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Dedicated to the memory of
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ON SOME CASES OF PLANE ORTHOTROPY

Summary

There are considered some cases of plane orthotropy in the absence of body forces. Then every function from a pair-solution of the equilibrium system of equations with respect to displacements satisfies the elliptic fouth-order equation of the type:

$$\left(\alpha_1 \frac{\partial^4}{\partial x^4} + \alpha_2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x,y) = 0,$$

with certain real $\alpha_k \neq 0$, $k = 1, 2$.

Keywords and phrases: the generalized Hook's law, a plane orthotropy, the equilibrium system

1. Introduction

As well-known (cf., e.g., [1, 2, 3]), in the case of isotropic plane deformations with the absence of body forces a function (displacement) u or v from a pair-solution $(u(x,y), v(x,y))$ of the equilibrium system of equations in displacements

$$\begin{cases} (\lambda+\mu)\left(\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial x \partial y}\right) + \mu \Delta_2 u(x,y) = 0, \\ (\lambda+\mu)\left(\frac{\partial^2 u(x,y)}{\partial x \partial y} + \frac{\partial^2 v(x,y)}{\partial y^2}\right) + \mu \Delta_2 v(x,y) = 0 \, \forall (x,y) \in D, \end{cases} \tag{1}$$

as well as the stress Airy's function, satisfies the biharmonic equation: $(\Delta_2)^2 w(x,y) = 0$, where $\Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the 2-D Laplasian, D is a domain of the Cartesian plane xOy, λ and μ are the Lamé constants.

Similar results for anisotrophic solid body are not well-known. One of the reason of this fact is a difficulty (due to a variety of cofficients) of the generalized Hooke's law expressing strains via stresses in a linear form.

The aim of this paper is to prove analogous (to the isotropic case) statements for some cases of an elastic anisotrophic homogeneous plane solid body — a plane orthotropic body, or briefly, a plane orthotropy. We will restrict our attention on some simple but interesting cases of orthotropy.

2. Notations and preliminaries

Let $\mathbb{R}^{3\times 3}$ be a set of all real 3×3 matrices, $A \in \mathbb{R}^{3\times 3}$, $\det A$ is a determinant of A. If $\det A \neq 0$ then there exists the inverse matrix $B = A^{-1}$ such that $AB = BA = 1$, where 1 is the unity matrix. By $\mathbb{R}^{3\times 3}_{+}$ we define all matrices of $\mathbb{R}^{3\times 3}$ which are symmetric and positive defined. A symbol $\overleftarrow{\vartheta}$ defines a vector-column having three real coordinates ϑ_k, $k = 1, 2, 3$.

Let a model of an elastic anisotropic medium occupied a domain D of the Cartesian plane xOy be a homogeneous (cf., e.g., [4, p. 25]) plane orthotropic (cf., e.g., [4, p. 35]) body.

Let $\overleftarrow{\varepsilon}$ has coordinates equal to strains (cf., e.g., [4, p. 18]):

$$\varepsilon_1 := \varepsilon_x,\ \varepsilon_2 := \varepsilon_x,\ \varepsilon_3 = \gamma_{xy}.$$

Let $\overleftarrow{\sigma}$ has coordinates equal to stresses (cf., e.g., [4, p. 16]):

$$\sigma_1 := \sigma_x,\ \sigma_2 := \sigma_y,\ \sigma_3 := \tau_{xy}.$$

The generalized Hooke's law for our model has two equivalent forms (cf., e.g., [4, § 3], [5, § 4.1.3]):

$$\overleftarrow{\varepsilon} = A\overleftarrow{\sigma},\ \overleftarrow{\sigma} = A^{-1}\overleftarrow{\varepsilon}, \tag{2}$$

with $A \in \mathbb{R}^{3\times 3}_{+}$ of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{pmatrix}, \tag{3}$$

where

$$a_{11} > 0,\ a_{11}a_{22} - (a_{12})^2 > 0,\ a_{66} > 0. \tag{4}$$

Unequalities (4) follows from the Sylvester's criterion of positive definiteness of the matrix (3).

A numbers a_{ij} and A_{ij} , $k \leq m$, $k, m = 1, 2, 6$, are called *elastic constants* ([4, p. 27]). They are constants in D due to the homogeneity of the solid body.

Consider notations for elements of A^{-1}:

$$A^{-1} =: \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{pmatrix}, \tag{5}$$

where A_{km} satisfy (4) with $a_{km} := A_{km}$, $k \leq m$, $k, m = 1, 2, 6$.

A *stress function* ([6, p. 21] with $\overline{U} \equiv 0$) is a function w satisfying relations:

$$\sigma_x(x_0, y_0) = \frac{\partial^2 w}{\partial y^2}(x_0, y_0),\ \sigma_y(x_0, y_0) = \frac{\partial^2 w}{\partial x^2}(x_0, y_0),$$

$$\tau_{xy}(x_0, y_0) = -\frac{\partial^2 w}{\partial x \partial y}(x_0, y_0)\,\forall (x_0, y_0) \in D.$$

In the absence of body forces, the stress function $w(x, y)$ satisfies the elliptic fouth-order equation (" the stress equation", cf., e.g., [6, p. 27] with $a_{16} = a_{26} = 0$):

$$\left(a_{22}\frac{\partial^4}{\partial x^2} + (2a_{12} + a_{66})\frac{\partial^4}{\partial x^2 \partial y^2} + a_{11}\frac{\partial^4}{\partial y^4} \right) w(x, y) = 0. \tag{6}$$

The equilibrium system of equations with respect to the displacement vector $(u(x, y), v(x, y))$ has a form (cf., e.g., [4, p. 75]):

$$\begin{cases} \left(A_{11}\frac{\partial^2}{\partial x^2} + A_{66}\frac{\partial^2}{\partial y^2} \right) u(x, y) + (A_{12} + A_{66})\frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ \left(A_{66}\frac{\partial^2}{\partial x^2} + A_{22}\frac{\partial^2}{\partial y^2} \right) v(x, y) + (A_{12} + A_{66})\frac{\partial^2 u(x,y)}{\partial x \partial y} = 0, \end{cases} \tag{7}$$

where all $(x, y) \in D$; A_{km}, $k \leq m$, $k, m = 1, 2, 6$, are defined in (5).

3. Cases of orthotropy and solutions of their equilibrium systems and stress equation

Consider the following equation (particular case of (6)):

$$l_{0,p}w(x, y) \equiv$$

$$\equiv \left((2p - 1)\frac{\partial^4}{\partial x^4} + 2p\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) w(x, y) = 0\ \forall (x, y) \in D, \tag{8}$$

where $p \neq 1$ is a real parameter.

Consider an orthotropy with

$$a_{11} = a_{12} = 1,\ a_{22} = 2p - 1,\ a_{16} = a_{26} = 0,\ a_{66} = 2(p - 1). \tag{9}$$

Then the equation (8) is a stress equation. It is easy to check that the matrix (3) is positive defened only for $p > 1$. So a case $p < 1$ has no elastic meaning and we are to investigate a case $p > 1$. Calculating the inverse matrix A^{-1} we find:

$$A_{11} = \frac{2p - 1}{2(p - 1)},\ A_{12} = -\frac{1}{2(p - 1)},\ A_{22} = A_{66} = -A_{12}. \tag{10}$$

Since $A_{12} + A_{66} = 0$ a system (7) takes a form

$$\begin{cases} \frac{1}{2(p-1)}\, l_{1,p} u(x,y) = 0, \\ \frac{1}{2(p-1)}\, \Delta_2 v(x,y) = 0\,\forall (x,y) \in D, \end{cases} \tag{11}$$

where $l_{1,p} := (2p-1)\, \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Taking into account that the operator (8) can be factorisated in the form:

$$l_{1,p} = l_{1,p} \circ \Delta_2 = \Delta_2 \circ l_{1,p}$$

($l_1 \circ l_2$ is a symbol of composition of operators l_1 and l_2), we see that if a pair (u, v) is a solution of (11) then $w := u$ or $w := v$ is a solution of the equation (8). So we proved the following theorem.

Theorem 1. *Let $p > 1$, an orthotropy is defined by* (2), (9). *Then every displacement-function from a pair of solution of the eqilibrium system* (11) *satisfies the equation* (8).

Now consider another cases of orthotropy for which an equilibrium equation splits onto two equations containing except of operators of the type $l_{1,p}$ an extra term-operator $\frac{\partial^2}{\partial x \partial y}$ acted to another unknown function and has a non-zero coefficient.

Let p be an arbitrary fixed number: $0 < p < 1$.

Take into consideration the plane orthotropy:

$$a_{11} = a_{22} = 1,\ a_{16} = a_{26} = 0,\ a_{66} = 2(p - a_{12}),\ -1 < a_{12} < p. \tag{12}$$

An a_{12} belongs to such measures due to the positiveness of the matrix (3). Therefore, we have:

$$A_{11} = A_{22} = \frac{1}{1 - a_{12}^2},\ A_{21} = A_{12} = -\frac{a_{12}}{1 - a_{12}^2},\ A_{66} = \frac{1}{2(p - a_{12})}.$$

The equilibrium system (7) gets a form:

$$\begin{cases} \frac{1}{1-a_{12}^2} \frac{\partial^2}{\partial x^2} u(x,y) + \frac{1}{2(p-a_{12})} \frac{\partial^2}{\partial y^2} u(x,y) + \\ + \left(-\frac{a_{12}}{1-a_{12}^2} + \frac{1}{2(p-a_{12})} \right) \frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ \frac{1}{2(p-a_{12})} \frac{\partial^2}{\partial x^2} u(x,y) + \frac{1}{1-a_{12}^2} \frac{\partial^2}{\partial y^2} v(x,y) + \\ + \left(-\frac{a_{12}}{1-a_{12}^2} + \frac{1}{2(p-a_{12})} \right) \frac{\partial^2 u(x,y)}{\partial x \partial y} = 0, \end{cases} \tag{13}$$

where all $(x, y) \in D$.

Consider the following (“stress”) equation:

$$l_{2,p} w(x,y) \equiv$$

$$\equiv \left(\frac{\partial^4}{\partial x^4} + 2p \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \right) w(x,y) = 0\,\forall (x,y) \in D. \tag{14}$$

The system (13) is equivalent to the following system:

$$\begin{cases} B_{11}\frac{\partial^2 u(x,y)}{\partial x^2} + B_{12}\frac{\partial^2 u(x,y)}{\partial y^2} + \frac{\partial^2 v(x,y)}{\partial x \partial y} = 0, \\ B_{21}\frac{\partial^2 v(x,y)}{\partial x^2} + B_{22}\frac{\partial^2 v(x,y)}{\partial y^2} + \frac{\partial^2 u(x,y)}{\partial x \partial y} = 0 \, \forall (x,y) \in D. \end{cases} \tag{15}$$

where

$$B_{11} = B_{22} := \frac{2\,(p - a_{12})}{(a_{12} - p)^2 + 1 - p^2},$$

$$B_{12} = B_{21} := \frac{1 - (a_{12})^2}{(a_{12} - p)^2 + 1 - p^2}.$$

Theorem 2. *Let* $0 < p < 1$*, an orthotropy is defined by* (2), (12)*. Then every displacement-function from a pair of solution of the eqilibrium system* (15) *satisfies the equation* (14).

Proof. Acting by the differential operator $\frac{\partial^2}{\partial x \partial y}$ on the second equation of (15) and substituting to the obtained equation an expression of $\frac{\partial^2 v}{\partial x \partial y}$, we arrive at the equation:

$$\frac{\partial^4 u(x,u)}{\partial x^4} + C_2 \frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + C_3 \frac{\partial^4 u(x,y)}{\partial y^4} \, \forall (x,y) \in D, \tag{16}$$

where

$$C_3 = \frac{B_{22}B_{12}}{B_{11}B_{21}} \equiv 1, \; C_2 := \frac{B_{11}B_{22} + B_{12}B_{21} - 1}{B_{11}B_{21}}.$$

So, to prove Theorem we need to check the equality $C_2 = 2p$. In terms of p and a_{12} the relation $C_2 = 2p$ can be rewritten in the form:

$$\alpha^2 + \beta^2 + 2p\alpha\beta = (\alpha + a_{12}\beta)^2\,,$$

where $\alpha := 1 - a_{12}^2$, $\beta := 2(a_{12} - p)$. By doing simple algebraic transformation, the last one is equivalent to the relation

$$1 - a_{12}^2 = 2(a_{12} - p)\frac{\alpha}{\beta},$$

which with use of the definitions of α and β is an identity. So, we proved that if (u, v) is a solution of (15) then v satisfies the equation (14).

A similar statement for v can be proved analogously. The theorem is proved. □

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O PEWNYCH PRZYPADKACH PŁASKIEJ ORTHOTROPII

S t r e s z c z e n i e

Rozpatrywane są pewne przypadki płaskiej orthotropii przy założeniu braku oddziaływania sił ciała. Wówczas każda funkcja z pary rozwiązań układu równowagi równań ze względu na przemieszczenia spełnia równanie eliptyczne czwartego rzędu typu:

$$\left(\alpha_1 \frac{\partial^4}{\partial x^4} + \alpha_2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) w(x,y) = 0,$$

z pewnymi rzeczywistymi stałymi $\alpha_k \neq 0$, $k = 1, 2$.

Słowa kluczowe: uogólnione prawo Hooke'a, orthotropia płaska, układ równowagi