{k=1}^{\infty} e_k \int\limits_{\Gamma} U_k(x,s,y,z)dx + i\sum_{k=1}^{\infty} e_k \int\limits_{\Gamma} U_k(x,s,y,z)ds +$$

$$+\sum_{k=1}^{\infty} e_2 e_k \int\limits_{\Gamma} U_k(x,s,y,z)dy + \sum_{k=1}^{\infty} e_3 e_k \int\limits_{\Gamma} U_k(x,s,y,z)dz \tag{8}$$

in the case where the series on the right-hand side of the equality are elements of the algebra $\mathbb{F}_{\mathbb{C}}$.

Theorem 2. *Let $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ be a monogenic function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then for every closed Jordan rectifiable curve $\Gamma_\xi \subset Q_\xi$ homotopic to a point in Q_ξ, the following equality holds:*

$$\int\limits_{\Gamma_\xi} \Phi(\xi)d\xi = 0\,. \tag{9}$$

Proof. Using the Stokes formula and the equalities (5), we obtain the equality

$$\int\limits_{\partial\triangle_\xi} \Phi(\xi)d\xi = 0 \tag{10}$$

for the boundary $\partial\triangle_\xi$ of every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$. Now, we can complete the proof similarly to the proof of Theorem 3.2 [11]. The theorem is proved. □

For functions $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ the following Morera theorem can be established in the usual way.

Theorem 3. *If a function $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ is continuous in a domain Q_ξ and satisfies the equality* (10) *for every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$, then the function Φ is monogenic in the domain Q_ξ.*

Let $\tau := we_1 + \hat{y}e_2 + \hat{z}e_3$ where $w \in \mathbb{C}$ and $\hat{y}\,, \hat{z} \in \mathbb{R}$. Generalizing a resolvent resolution (cf. the equality (5) in [8]), we obtain

$$(\tau-\xi)^{-1} = \frac{1}{\sqrt{(w-\tau_1)(w-\tau_2)}}\Bigg(e_1 + \sum_{k=1}^{\infty} i^k \left(u_2^{-k} + u_1^k\right) e_{2k+1} +$$

$$+ \sum_{k=1}^{\infty} i^{k-1} \left(u_2^{-k} - u_1^k\right) e_{2k}\Bigg), \qquad w \notin s[\tau_1,\tau_2], \tag{11}$$

where

$$\tau_1 := x + is - i\sqrt{(y-\hat{y})^2 + (z-\hat{z})^2}, \qquad \tau_2 := x + is + i\sqrt{(y-\hat{y})^2 + (z-\hat{z})^2},$$

$$u_1 := \frac{(w-x-is) - \sqrt{(w-\tau_1)(w-\tau_2)}}{(y-\hat{y}) + i(z-\hat{z})},$$

$$u_2 := \frac{(w-x-is) + \sqrt{(w-\tau_1)(w-\tau_2)}}{(y-\hat{y}) + i(z-\hat{z})},$$

$s[\tau_1, \tau_2]$ is the segment connecting the points τ_1, τ_2, and $\sqrt{(w-\tau_1)(w-\tau_2)}$ is that continuous branch of the function

$$G(w) = \sqrt{(w-\tau_1)(w-\tau_2)}$$

analytic outside of the cut along the segment $s[\tau_1, \tau_2]$ for which $G(w) > 0$ for any $w > x$. Let us note that one should to set $u_1^k = 0$ and $u_2^{-k} = 0$ by continuity in the equality (11) for that $w \notin s[\tau_1, \tau_2]$ for which $\hat{y} = y$ and $\hat{z} = z$.

Thus, for every ξ the element $\left(\tau - \xi\right)^{-1}$ exists for all

$$\tau \notin S(\xi) := \Big\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 :$$

$$\operatorname{Re} w = x,\ |\operatorname{Im} w - s| \le \sqrt{(y-\hat{y})^2 + (z-\hat{z})^2}\Big\}.$$

Now, the next theorem can be proved similarly to Theorem 5 [12].

Theorem 4. *Suppose that Q is a domain convex in the direction of the axes Oy, Oz. Suppose also that $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ is a monogenic function in the domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then for every point $\xi \in Q_\xi$ the following equality is true:*

$$\Phi(\xi) = \frac{1}{2\pi i} \int\limits_{\Gamma_\xi} \Phi(\tau)\big(\tau - \xi\big)^{-1} d\tau\,, \tag{12}$$

where Γ_ξ is an arbitrary closed Jordan rectifiable curve in Q_ξ, which surrounds once the set $S(\xi)$ and is homotopic to the circle $\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 : |w - x - is| = R,\ \hat{y} = y, \hat{z} = z\}$ contained completely in Ω_ξ.

Using the formula (12), we obtain the Taylor expansion of monogenic function $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ in the usual way (see., for example, [13, p. 107]) in the case where the conditions of Theorem 4 are satisfied. Thus, in this case, $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ is an analytic function. In addition, in this case, an uniqueness theorem for monogenic functions can also be proved in the same way as for holomorphic functions of the complex variable (cf. [13, p. 110]).

Thus, the following theorem is true:

Theorem 5. *Let $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ be a continuous function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Then the function Φ is monogenic in Q_ξ if and only if one of the following conditions is satisfied:*

(I) *the conditions* (5) *are satisfied in Q_ξ and the relations* (6), (7) *are fulfilled in Q;*

(II) *the function Φ satisfies the equality* (10) *for every triangle $\triangle_\xi$ such that $\overline{\triangle_\xi} \subset Q_\xi$;*

(III) *the function Φ is analytic in the domain Q_ξ.*

5. Monogenic functions with values in a topological vector space $\widetilde{\mathbb{F}}_{\mathbb{C}}$ containing the algebra $\mathbb{F}_{\mathbb{C}}$

Let us insert the algebra $\mathbb{F}_{\mathbb{C}}$ in the topological vector space

$$\widetilde{\mathbb{F}}_{\mathbb{C}} := \Big\{ g = \sum_{k=1}^{\infty} c_k e_k : c_k \in \mathbb{C} \Big\}$$

with the topology of coordinate-wise convergence. Note that $\widetilde{\mathbb{F}}_{\mathbb{C}}$ is not an algebra because the product of elements $g_1, g_2 \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ is defined not always. At the same time, for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ and $\xi = (x + is)e_1 + ye_2 + ze_3$ with $x, s, y, z \in \mathbb{R}$ it is easy to define the product

$$g\,\xi \equiv \xi\, g := (x+is)\sum_{k=1}^{\infty} c_k e_k + y\Biggl(-\frac{c_2}{2}\,e_1 + \Bigl(c_1-\frac{c_5}{2}\Bigr)\,e_2 - \frac{c_4}{2}\,e_3 +$$

$$+\frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-1}-c_{2k+3})\,e_{2k} - \frac{1}{2}\sum_{k=2}^{\infty}(c_{2k-2}+c_{2k+2})\,e_{2k+1}\Biggr)+$$

$$+z\Biggl(-\frac{c_3}{2}\,e_1 - \frac{c_4}{2}\,e_2 + \Bigl(c_1-\frac{c_5}{2}\Bigr)\,e_3 + \frac{1}{2}\sum_{k=4}^{\infty}(c_{k-2}-c_{k+2})\,e_k\Biggr).$$

In the paper [8], we proved that monogenic functions given in domains of the linear manifold $\{\zeta = xe_1 + ye_2 + ze_3 \,:\, x,y,z \in \mathbb{R}\}$ and taking values in the space $\widetilde{\mathbb{F}}_{\mathbb{C}}$ can be extended to monogenic functions given in domains of the linear manifold E_4.

We shall consider functions $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ for which the functions $U_k : Q \to \mathbb{C}$ in the decomposition (4) are $\mathbb{R}$-differentiable in the domain Q. Such a function Φ is continuous in Q_ξ and, therefore, we call Φ a *monogenic* function in Q_ξ if $\Phi'(\xi) \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ in the equality (3).

The next theorem is similar to Theorem 1, where the necessary and sufficient conditions for a function $\Phi : Q_\xi \to \mathbb{F}_{\mathbb{C}}$ to be monogenic include additional relations (6), (7) conditioned by the norm of absolute convergence in the algebra $\mathbb{F}_{\mathbb{C}}$.

Theorem 6. *Let a function $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ be of the form* (4) *and the functions $U_k : Q \to \mathbb{C}$ be $\mathbb{R}$-differentiable in Q. In order the function Φ be monogenic in the domain Q_ξ, it is necessary and sufficient that the conditions* (5) *be satisfied in Q_ξ.*

For a continuous function $\Phi : \Gamma_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ of the form (4), we define an integral along a Jordan rectifiable curve Γ_ξ by the equality (8) in the case where the series on the right-hand side of this equality are elements of the space $\widetilde{\mathbb{F}}_{\mathbb{C}}$.

In the next theorem, for the sake of simplicity, we suppose that the curve Γ_ξ is the piece-smooth edge of a piece-smooth surface. In this case the following statement is a result of the Stokes formula and the equalities (5).

Theorem 7. *Suppose that $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ is a monogenic function in a domain Q_ξ and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *have continuous partial derivatives in Q. Suppose also that Σ is a piece-smooth surface in Q with the piece-smooth edge Γ. Then the equality* (9) *holds.*

Let us define the product $gh \equiv hg$ for each $g = \sum\limits_{k=1}^{\infty} c_k e_k \in \widetilde{\mathbb{F}}_{\mathbb{C}}$ and $h =$

$\sum\limits_{k=1}^{\infty} t_k e_k \in \mathbb{F}_{\mathbb{C}}$ in the case where the sequence $\{c_k\}_{k=1}^{\infty}$ is bounded:

$$gh \equiv hg := \left(c_1 t_1 + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{[k/2]} c_k t_k \right) e_1 +$$

$$+ \left(c_2 t_1 + \left(c_1 + \frac{c_5}{2} \right) t_2 + \frac{-c_4}{2}\, t_3 + \frac{1}{2} \sum_{k=4}^{\infty} (-1)^{\left[\frac{k-1}{2}\right]} \left(c_{k-2+(-1)^k} + c_{k+2+(-1)^k} \right) \right) e_2 +$$

$$+ \left(c_3 t_1 + \frac{-c_4}{2}\, t_2 + \left(c_1 - \frac{c_5}{2} \right) t_3 + \frac{1}{2} \sum_{k=4}^{\infty} (-1)^{\left[\frac{k-2}{2}\right]} \left(c_{k-2} - c_{k+2} \right) \right) e_3 +$$

$$+ \sum_{m=4}^{\infty} \Upsilon_m \, e_m \,,$$

where the constants Υ_m are defined by the next relations in four following cases:

1) if m is of the form $m = 4r$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-1} \left(c_{m-k+1} + (-1)^{\left[\frac{k-1}{2}\right]} c_{m+k+(-1)^k} \right) t_k +$$
$$+ \left(c_1 - \frac{c_{2m+1}}{2} \right) t_m + \frac{c_{2m}}{2}\, t_{m+1} +$$
$$+ \frac{1}{2} \sum_{k=m+2}^{\infty} (-1)^{\left[\frac{k+1}{2}\right]} \left(c_{k-m+(-1)^k} - c_{k+m+(-1)^k} \right) t_k \,;$$

2) if m is of the form $m = 4r - 1$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-2} \left((-1)^{k-1} c_{m-k-(-1)^k} + (-1)^{\left[\frac{k}{2}\right]} c_{m+k-1} \right) t_k -$$
$$- \frac{c_{2m-2}}{2}\, t_{m-1} + \left(c_1 - \frac{c_{2m-1}}{2} \right) t_m +$$
$$+ \frac{1}{2} \sum_{k=m+1}^{\infty} (-1)^{\left[\frac{k-2}{2}\right]} \left(c_{k-m+1} - c_{k+m-1} \right) t_k \,;$$

3) if m is of the form $m = 4r - 2$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2} \sum_{k=2}^{m-1} \left(c_{m-k+1} + (-1)^{\left[\frac{k-1}{2}\right]} c_{m+k+(-1)^k} \right) t_k +$$
$$+ \left(c_1 + \frac{c_{2m+1}}{2} \right) t_m - \frac{c_{2m}}{2}\, t_{m+1} +$$
$$+ \frac{1}{2} \sum_{k=m+2}^{\infty} (-1)^{\left[\frac{k-1}{2}\right]} \left(c_{k-m+(-1)^k} + c_{k+m+(-1)^k} \right) t_k \,;$$

4) if m is of the form $m = 4r - 3$ with natural r, then

$$\Upsilon_m = c_m t_1 + \frac{1}{2}\sum_{k=2}^{m-2}\Big((-1)^{k-1}c_{m-k-(-1)^k} + (-1)^{\left[\frac{k}{2}\right]}c_{m+k-1}\Big)\,t_k +$$

$$+\frac{c_{2m-2}}{2}\,t_{m-1} + \Big(c_1 + \frac{c_{2m-1}}{2}\Big)\,t_m + \frac{1}{2}\sum_{k=m+1}^{\infty}(-1)^{\left[\frac{k}{2}\right]}\left(c_{k-m+1} + c_{k+m-1}\right)t_k\,.$$

In the case where Γ is a piece-smooth curve (or Σ is a piece-smooth surface) in $\mathbb{R}^4$ we shall say that Γ_ξ is also a piece-smooth curve (or Σ_ξ is also a piece-smooth surface, respectively). We say that a domain $Q \subset \mathbb{R}^4$ is *convex in the direction of the plane* $\{(\hat{x}, \hat{s}, \hat{y}, \hat{z}) : \hat{x}, \hat{s} \in \mathbb{R},\ \hat{y} = y, \hat{z} = z\}$ if Q contains any segment that is parallel to the mentioned plane and connects two points of the domain Q.

The next theorem can be proved similarly to Theorem 5 in [12].

Theorem 8. *Suppose that Q is a domain convex in the direction of the plane $\{(\hat{x}, \hat{s}, \hat{y}, \hat{z}) : \hat{x}, \hat{s} \in \mathbb{R},\ \hat{y} = y, \hat{z} = z\}$. Suppose also that $\Phi : Q_\xi \to \widetilde{\mathbb{F}}_{\mathbb{C}}$ is a monogenic function in the domain Q_ξ, and the functions $U_k : Q \to \mathbb{C}$ from the decomposition* (4) *form an uniformly bounded family and have continuous partial derivatives in Q. Then for every point $\xi \in Q_\xi$ the equality* (12) *holds, where Γ_ξ is a piece-smooth curve that surrounds once the set $S(\xi)$ and, in addition, Γ_ξ and the circle $\{\tau = we_1 + \hat{y}e_2 + \hat{z}e_3 : |w - x - is| = R,\ \hat{y} = y, \hat{z} = z\}$ are edges of a piece-smooth surface Σ_ξ contained completely in Ω_ξ.*

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TWIERDZENIA CAŁKOWE DLA FUNKCJI MONOGENICZNYCH W PRZESTRZENI NIESKOŃCZENIE-WYMIAROWEJ Z MNOŻENIEM PRZEMIENNYM

S t r e s z c z e n i e

Rozpatrujemy funkcje o wartościach w wektorowej przestrzeni topologicznej będącej rozszerzeniem pewnej nieskończenie-wymiarowej przemiennej algebry Banacha stowarzyszonej z trójwymiarowym równaniem Laplace'a. Uzyskujemy twierdenia całkowe dla funkcji monogenicznych o wartościach we wspomnianej algebrze i we wspomnianej wektorowej przestrzeni topologicznej.

Słowa kluczowe: równanie Laplace'a, potencjaly przestrzenne, algebra harmoniczna, przestrzeń wektorowa topologiczna, różniczkowalność w sensie Gâteaux, funkcja monogeniczna, warunki Cauchy'ego-Riemanna