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Dedicated to the memory of
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WEIGHTED BERGMAN SPACES AND THE BERGMAN PROJECTION

Summary

It is well known that if $-1 < q, \beta < \infty$ and $1 \leq p < \infty$ then the Bergman projection P_β is a bounded operator from $L^p(\mathbb{D}, dA_q)$ onto the Bergman space $\mathcal{A}^p_q$ if and only if $q + 1 < (\beta + 1)p$. In this paper we study the Bergman operator P_β from $L^p(\mathbb{D}, dA_q)$ in the weighted Bergman space ${}_s\mathcal{A}^p_q$ and it is proved that P_β is a bounded operator for certain values of β, p, q and s, that in particular satisfy $q + 1 \geq (\beta + 1)p$.

Keywords and phrases: Bloch space, Bergman projection, ${}_s\mathcal{A}^p_q$ weighted space

1. Introduction

Let $\varphi_z : \mathbb{C} \setminus \{\frac{1}{\overline{z}}\} \to \mathbb{C}$ be the Möbius transformation

$$\varphi_z(w) = \frac{z - w}{1 - \overline{z}w},$$

with pole at $w = \frac{1}{\overline{z}}$, which verifies $\varphi_z^{-1} = \varphi_z$ and

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{z}w|^2} = (1 - |w|^2)|\varphi_z'(w)| \ . \tag{1.1}$$

Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk and denote by $\mathcal{H}$ the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$. Let $-1 < q < \infty$, $0 \leq p < \infty$. We recall that f belongs to the Bergman space $\mathcal{A}^p_q$ if $f \in \mathcal{H} \cap L^p(\mathbb{D}, dA_q)$, where $dA_q(w) = (q + 1)(1 - |w|^2)^q \, dA$, see [4]. If f is in

$L^p(\mathbb{D}, dA_q)$, we write

$$\|f\|_{p,q} = \left(\int_{\mathbb{D}} |f(z)|^p dA_q(z)\right)^{1/p} .$$

When $1 \leq p < \infty$, the space $L^p(\mathbb{D}, dA_q)$ is a Banach space with the above norm; when $0 < p < 1$, the space $L^p(\mathbb{D}, dA_q)$ is a complete metric space with the metric defined by

$$d(f,g) = \|f-g\|^p_{p,q} .$$

Let $0 < s < \infty$ be fixed and we add the weight $(1-|\varphi_z(w)|^2)^s$ in the integral definition of the Bergman space, so we have for each $f \in \mathcal{A}^p_q$

$$\int_{\mathbb{D}} |f(w)|^p (1-|\varphi_z(w)|^2)^s \, dA_q(w) \leq \int_{\mathbb{D}} |f(w)|^p \, dA_q(w) < \infty , \tag{1.2}$$

that is, the Bergman space $\mathcal{A}^p_q$ is a subspace of each member of the two parameter family of spaces $L^p(\mathbb{D}, d\mu_q)$, with $d\mu_q(w) = d\mu_q(s,z)(w) = (1-|\varphi_z(w)|^2)^s \, dA_q(w)$, $0 < s < \infty$ and $z \in \mathbb{D}$. In particular

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f(w)|^p (1-|\varphi_z(w)|^2)^s \, dA_q(w) \leq \int_{\mathbb{D}} |f(w)|^p \, dA_q(w) ,$$

for each $0 < s < \infty$. The previous discussion motivates the following definition.

For $0 < p < \infty$, $-1 < q < \infty$, $0 \leq s < \infty$ and $f \in \mathcal{H}$ define

$$l_{p,q,s}(f)(z) := \int_{\mathbb{D}} |f(w)|^p (1-|\varphi_z(w)|^2)^s \; dA_q(w) . \tag{1.3}$$

The q,s-weighted p-Bergman space ${}_s\mathcal{A}^p_q$ is defined by

$${}_s\mathcal{A}^p_q := \{f \in \mathcal{H} : \sup_{z\in\mathbb{D}} l_{p,q,s}(f)(z) < \infty\}$$

and for $0 < s < \infty$ its associated little space is

$${}_{s,0}\mathcal{A}^p_q := \{f \in \mathcal{H} : \lim_{|z|\to 1^-} l_{p,q,s}(f)(z) = 0\} .$$

We observe that ${}_0\mathcal{A}^p_q = \mathcal{A}^p_q$.

With the previous definitions, from (1.2) we get

$$\mathcal{A}^p_q \subset {}_s\mathcal{A}^p_q \subset \bigcap_{z\in\mathbb{D}} L^p(\mathbb{D}, d\mu_q(s,z)) . \tag{1.4}$$

Thus each Bergman space $\mathcal{A}^p_q$ can be included in each space ${}_s\mathcal{A}^p_q$ in a natural way. If $f \in {}_s\mathcal{A}^p_q$ we write

$$\| f \|_{\varphi} = \sup_{z\in\mathbb{D}} (l_{p,q,s}(f)(z))^{\frac{1}{p}} .$$

Let $0 < \alpha < \infty$. We say that $f \in \mathcal{H}$ belongs to the α-growth space (or α-type Bloch space), denoted by $\mathcal{A}^{-\alpha}$ (see [4]), if

$$\| f \|_{-\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha} |f(z)| < \infty$$

and belongs to the little α-growth space, denoted by $\mathcal{A}^{-\alpha,0}$, if

$$\| f \|_{-\alpha} = \lim_{|z|\to 1^-} (1-|z|^2)^{\alpha} |f(z)| = 0 \ .$$

It is clear that $\mathcal{A}^{-\alpha,0} \subset \mathcal{A}^{-\alpha}$; moreover with the definitions of $\| f \|_{\varphi}$ and $\| f \|_{-\alpha}$, each one of the previous spaces are complete spaces, see [3] and [4]. In fact, for $1 \le p < \infty$ they are Banach spaces.

Let $-1 < q < \infty$. For each $f \in L^1(\mathbb{D}, dA_q)$, the Bergman projection of f is defined as

$$P_q f(z) = \int_{\mathbb{D}} \frac{f(w)\, dA_q(w)}{(1-z\overline{w})^{2+q}} \ .$$

In this article we study first several properties of the Banach spaces ${}_s\mathcal{A}^p_q$ and the Bergman projection in the growth and ${}_s\mathcal{A}^p_q$ spaces.

Now, from the well known result (see [4]):

Theorem 1.1. *Suppose $-1 < q,\ \beta < \infty$ and $1 \le p < \infty$. Then P_q is a bounded projection from $L^p(\mathbb{D}, dA_q)$ onto $\mathcal{A}^p_q$ if and only if $q+1 < (\beta+1)p$,*

we see that is worthy of study the Bergman projection in the spaces ${}_s\mathcal{A}^p_q$ when $q+1 \ge (\beta+1)p$ for certain values of $p,\ q,\ s$ and β, see Theorems 4.5, 4.6, 4.7 and 4.8, where in fact, we get some extensions of Theorem 1.1.

2. Some properties of the Bergman spaces ${}_s\mathcal{A}^p_q$.

In this section we give some properties of the weighted Bergman spaces ${}_s\mathcal{A}^p_q$ an we prove that the integral operator defined by the formula of the Bergman projection is a bounded operator in the growth spaces $\mathcal{A}^{-\alpha}$.

We will use the following results.

Theorem 2.1 ([4])**.** *Let $t > -1$, $c \in \mathbb{R}$. Define $I_{t,c} : \mathbb{D} \to \mathbb{R}$ by*

$$I_{t,c}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\overline{z}w|^{2+t+c}}\, dA(w)$$

and $J_c : \mathbb{D} \to \mathbb{R}$ by

$$J_c(z) = \int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^{1+c}} \ .$$

Then

$$I_{t,c}(z) \approx J_c(z) \approx h_c(z) = \begin{cases} 1 & \text{if } c < 0 \\ \ln \dfrac{1}{1-|z|^2} & \text{if } c = 0 \\ \dfrac{1}{(1-|z|^2)^c} & \text{if } c > 0 \ . \end{cases}$$

as $|z| \to 1^-$.

Let $0 < R < 1$. The pseudohyperbolic disk is defined by

$$D(z,R) := \varphi_z(\mathbb{D}_R) = \{\ w \in \mathbb{D}\ :\ |\varphi_z(w)| < R\ \}\ .$$

In fact $D(z,R)$ is an Euclidean disk with center and radius given by

$$c = \frac{1-R^2}{1-R^2|z|^2}z, \qquad r = \frac{1-|z|^2}{1-R^2|z|^2}R \tag{2.5}$$

and we denote by $|D(z,R)|$ its area.

Proposition 2.1. *Let* $0 < r < 1$ *and* $0 < R < 1$*. Then there exist* $\rho > 0$ *such that if* $\rho < |z| < 1$*, we get*

$$D(z,R) \subset \mathbb{A}_r := \mathbb{D} \setminus \mathbb{D}_r\ .$$

The following results were proved in Lemma 2.2, Corollary 2.5 and Theorems 2.4 and 3.4 of [3]. In particular Theorem 2.3 improves (1.4).

Lemma 2.1. *Let* $-2 < q < \infty$ *and* $0 < s < \infty$*. Then*

$$\lim_{|z|\to 1^-} \int_{\mathbb{D}} (1-|\varphi_z(w)|^2)^s\, dA_q(w) = 0\ .$$

Corollary 2.1. *Let* $0 < p < \infty$, $-2 < q < \infty$*. Then* $\mathcal{A}^p_q \subset \mathcal{A}^{-\frac{q+2}{p},0}$.

Theorem 2.2. *Let* $0 < p < \infty$, $-2 < q < \infty$ *and* $1 < s < \infty$*. Then* ${}_s\mathcal{A}^p_q = \mathcal{A}^{-\frac{q+2}{p}}$.

Theorem 2.3. *Let* $0 < p < \infty$, $-1 < q < \infty$*. Then*

$$\mathcal{A}^p_q \subset \bigcap_{0<s<\infty} {}_{s,0}\mathcal{A}^p_q \subset \bigcap_{0<s<\infty} {}_s\mathcal{A}^p_q\ .$$

Proof. We prove the first inclusion. Let $f \in \mathcal{A}^p_q$, $1 \le s < \infty$ and $\varepsilon > 0$. By Corollary 2.1, there exists $0 < R < 1$ such that

$$(1-|w|^2)^{q+2}|f(w)|^p < \varepsilon \qquad \text{for all} \qquad w \in \mathbb{A}_R \tag{2.6}$$

and

$$\int_{\mathbb{A}_R} |f(w)|^p\, dA_q(w) < \varepsilon\ . \tag{2.7}$$

by absolute continuity of the integral. We split the integral

$$\begin{aligned} &h_{p,q,s}(f)(z) \\ &\quad = \int_{\mathbb{D}_R} |f(w)|^p (1-|\varphi_z(w)|^2)^s\, dA_q(w) + \int_{\mathbb{A}_R} |f(w)|^p (1-|\varphi_z(w)|^2)^s\, dA_q(w)\ . \end{aligned}$$

By Lemma 2.1 the first integral goes to 0 when $|z| \to 1^-$. We split again the second integral: By Proposition 2.1 we can choose R' fix, such that $\sqrt{1-e^{-\frac{1}{\pi}}} < R < R' <$

$|z| < 1$ with $D(z, R') \subset \mathbb{A}_R$, and

$$\begin{aligned}
&\int_{\mathbb{A}_R} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) \\
&= \int\limits_{\mathbb{A}_R \setminus D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) + \int\limits_{D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) \ .
\end{aligned}$$

Now by (2.7) we have

$$\int\limits_{\mathbb{A}_R \setminus D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) \leq \int\limits_{\mathbb{A}_R \setminus D(z,R')} |f(w)|^p \, dA_q(w) < \varepsilon.$$

Otherwise, we have by Theorem 2.1

$$\begin{aligned}
\int\limits_{D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) &\leq \int\limits_{D(z,R')} \frac{\varepsilon}{(1 - |w|^2)^{q+2}} (1 - |\varphi_z(w)|^2)^s \, dA_q(w) \\
&\leq \varepsilon (1 - |z|^2)^s \int_{\mathbb{D}} \frac{(1 - |w|^2)^{s-2}}{|1 - z\overline{w}|^{2s}} \, dA(w) \\
&< \varepsilon
\end{aligned}$$

since $1 < s < \infty$.

For $s = 1$, by (2.6) and the change of variable $w = \varphi_z(\zeta)$ we have

$$\begin{aligned}
\int\limits_{D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) &\leq \int\limits_{D(z,R')} \frac{\varepsilon}{(1 - |w|^2)^2} (1 - |\varphi_z(w)|^2) \, dA(w) \\
&= \varepsilon \int_{\mathbb{D}_{R'}} \frac{(1 - |\zeta|^2)}{(1 - |\varphi_z(\zeta)|^2)^2} \frac{(1 - |z|^2)^2}{|1 - z\overline{\zeta}|^4} \, dA(\zeta) \\
&= \varepsilon \int_{\mathbb{D}_{R'}} \frac{1}{1 - |\zeta|^2} \, dA(\zeta) \\
&< -\varepsilon \, \pi \ln(1 - R'^2) \\
&< \varepsilon \ .
\end{aligned}$$

Thus

$$\int\limits_{D(z,R')} |f(w)|^p (1 - |\varphi_z(w)|^2)^s \, dA_q(w) < \varepsilon$$

for all $1 \leq s < \infty$.

On the other hand, let $0 < s, s' < 1$ with $s + s' = 1$. By Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{D}} |f(w)|^p \, (1-|\varphi_z(w)|^2)^s \, dA_q(w) \\
&\quad = \int_{\mathbb{D}} |f(w)|^{ps} \, (1-|\varphi_z(w)|^2)^s \, |f(w)|^{p-ps} \, dA_q(w) \\
&\quad \le \left[\int_{\mathbb{D}} \left(|f(w)|^{ps} \, (1-|\varphi_z(w)|^2)^s\right)^{\frac{1}{s}} \, dA_q(w)\right]^s \left[\int_{\mathbb{D}} \left(|f(w)|^{p-ps}\right)^{\frac{1}{1-s}} \, dA_q(w)\right]^{1-s} \\
&\quad = \left[\int_{\mathbb{D}} |f(w)|^p \, (1-|\varphi_z(w)|^2) \, dA_q(w)\right]^s \left[\int_{\mathbb{D}} |f(w)|^p \, dA_q(w)\right]^{1-s} ,
\end{aligned}$$

and the fact that $\mathcal{A}^p_q \subset {}_{1,0}\mathcal{A}^p_q$ we get

$$\lim_{|z|\to 1^-} \int_{\mathbb{D}} |f(w)|^p \, (1-|\varphi_z(w)|^2)^s \, dA_q(w) = 0 \, .$$

Thus $f \in {}_{s,0}\mathcal{A}^p_q$ for all $s > 0$, and the proof is complete. □

The following lemma is used to give another proof (see Theorem 4.2 in [3]) of a different characterization of the q, s-weighted p-Bergman spaces. This characterization is related with classic theory of $\mathcal{Q}_p$ spaces started by R. Aulaskari and P. Lappan [1] and developed by many others [2], [8], [7] etc.

We need the following notation. Let $p, \; q, \ldots \in \mathbb{R}$ fixed. We say that two quantities $A(p, q, \ldots)$ and $B(p, q, \ldots)$ are comparable if there exists a constant $C > 0$ possibly depending on $p, \; q, \ldots$ such that

$$\frac{A}{C} \le B \le AC$$

and write $A \approx B$. In analogous form we define $B \preceq A$ if $B \le AC$.

Lemma 2.2 ([7]). *Let $q(r)$ and $p(r)$ be two integrable and nonnegative functions on $[0, 1)$, $p(r) > 0$. If there exists τ' with $0 < \tau' < 1$ fixed and C a positive constant such that $q(r) \le Cp(r)$ for all $r \in [\tau', 1)$, then for all τ with $\tau' < \tau \le 1$ and all $h(r)$ a nondecreasing and nonnegative function on $[0, 1)$, there exists a constant $K = K(\tau) \ge C$, independent of τ' and h, such that*

$$\int_0^\tau h(r)q(r)\,dr \le K \int_0^\tau h(r)p(r)\,dr \; ,$$

that is

$$\int_0^\tau h(r)q(r)\,dr \preceq \int_0^\tau h(r)p(r)\,dr \; .$$

Theorem 2.4 ([3]). *Let $0 < p < \infty$, $-1 < q < \infty$ and $0 \le s < \infty$. Then $f \in {}_s\mathcal{A}^p_q$ if and only if*

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f(w)|^p g^s(w, z)\,dA_q(w) < \infty \; ,$$

where $g(w,z)$ is the Green's function of $\mathbb{D}$, given by

$$g(w,z) = \ln \frac{|1-\overline{z}w|}{|z-w|} = \ln \frac{1}{|\varphi_z(w)|} \ .$$

Proof. We need to prove that

$$\int_{\mathbb{D}} |f(w)|^p \ln^s \frac{1}{|\varphi_z(w)|} \ dA_q(w) \approx \int_{\mathbb{D}} |f(w)|^p (1-|\varphi_z(w)|^2)^s \ dA_q(w) \qquad (2.8)$$

and the constant of comparability does not depend on z. In order to do this, we use the change of variable $w = \varphi_z(\lambda)$ and so, we have to prove that

$$\int_{\mathbb{D}} |f(\varphi_z(\lambda))|^p (1-|\varphi_z(\lambda)|^2)^q \ \frac{(1-|z|^2)^2}{|1-\overline{z}\lambda|^4} \ \ln^s \frac{1}{|\lambda|} \ dA(\lambda)$$
$$\approx \int_{\mathbb{D}} |f(\varphi_z(\lambda))|^p (1-|\varphi_z(\lambda)|^2)^q \ \frac{(1-|z|^2)^2}{|1-\overline{z}\lambda|^4} \ (1-|\lambda|^2)^s \ dA(\lambda) \ .$$

By (1.1), we rewrite the previous expression as

$$\int_{\mathbb{D}} |f(\varphi_z(\lambda))|^p (1-|\lambda|^2)^q \ \frac{(1-|z|^2)^{q+2}}{|1-\overline{z}\lambda|^{4+2q}} \ \ln^s \frac{1}{|\lambda|} \ dA(\lambda)$$
$$\approx \int_{\mathbb{D}} |f(\varphi_z(\lambda))|^p (1-|\lambda|^2)^{q+s} \ \frac{(1-|z|^2)^{q+2}}{|1-\overline{z}\lambda|^{4+2q}} \ dA(\lambda) \ . \qquad (2.9)$$

Since $g(\lambda) = \dfrac{f(\varphi_z(\lambda))}{(1-\overline{z}\lambda)^{\frac{4+2q}{p}}}$ is holomorphic in $\mathbb{D}$, then the function $H:\mathbb{D}\to\mathbb{R}$ given by

$$H(\lambda) = |f(\varphi_z(\lambda))|^p \ \frac{(1-|z|^2)^{q+2}}{|1-\overline{z}\lambda|^{4+2q}}$$

is subharmonic. Using this notation and polar coordinates in (2.9) we have to prove that

$$\int_0^1 (1-r^2)^q \ r \ln^s \frac{1}{r} \ \int_0^{2\pi} H(re^{i\theta}) \ d\theta \ dr \approx \int_0^1 (1-r^2)^{q+s} r \ \int_0^{2\pi} H(re^{i\theta}) \ d\theta \ dr.$$

Since $H(re^{i\theta})$ is a nonnegative subharmonic function, we have that

$$h(r) = \int_0^{2\pi} H(re^{i\theta}) \ d\theta$$

is a nondecreasing and nonnegative function. Moreover $q(r) = (1-r^2)^q \ r \ln^s \frac{1}{r}$ and $p(r) = (1-r^2)^{q+s} \ r$ are continuous functions on $[0,1)$ (we define $q(0) = \lim_{r\to 0^+} q(r) = 0$). Let $\tau' = 0.450754...$ be a root of the equation $1-x^2 = -\ln x$. Thus $q(r) \le p(r)$ if $r \in [\tau', 1)$, and since $1-x^2 \le -2\ln x$ for all $x \in [0,1)$ then $p(r) \le 2q(r)$ if $r \in [\tau', 1)$. So the conditions of Lemma 2.2 are satisfied, and we verify (2.8). □

We recall that each ${}_s\mathcal{A}^p_q$ is a complete space by itself and ${}_s\mathcal{A}^p_q \subset {}_{s'}\mathcal{A}^p_q$ if $0 < s < s' < \infty$. However, we will prove that ${}_s\mathcal{A}^p_q$ is not a closed subspace of ${}_{s'}\mathcal{A}^p_q$.

For $n \in \mathbb{N}$, define

$$I_n = \{\ k \in \mathbb{N}\ :\ 2^n \le k < 2^{n+1}\}\ .$$

The following lemma was proved by Mateljevic and Pavlovic.

Lemma 2.3. *Let $0 < \alpha < \infty$ and $0 < p < \infty$. Let $f(x) = \sum_{n=1}^{\infty} a_n x^n$, with $0 \le x < 1$, $0 \le\ a_n < \infty$ for each $n \in \mathbb{N}$. Then*

$$\sum_{n=0}^{\infty} \frac{t_n^p}{2^{n\alpha}} \approx \int_0^1 (1-x)^{\alpha-1} f(x)^p\, dx,$$

where $t_n = \sum_{k\in I_n} a_k$.

Lemma 2.4. *Let $0 < p < \infty$, $-1 < q < \infty$, $0 < s < \infty$ and $f(w) = \sum_{k=0}^{\infty} a_k w^k$. Then there exists a constant $C = C(p,q,s)$ such that*

$$\int_{\mathbb{D}} \Big(\sum_{k=0}^{\infty} |a_k||w|^k\Big)^p \big(1-|\varphi_z(w)|^2\big)^s\ dA_q(w) \le C(p,q,s) \sum_{k=0}^{\infty} \frac{t_n^p}{2^{n(q+s+1)}}\ ,$$

where $t_n = \sum_{k\in I_n} |a_k|$.

Proof. By using polar coordinates, we have

$$\begin{aligned}
I(z) &= \int_{\mathbb{D}} \Big(\sum_{k=0}^{\infty} |a_k||w|^k\Big)^p \big(1-|\varphi_z(w)|^2\big)^s (1-|w|^2)^q\ dA(w)\\
&= \int_0^1 \int_0^{2\pi} \Big(\sum_{k=0}^{\infty} |a_k| r^k\Big)^p (1-r^2)^q\, r\, \frac{(1-|z|^2)^s(1-r^2)^s}{|1-zre^{-i\theta}|^{2s}}\ d\theta\ dr\\
&\le 2^s \int_0^1 \Big(\sum_{k=0}^{\infty} |a_k| r^k\Big)^p (1-r^2)^{q+s}\, r \int_0^{2\pi} \frac{1}{|1-(zr)e^{-i\theta}|^s}\ d\theta\ dr \qquad (2.10)\\
&\le\ C_1(s) \int_0^1 \Big(\sum_{k=0}^{\infty} |a_k| r^k\Big)^p (1-r^2)^{q+s}\ dr\ ,
\end{aligned}$$

where we get the last inequality by Theorem 2.1. By Lemma 2.3 there is a constant $C_2(p,q,s)$ such that

$$I(z) \le C_1(s)\cdot C_2(p,q,s) \sum_{n=0}^{\infty} \frac{t_n^p}{2^{n(q+s+1)}}\ .$$

□

The previous lemma is used to prove the following result.

Theorem 2.5. *Let $0 < p < \infty$, $-1 < q < \infty$ and $0 < t < s < 1$. Then the subspace ${}_t\mathcal{A}^p_q$ is not a closed subspace of ${}_s\mathcal{A}^p_q$.*

Proof. It is known that ${}_t\mathcal{A}^p_q \subset {}_s\mathcal{A}^p_q$, see [3]. Consider the Lacunary series and its partial sums

$$f(z) = \sum_{n=0}^{\infty} 2^{\frac{n(q+t+1)}{p}} z^{2^n} \quad \text{and} \quad f_n(z) = \sum_{k=0}^{n} 2^{\frac{k(q+t+1)}{p}} z^{2^k} \ ,$$

then $\{f_n\} \subset {}_t\mathcal{A}^p_q \cap {}_s\mathcal{A}^p_q$ and converges to the function f in the norm $\|\cdot\|_\varphi$. Indeed, by Lemma 2.4, for $0 < s < 1$ there is a constant $C(p, q, s)$ such that

$$\begin{aligned} I(z) &= \int_{\mathbb{D}} |f(w) - f_n(w)|^p (1 - |\varphi_z(w)|^2)^s \ dA_q(w) \\ &\le \int_{\mathbb{D}} \Big(\sum_{k=n+1}^{\infty} 2^{\frac{k(q+t+1)}{p}} |w|^{2^k} \Big)^p (1 - |\varphi_z(w)|^2)^s dA_q(w) \\ &\le C(p, q, s) \sum_{k=n+1}^{\infty} \frac{2^{k(q+t+1)}}{2^{k(q+s+1)}} \\ &= C(p, q, s) \sum_{k=n+1}^{\infty} \frac{1}{2^{k(s-t)}} \ . \end{aligned}$$

Since $t < s$, then $\sum_{k=n+1}^{\infty} \frac{1}{2^{k(s-t)}}$ is a convergent series and thus f_n converges to f in the mentioned norm. In particular $\{f_n\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_\varphi$. By Theorem 5.5 in [3], $f \notin {}_t\mathcal{A}^p_q$ since

$$\sum_{k=0}^{\infty} \frac{2^{n(q+t+1)}}{2^{n(q+t+1)}} = \infty \ .$$

□

We present now two immediate results about the integral operator defined by the formula of the Bergman projection.

Theorem 2.6. *Let* $1 < p < \infty$, $-1 < q$, $\beta < \infty$ *and* $0 \le s < \infty$. *Then* $P_\beta : L^p(\mathbb{D}, dA_q) \to {}_s\mathcal{A}^p_q$ *is a bounded operator if* $q + 1 < (\beta + 1)p$.

Proof. By (2.8) there exists $C > 0$ such that

$$\begin{aligned} \sup_{z\in\mathbb{D}} \Big\{ \int_{\mathbb{D}} |P_\beta f(w)|^p (1 - |\varphi_z(w)|^2)^s \ dA_q(w) \Big\}^{1/p} &\le \Big\{ \int_{\mathbb{D}} |P_\beta f(w)|^p \ dA_q(w) \Big\}^{1/p} \\ &= \|P_\beta f\|_{p,q} \\ &\preceq \|f\|_{p,q} \ . \end{aligned}$$

We get the last inequality by Theorem 1.10 in [4]. □

The formula of the Bergman projection gives a bounded operator into the growth spaces.

Lemma 2.5. *Let* $-1 < q,\ \beta < \infty$ *and* $1 < p < \infty$. *If* $\alpha > \frac{q+2}{p}$, *then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \to \mathcal{A}^{-\alpha,0}$$

is a bounded operator.

If $\alpha = \frac{q+2}{p}$, *then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \to \mathcal{A}^{-\frac{q+2}{p}}$$

is a bounded operator. (Recall that if $1 < s < \infty$ *then* $\mathcal{A}^{-\frac{q+2}{p}} = {}_s\mathcal{A}^p_q$*).*

Proof. By the Hölder inequality we get the estimation

$$\begin{aligned}
(1-|z|^2)^\alpha\,|P_\beta f(z)| &= (1-|z|^2)^\alpha \left| \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\beta}}\ dA_\beta(w) \right| \\
&= (1-|z|^2)^\alpha \left| \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta-q}}{(1-z\overline{w})^{2+\beta}} f(w)\ dA_q(w) \right| \\
&\le (1-|z|^2)^\alpha \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta-q}}{|1-z\overline{w}|^{2+\beta}} |f(w)|\ dA_q(w) \\
&\le (1-|z|^2)^\alpha \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{(\beta-q)p^*}}{|1-z\overline{w}|^{(2+\beta)p^*}}\ dA_q(w) \right)^{1/p^*} \\
&\quad \cdot \left(\int_{\mathbb{D}} |f(w)|^p\ dA_q(w) \right)^{1/p} \\
&= (1-|z|^2)^\alpha \|f\|_{p,q} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{q+(\beta-q)p^*}}{|1-z\overline{w}|^{(2+\beta)p^*}}\ dA(w) \right)^{1/p^*}
\end{aligned}$$

As $p^* = \frac{p}{p-1}$, by Theorem 2.1 there is a constant $K > 0$ such that

$$\begin{aligned}
(1-|z|^2)^\alpha \big|P_\beta f(z)\big| &\le K\ (1-|z|^2)^\alpha \|f\|_{p,q} \left(\frac{1}{(1-|z|^2)^{\frac{q+2}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\le K\ (1-|z|^2)^{\alpha - \frac{q+2}{p}} \|f\|_{p,q}
\end{aligned}$$

Thus $P_\beta f \in \mathcal{A}^{-\alpha,0}$ if $\alpha > \frac{q+2}{p}$ and if $\alpha = \frac{q+2}{p}$ then $P_\beta f \in \mathcal{A}^{-\frac{q+2}{p}}$.

□

3. An integral estimation

In order to study the integral operator defined by the formula of the Bergman projection into the spaces ${}_s\mathcal{A}^p_q$, we need to estimate an integral that is cited without proof in [5]. We need several preliminaries to give a proof following the ideas in [6]. As we will see the proof is far to be straightforward.

Definition 3.1. *For $z,\ \zeta \in \mathbb{C}$ let $d(\zeta, z) = |\overline{z}(z-\zeta)| + |\overline{\zeta}(\zeta - z)|$ be a non isotropic pseudo-distance.*

Proposition 3.1. *There exists a constant $C > 0$ such that*

$$d(\zeta, z) \leq C\,(d(\zeta, w) + d(w, z)) \tag{3.11}$$

for all $\zeta,\ z,\ w \in \mathbb{D}$, that is $d(\zeta, z) \preceq d(\zeta, w) + d(w, z)$.

Proof. Suppose that for each $n \in \mathbb{N}$ there are $z_n, w_n, \zeta_n \in \overline{\mathbb{D}}$ such that

$$d(z_n, \zeta_n) > n\,(d(z_n, w_n) + d(w_n, \zeta_n))\ .$$

By Bolzano-Weierstrass, we can assume that $z_n \to z$, $w_n \to w$ and $\zeta_n \to \zeta$. Since

$$d(z_n, \zeta_n) > n \max\{d(z_n, w_n),\ d(w_n, \zeta_n)\}$$

then $z = \zeta = w$. Now, without loss of generality, suppose that $|z| = R$. Then there exists $N > 3$ such that for $n \geq N$

$$\begin{aligned} 3R\,|\zeta_n - z_n| \geq d(z_n, \zeta_n) &\geq n\Big((|z_n| + |w_n|)|z_n - w_n| + (|w_n| + |\zeta_n|)|w_n - \zeta_n|\Big) \\ &\geq n\ R\Big(|z_n - w_n| + |w_n - \zeta_n|\Big)\ . \end{aligned}$$

Thus

$$3\Big(|z_n - w_n| + |w_n - \zeta_n|\Big) \geq 3|\zeta_n - z_n| \geq n\Big(|z_n - w_n| + |w_n - \zeta_n|\Big)$$

and we get a contradiction. □

Given $\zeta,\ z \in \mathbb{D}$ and $C > 0$ as in Proposition 3.1 we define

$$\Omega = \left\{\eta \in \mathbb{D} : d(\eta, z) \leq \frac{d(\zeta, z)}{2C}\right\}.$$

In particular we obtain the partition $\Omega \cup (\mathbb{D} \setminus \Omega)$ of the unit disk $\mathbb{D}$.

Lemma 3.1. *With the above definition of Ω, it holds*

$$|1 - \overline{\eta}z| \preceq |1 - \overline{\zeta}z| \preceq |1 - \overline{\eta}\zeta|, \quad \textit{for each } \eta \in \Omega.$$

Proof. First we observe that

$$|1 - \overline{\zeta}z| \approx 1 - |\zeta|^2 + d(\zeta, z) \approx 1 - |z|^2 + d(\zeta, z) \tag{3.12}$$

for every $\zeta, z \in \mathbb{D}$. Indeed, we have

$$\begin{aligned} |1 - \overline{\zeta}z| &= |1 - \overline{\zeta}\zeta + \overline{\zeta}\zeta - \overline{\zeta}z| \leq |1 - \overline{\zeta}\zeta| + |\overline{\zeta}(z - \zeta)| \\ &= |1 - |\zeta|^2| + |\zeta||\zeta - z| \leq 1 - |\zeta|^2 + (|\zeta| + |z|)|\zeta - z| \\ &= 1 - |\zeta|^2 + d(\zeta, z)\ . \end{aligned}$$

Otherwise, $1-|z|^2 \le 2(1-|z|) \le 2|1-\overline{\zeta}z|$. Moreover

$$\begin{aligned}|z-\zeta| &= |z - z\zeta\overline{z} + z\zeta\overline{z} - \zeta| = |z(1-\overline{z}\zeta) + \zeta(|z|^2-1)| \\ &\le |z||1-\overline{z}\zeta| + |\zeta|(1-|z|^2) \\ &\le 3|1-\overline{z}\zeta|\end{aligned}$$

and so we have proved (3.12).

Now, we will prove

$$|1-\overline{\eta}z| \preceq |1-\overline{\zeta}z| \preceq |1-\overline{\eta}\zeta|, \quad \text{for each } \eta \in \Omega\ .$$

Since $\eta \in \Omega$, by (3.12) we have

$$|1-\overline{\eta}z| \approx 1-|z|^2 + d(\eta,z) \preceq 1-|z|^2 + d(z,\zeta) \approx |1-\overline{\zeta}z|\ .$$

On the other hand, we observe that

$$d(z,\zeta) \le C(d(z,\eta)+d(\eta,\zeta)) \le C\left(\frac{d(z,\zeta)}{2C} + d(\eta,\zeta)\right)$$

and from here

$$d(z,\zeta) \le 2Cd(\eta,\zeta)\ .$$

Thus

$$|1-\overline{\zeta}z| \approx 1-|\zeta|^2 + d(z,\zeta) \preceq 1-|\zeta|^2 + d(\eta,\zeta) \approx |1-\overline{\eta}\zeta|$$

and we finished the proof. □

Lemma 3.2. *Assume that* $-1<t_2<\infty$, $0 \le t_1 < 2+t_2 < \infty$ *and* $-1 \le t_0 < t_2 < t_0+t_1 < \infty$. *Then*

$$\int_{\mathbb{D}} \frac{(1-|\eta|^2)^{t_2}}{|1-\overline{\eta}z|^{2+t_0}|1-\overline{\eta}\zeta|^{t_1}}\ dA(\eta) \preceq \frac{1}{|1-\overline{\zeta}z|^{t_0+t_1-t_2}}$$

Proof. Let $z,\ \zeta \in \mathbb{D}$ and $\eta \in \Omega$. By the definition 3.1 and Lemma 3.1 we have

$$|1-\overline{\zeta}z| + 1-|\eta|^2 \preceq |1-\overline{\eta}\zeta|,$$

since $|1-\overline{\zeta}z| \preceq |1-\overline{\eta}\zeta|$ for all $\eta\in\Omega$ and $1-|\eta|^2 \le 2|1-\overline{\eta}\zeta|$. Now $|1-\overline{\zeta}z| \preceq |1-\overline{\eta}z|$ for all $\eta \in \mathbb{D}\setminus\Omega$ and $1-|\eta|^2 \le 2|1-\overline{\eta}\zeta|$ then

$$\left(|1-\overline{\zeta}z| + |1-\overline{\eta}z|\right)^{2+t_0}\left(1-|\eta|^2\right)^{t_1} \preceq |1-\overline{\eta}z|^{2+t_0}|1-\overline{\eta}\zeta|^{t_1}\ .$$

Thus we split the integral to obtain the estimation

$$\begin{aligned}I(z,\zeta) &:= \int_{\mathbb{D}} \frac{(1-|\eta|^2)^{t_2}}{|1-\overline{\eta}z|^{2+t_0}|1-\overline{\eta}\zeta|^{t_1}}\ dA(\eta) \\ &\preceq \int_{\Omega} \frac{(1-|\eta|^2)^{t_2}}{|1-\overline{\eta}z|^{2+t_0}(|1-\overline{\zeta}z|+1-|\eta|^2)^{t_1}}\ dA(\eta) \\ &\quad + \int_{\mathbb{D}\setminus\Omega} \frac{(1-|\eta|^2)^{t_2-t_1}}{\left(|1-\overline{\zeta}z|+|1-\overline{\eta}z|\right)^{2+t_0}}\ dA(\eta)\ .\end{aligned}$$

We change to polar coordinates, so

$$I(z,\zeta) \preceq \int_0^1 \int_0^{2\pi} \frac{(1-r^2)^{t_2}\ r}{|1-rze^{-i\theta}|^{2+t_0}(|1-\overline{\zeta}z|+1-r^2)^{t_1}}\ d\theta\ dr$$
$$+ \int_0^1 \int_0^{2\pi} \frac{(1-r^2)^{t_2-t_1}\ r}{\big(|1-\overline{\zeta}z|+|1-rze^{-i\theta}|\big)^{2+t_0}}\ d\theta\ dr\ .$$

By Theorem 2.1

$$\int_0^{2\pi} \frac{d\theta}{|1-rze^{-i\theta}|^{2+t_0}} \approx \frac{1}{(1-r^2|z|^2)^{1+t_0}}$$

and since

$$(1+|1-\overline{\zeta}z|)\left|1-\frac{rz}{1+|1-\overline{\zeta}z|}e^{-i\theta}\right| = |1+|1-\overline{\zeta}z|-rze^{-i\theta}| \le |1-\overline{\zeta}z|+|1-rze^{i\theta}|$$

we have

$$\int_0^{2\pi} \frac{d\theta}{\big(|1-\overline{\zeta}z|+|1-rze^{-i\theta}|\big)^{2+t_0}}$$
$$\le \frac{1}{(1+|1-\overline{\zeta}z|)^{2+t_0}} \int_0^{2\pi} \frac{d\theta}{\left|1-\dfrac{rz}{1+|1-\overline{\zeta}z|}e^{-i\theta}\right|^{2+t_0}}$$
$$\preceq \frac{1}{(1+|1-\overline{\zeta}z|)^{2+t_0}} \frac{1}{\left(1-\dfrac{r^2|z|^2}{(1+|1-\overline{\zeta}z|)^2}\right)^{1+t_0}}$$
$$\preceq \frac{1}{\big((1+|1-\overline{\zeta}z|)^2-r^2|z|^2\big)^{1+t_0}}$$
$$\le \frac{1}{\big(1+|1-\overline{\zeta}z|-r^2|z|^2\big)^{1+t_0}}\ .$$

Therefore

$$I(z,\zeta) \preceq \int_0^1 \frac{(1-r^2)^{t_2}\ r}{(1-|z|^2r^2)^{1+t_0}(|1-\overline{\zeta}z|+1-r^2)^{t_1}}\ dr$$
$$+ \int_0^1 \frac{(1-r^2)^{t_2-t_1}\ r}{\big(|1-\overline{\zeta}z|+1-|z|^2r^2\big)^{1+t_0}}\ dr\ .$$

Moreover

$$1-|z|^2+1-r^2 < 1-|z|^2r^2+1-|z|^2r^2 < 2(1-|z|^2r^2)$$

and

$$|1-\overline{\zeta}z|+1-r^2 < |1-\overline{\zeta}z|+1-|z|^2r^2\ .$$

So

$$I(z,\zeta) \preceq \int_0^1 \frac{(1-r^2)^{t_2}\ r}{(1-|z|^2+1-r^2)^{1+t_0}(|1-\overline{\zeta}z|+1-r^2)^{t_1}}\ dr$$
$$+\int_0^1 \frac{(1-r^2)^{t_2-t_1}\ r}{\left(|1-\overline{\zeta}z|+1-r^2\right)^{1+t_0}}\ dr\ .$$

Taking the change of variable $u = 1 - r^2$ we have

$$I(z,\zeta) \preceq \int_0^1 \frac{u^{t_2}}{(1-|z|^2+u)^{1+t_0}(|1-\overline{\zeta}z|+u)^{t_1}}\ du + \int_0^1 \frac{u^{t_2-t_1}}{\left(|1-\overline{\zeta}z|+u\right)^{1+t_0}}\ du.$$

We now estimate the integral

$$H_1 := \int_0^1 \frac{u^{t_2}}{(1-|z|^2+u)^{1+t_0}(|1-\overline{\zeta}z|+u)^{t_1}}\ du\ .$$

We note that

$$H_1 = \int_0^{|1-\overline{\zeta}z|} \frac{u^{t_2}\ \ du}{(1-|z|^2+u)^{1+t_0}(|1-\overline{\zeta}z|+u)^{t_1}}$$
$$+\int_{|1-\overline{\zeta}z|}^1 \frac{u^{t_2}\ \ du}{(1-|z|^2+u)^{1+t_0}(|1-\overline{\zeta}z|+u)^{t_1}}$$

Since $u < u+1-|z|^2$ and $|1-\overline{\zeta}z| < |1-\overline{\zeta}z|+u$ for the first integral and for the second one $u < |1-\overline{\zeta}z|+u$ and $|1-\overline{\zeta}z|+u < 2(u+1-|z|^2)$ we have

$$H_1 \preceq \frac{1}{|1-\overline{\zeta}z|^{t_1}} \int_0^{|1-\overline{\zeta}z|} u^{t_2-1-t_0}\ du + \int_{|1-\overline{\zeta}z|}^1 (u+|1-\overline{\zeta}z|)^{t_2-t_0-t_1-1}\ du$$
$$\preceq \frac{1}{|1-\overline{\zeta}z|^{t_0+t_1-t_2}}\ .$$

We now estimate the integral

$$H_2 = \int_0^{|1-\overline{\zeta}z|} \frac{u^{t_2-t_1}}{\left(|1-\overline{\zeta}z|+u\right)^{1+t_0}}\ du + \int_{|1-\overline{\zeta}z|}^1 \frac{u^{t_2-t_1}}{\left(|1-\overline{\zeta}z|+u\right)^{1+t_0}}\ du\ .$$

As $|1-\overline{\zeta}z| < |1-\overline{\zeta}z|+u$ for the first integral and $u < |1-\overline{\zeta}z|+u$ for the second integral we obtain

$$H_2 \preceq \frac{1}{|1-\overline{\zeta}z|^{1+t_0}} \int\limits_0^{|1-\overline{\zeta}z|} u^{t_2-t_1}\,du + \int\limits_{|1-\overline{\zeta}z|}^{1} (u+|1-\overline{\zeta}z|)^{t_2-t_0-t_1-1}\,du$$

$$\preceq \frac{1}{|1-\overline{\zeta}z|^{t_0+t_1-t_2}}\ .$$

Thus

$$I(z,\zeta) \le H_1 + H_2 \le \frac{C}{|1-\overline{\zeta}z|^{t_0+t_1-t_2}}\ .$$

□

4. The Bergman Projection in ${}_s\mathcal{A}^p_q$

We recall the well known result, see [4].

Theorem 4.1. *Suppose* $-1 < q,\ \beta < \infty$ *and* $1 \le p < \infty$*. Then* P_β *is a bounded projection from* $L^p(\mathbb{D}, dA_q)$ *onto* $\mathcal{A}^p_q$ *if and only if* $q+1 < (\beta+1)p$*.*

In this section we study the integral operator defined by the formula of the Bergman projection into the spaces ${}_s\mathcal{A}^p_q$ when $q+1 \ge (\beta+1)p$ for certain values of the parameters $p,\ q,\ s$ and β.

The case $p = 1$ is treated separately.

Theorem 4.2. *Suppose* $-1 < q < \infty$*,* $0 < s < 1$*,* $p = 1$ *and* $(\beta+1)p = \beta+1 = q+1$*. Then* $P_\beta = P_q$ *is a bounded operator from* $L^1(\mathbb{D}, dA_q)$ *in* ${}_s\mathcal{A}^1_q(\mathbb{D})$ *.*

Proof. As $p = 1$ we have $\beta = q$, thus

$$P_\beta f(z) = P_q f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+q}}\ dA_q(w)\ .$$

By Fubini's theorem we have:

$$\begin{aligned}
&l_{1,q,s}(P_q f)(a)\\
&\qquad = \int_{\mathbb{D}} |P_q f(z)|(1-|\varphi_a(z)|^2)^s\ dA_q(z)\\
&\qquad = \int_{\mathbb{D}} \Big| \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+q}}\ dA_q(w)\Big| (1-|\varphi_a(z)|^2)^s\ dA_q(z)\\
&\qquad \le \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^q}{|1-z\overline{w}|^{2+q}}\ |f(w)|\ dA(w)\ \frac{(1-|z|^2)^s(1-|a|^2)^s}{|1-z\overline{a}|^{2s}}\ dA_q(z)\\
&\qquad = \int_{\mathbb{D}} (1-|w|^2)^q\ (1-|a|^2)^s\ |f(w)| \int_{\mathbb{D}} \frac{(1-|z|^2)^{s+q}}{|1-z\overline{w}|^{2+q}|1-z\overline{a}|^{2s}}\ dA(z)\ dA(w)
\end{aligned}$$

Writing $t_0 = q$, $t_1 = 2s$ and $t_2 = s+q$ we get $-1 < t_3$, $0 < t_1 < 2 + t_3$ and $-1 < t_0 < t_3 < t_0 + t_1$. Then by Lemma 3.2 we have

$$\begin{aligned} l_{1,q,s}(P_q f)(a) &\preceq \int_{\mathbb{D}} (1-|w|^2)^q \ (1-|a|^2)^s \ |f(w)| \frac{1}{|1-a\overline{w}|^s} \ dA(w) \\ &\preceq \int_{\mathbb{D}} (1-|w|^2)^q \ |f(w)| \frac{(1-|a|)^s(1+|a|)^s}{(1-|a|)^s} \ dA(w) \\ &\preceq \int_{\mathbb{D}} \ |f(w)| \ dA_q(w) \\ &\preceq \|f\|_{1,q} \end{aligned}$$

and the proof follows from this estimation. □

Following the same idea of the previous proof we obtain

Theorem 4.3. *Suppose $-1 < q < \infty$, $0 < s < 1$, $p = 1$ and $(\beta+1)p = \beta+1 < q+1$. Then P_β is a bounded operator from $L^1(\mathbb{D}, dA_q)$ in ${}_sA^1_q$ if $q < \beta + s$.*

We analyze the case $1 < p < \infty$ and $(\beta+1)p \leq q+1$.

From Lemma 2.1 is immediate the following result.

Lemma 4.1. *Let $1 < p < \infty$, $-1 < q < \infty$ and $\beta \in \mathbb{R}$. Then*

$$I_{q,v}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^q}{|1-z\overline{w}|^{(2+\beta)\cdot \frac{p}{p-1}}} \ dA(w) \approx h_v(z) = \begin{cases} 1 & \text{if } v < 0 \\ \ln \dfrac{1}{1-|z|^2} & \text{if } v = 0 \\ \dfrac{1}{(1-|z|^2)^v} & \text{if } v > 0\ . \end{cases}$$

where

$$v = v(p,q,\beta) = \frac{(2+\beta)p - (2+q)(p-1)}{p-1}\ . \tag{4.13}$$

Let $-1 < \beta < \infty$, $1 < p < \infty$ and $0 < s < 1$. Then

$$\begin{aligned} I(a) &= \int_{\mathbb{D}} \Big| P_\beta f(z) \Big|^p (1-|\varphi_a(z)|^2)^s \ dA_q(z) \\ &= \int_{\mathbb{D}} \Big| \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{2+\beta}} \ dA_q(w) \Big|^p \frac{(1-|z|^2)^s(1-|a|^2)^s}{|1-a\overline{z}|^{2s}} \ dA_q(z)\ . \end{aligned}$$

We estimate the Bergman projection using the Hölder inequality. Thus

$$\begin{aligned} \left| \int_{\mathbb{D}} \frac{f(w)\ dA_q(w)}{(1-z\overline{w})^{2+\beta}} \right| &\leq \Big(\int_{\mathbb{D}} |f(w)|^p \ dA_q(w) \Big)^{1/p} \Big(\int_{\mathbb{D}} \frac{dA_q(w)}{|1-z\overline{w}|^{(2+\beta)p^*}} \Big)^{1/p^*} \\ &= \| \ f \ \|_{p,q} \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^q dA(w)}{|1-z\overline{w}|^{(2+\beta)p^*}} \ \right)^{1/p^*} \end{aligned}$$

where $\dfrac{1}{p}+\dfrac{1}{p^*}=1$. By Lemma 4.1 we have

$$I(a) \preceq \| \, f \, \|_{p,q}^p \int_{\mathbb{D}} h_v^{p/p^*}(z) \frac{(1-|z|^2)^s(1-|a|^2)^s}{|1-\overline{a}z|^{2s}} \; dA_q(z) \; . \tag{4.14}$$

We will estimate the last integral applying again Lemma 2.1 and Lemma 4.1, in particular each case originated by the sign of (4.13).

Theorem 4.4. *Let* $-1 < q < \infty$, $\dfrac{q+2}{q+1} < p < \infty$, $0 < s < 1$ *and* $-1 < \beta < \dfrac{q(p-1)-2}{p}$. *Then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \mapsto {}_s\mathcal{A}_q^p$$

is a bounded operator if $v = \dfrac{(2+\beta)p-(2+q)(p-1)}{p-1} \le 0$.

Proof. If $v \le 0$, by Lemmas 4.1 and 2.1 we get immediately

$$I(a) \preceq \| \, f \, \|_{p,q}^p$$

and the proof follows from this claim. ☐

We study now the integral (4.14) when $v > 0$, that is

$$\begin{aligned} \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{(2+\beta)p-(2+q)(p-1)}} \cdot \frac{(1-|z|^2)^{s+q}}{|1-z\overline{a}|^{2s}} \; dA(z) \\ = \int_{\mathbb{D}} \frac{(1-|z|^2)^{p(q-\beta)+s-2}}{|1-z\overline{a}|^{2s}} \; dA(z) \; . \end{aligned} \tag{4.15}$$

We estimate this integral applying Lemma 2.1. Then it is necessary to have $p(q-\beta)+s-2 > -1$, and this is equivalent to

$$\beta < \frac{pq+s-1}{p} \tag{4.16}$$

and so we obtain the following result.

Lemma 4.2. *Let* $1 < p < \infty$, $-1 < q < \infty$, $\beta \in \mathbb{R}$ *and* $p(q-\beta)+s-2 > -1$. *Then*

$$I_{t,L}(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^{p(q-\beta)+s-2}}{|1-z\overline{a}|^{2s}} \; dA(z) \approx h_L(z) = \begin{cases} 1 & \textit{if } L < 0 \\ \ln \dfrac{1}{1-|z|^2} & \textit{if } L = 0 \\ \dfrac{1}{(1-|z|^2)^L} & \textit{if } L > 0 \end{cases}$$

where $L = L(p,q,s) = p(\beta-q)+s$.

Thus we need to study the three cases associated to L. We consider first the case $L > 0$. The result is formulated in the following theorem, but we need the following straightforward result.

Lemma 4.3. *Let* $1 < p < \infty$, $-1 < q < \infty$ *and* $\frac{1}{2} < s < 1$. *Let*

$$a = \max\left\{-1\ ,\ \frac{q(p-1)-2}{p}\ ,\ \frac{pq-s}{p}\right\} \quad and \quad b = \min\left\{\frac{q+1-p}{p}\ ,\ \frac{pq+s-1}{p}\right\}.$$

Then the interval (a, b) *is notempty if and only if*

$$\frac{1-p-s}{p} < q < \frac{1-p+s}{p-1}\ .$$

Moreover

$$a = \begin{cases} -1 & if \quad q < \dfrac{s-p}{p} \\ \dfrac{pq-s}{p} & if \quad q \geq \dfrac{s-p}{p}\ . \end{cases}$$

and

$$b = \begin{cases} \dfrac{q+1-p}{p} & if \quad q > \dfrac{2-p-s}{p-1} \\ \dfrac{pq+s-1}{p} & if \quad q \leq \dfrac{2-p-s}{p-1}\ . \end{cases}$$

Theorem 4.5. *Let* $1 < p < \infty$, $\frac{1}{2} < s < 1$, $\frac{1-p-s}{p} < q < \frac{1-p+s}{p-1}$, $a < \beta < b$, *with* a *and* b *as in the previous lemma and* $v > 0$ *(see Lemma 4.1). Then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \mapsto {}_s\mathcal{A}^p_q$$

is a bounded operator. Moreover

$$I(a) \preceq (1-|a|^2)^{-p(\beta-q)} \parallel f \parallel^p_{p,q}\ .$$

Proof. By hypothesis $-1 < \beta < \frac{q+1-p}{p}$. Since $v > 0$, then $\beta > \frac{q(p-1)-2}{p}$ and we recall (4.15), then

$$I(a) \preceq C(1-|a|^2)^s \parallel f \parallel^p_{p,q} \int_{\mathbb{D}} \frac{(1-|z|^2)^{p(q-\beta)+s-2}}{|1-a\overline{z}|^{2s}}\, dA(z)\ .$$

By hypothesis $\beta < \frac{pq+s-1}{p}$ and $L = p(\beta - q) + s > 0$, and this is equivalent to $\frac{pq-s}{p} < \beta$. Then by Lemma 4.2 we have

$$\begin{aligned} I(a) &\preceq (1-|a|^2)^s \parallel f \parallel^p_{p,q} \frac{1}{(1-|a|^2)^{p(\beta-q)+s}} \\ &\preceq (1-|a|^2)^{p(q-\beta)} \parallel f \parallel^p_{p,q} \end{aligned}$$

and by hypothesis $\beta < q$ and we conclude the proof. □

We describe explicitly the interval (a, b) of the previous theorem.

Proposition 4.1. *With the hypothesis of the Lemma 4.3 we have*

i. $(a,b) = \left(-1, \dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{s}{2s-1} < p < \infty$ *and* $\dfrac{2-p-s}{p-1} < q < \dfrac{s-p}{p}$.

ii. $(a,b) = \left(-1, \dfrac{pq+s-1}{p}\right)$ *if and only if*

ii.1 $1 < p < \dfrac{s}{2s-1}$ *and* $\dfrac{1-p-s}{p} < q < \dfrac{s-p}{p}$ *or*

ii.2 $\dfrac{s}{2s-1} \le p$ *and* $\dfrac{1-p-s}{p} < q < \dfrac{2-p-s}{p-1}$

iii. $(a,b) = \left(\dfrac{pq-s}{p}, \dfrac{q+1-p}{p}\right)$, *with*

iii.1 $1 < p < \dfrac{s}{2s-1}$ *and* $\dfrac{2-p-s}{p-1} < q < \dfrac{1-p+s}{p-1}$ *or*

iii.2 $\dfrac{s}{2s-1} \le p$ *and* $\dfrac{s-p}{p} < q < \dfrac{1-p+s}{p-1}$

iv. $(a,b) = \left(\dfrac{pq-s}{p}, \dfrac{pq+s-1}{p}\right)$ *if and only if* $1 < p < \dfrac{s}{2s-1}$ *and* $\dfrac{s-p}{p} < q < \dfrac{2-p-s}{p-1}$.

Proof. To prove case i note that

$$-1 < \beta < \frac{q+1-p}{p}$$

if only if

$$q < \frac{s-p}{p}\ ; \qquad q > \frac{2-p-s}{p-1}\ ; \qquad \text{and} \qquad \frac{1-p-s}{p} < q < \frac{1-p+s}{p-1}\ ,$$

that is, q must satisfy

$$\max\left\{\frac{2-p-s}{p-1}\ ,\ \frac{1-p-s}{p}\right\} < q < \min\left\{\frac{s-p}{p}\ ,\ \frac{1-p+s}{p-1}\right\},$$

or equivalently

$$\begin{array}{rcccccl}
\dfrac{2-p-s}{p-1} & < & \dfrac{s-p}{p} & \Leftrightarrow & p & > & \dfrac{2s}{2s-1} \\
\dfrac{2-p-s}{p-1} & < & \dfrac{1-p+s}{p-1} & \Leftrightarrow & \dfrac{1}{2} & < & s \\
\dfrac{1-p-s}{p} & < & \dfrac{s-p}{p} & \Leftrightarrow & \dfrac{1}{2} & < & s \\
\dfrac{1-p-s}{p} & < & \dfrac{1-p+s}{p-1} & \Leftrightarrow & p & > & \dfrac{s-1}{2s-1}\ .
\end{array}$$

Since $\frac{1}{2} < s < 1$ then

$$\max\left\{\frac{2-p-s}{p-1}\ ,\ \frac{1-p-s}{p}\right\} = \frac{2-p-s}{p-1}\ ,$$

and

$$\min\left\{\frac{1-p+s}{p-1}\ ,\ \frac{s-p}{p}\right\} = \frac{s-p}{p}\ .$$

This proves the claim. The other cases are analogous. □

We now consider the case $L < 0$, which is divided into two cases: $1 < p \leq 2$ and $2 < p < \infty$. In the following theorem, the result is formulated for $1 < p \leq 2$ and we need the next straightforward result.

Lemma 4.4. *Let $1 < p \leq 2$, $-1 < q < \infty$ and $0 < s < 1$. Let*

$$a = \max\left\{-1\ ,\ \frac{q(p-1)-2}{p}\right\} \quad \text{and} \quad b = \min\left\{\frac{q+1-p}{p}\ ,\ \frac{pq+s-1}{p}\ ,\ \frac{pq-s}{p}\right\}.$$

Then the interval (a, b) is notempty if and only if

1. *For $0 < s \leq \frac{1}{2}$ we have $\frac{1-p-s}{p} < q < \infty$. Moreover*

$$a = \begin{cases} -1 & \text{if} \quad q < \dfrac{2-p}{p-1} \\ \dfrac{q(p-1)-2}{p} & \text{if} \quad q \geq \dfrac{2-p}{p-1}\ . \end{cases}$$

and

$$b = \begin{cases} \dfrac{q+1-p}{p} & \text{if} \quad q > \dfrac{2-p-s}{p-1} \\ \dfrac{pq+s-1}{p} & \text{if} \quad q \leq \dfrac{2-p-s}{p-1}\ . \end{cases}$$

or

2. *For $\frac{1}{2} < s < 1$ we have $\frac{s-p}{p} < q < \infty$. Moreover*

$$a = \begin{cases} -1 & \text{if} \quad q < \dfrac{2-p}{p-1} \\ \dfrac{q(p-1)-2}{p} & \text{if} \quad q \geq \dfrac{2-p}{p-1}\ . \end{cases}$$

and

$$b = \begin{cases} \dfrac{q+1-p}{p} & \text{if} \quad q > \dfrac{1-p+s}{p-1} \\[2ex] \dfrac{pq-s}{p} & \text{if} \quad q \leq \dfrac{1-p+s}{p-1} \end{cases} .$$

Theorem 4.6. *Let* $1 < p \leq 2$, $0 < s < 1$, $\max\left\{\dfrac{1-p-s}{p}\ ,\ \dfrac{s-p}{p}\right\} < q < \infty$ *and* $a < \beta < b$, *with* a *and* b *as in the previous lemma. Then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \mapsto {}_s\mathcal{A}^p_q$$

is a bounded operator. Moreover

$$I(a) \preceq (1-|a|^2)^s \parallel f \parallel^p_{p,q}$$

Proof. Observe that

$$\max\left\{\frac{1-p-s}{p}\ ,\ \frac{s-p}{p}\right\} = \begin{cases} \dfrac{1-p-s}{p} & \text{if} \quad 0 < s < \dfrac{1}{2} \\[2ex] \dfrac{s-p}{p} & \text{if} \quad \dfrac{1}{2} \leq s < 1 \end{cases}$$

and imitate the proof of Theorem 4.5. □

We describe explicitly the intervals (a, b) of the previous theorem.

Proposition 4.2.

I. For $1 < p \leq 2$, $0 < s \leq \dfrac{1}{2}$, $\dfrac{1-p-s}{p} < q < \infty$ *and* a, b *as in 1 from Lemma 4.4, we have that*

i. $(a, b) = \left(-1, \dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{2-p-s}{p-1} < q < \dfrac{2-p}{p-1}$.

ii. $(a, b) = \left(-1, \dfrac{pq+s-1}{p}\right)$ *if and only if* $\dfrac{1-p-s}{p} < q < \dfrac{2-p-s}{p-1}$.

iii. $(a, b) = \left(\dfrac{q(p-1)-2}{p}, \dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{2-p}{p-1} < q < \infty$.

iv. $(a, b) = \left(\dfrac{q(p-1)-2}{p}, \dfrac{pq+s-1}{p}\right) = \emptyset$.

II. For $1 < p \leq 2$, $\dfrac{1}{2} < s < 1$, $\dfrac{s-p}{p} < q < \infty$ *and* a, b *as in 2 from Lemma 4.4, we have that*

i. $(a, b) = \left(-1, \dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{1-p+s}{p-1} < q < \dfrac{2-p}{p-1}$.

ii. $(a,b)=\left(-1,\dfrac{pq-s}{p}\right)$ *if and only if* $\dfrac{s-p}{p}<q<\dfrac{1-p+s}{p-1}$.

iii. $(a,b)=\left(\dfrac{q(p-1)-2}{p},\dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{2-p}{p-1}<q<\infty$.

iv. $(a,b)=\left(\dfrac{q(p-1)-2}{p},\dfrac{pq-s}{p}\right)=\emptyset$.

Now the result is formulated for $L<0$ and $2<p<\infty$ and we will need the following straightforward result.

Lemma 4.5. *Let* $2<p<\infty$, $-1<q<\infty$ *and* $0<s<1$. *Let*

$$a=\max\left\{-1\ ,\ \frac{q(p-1)-2}{p}\right\}\quad \textit{and}\quad b=\min\left\{\frac{q+1-p}{p}\ ,\ \frac{pq+s-1}{p}\ ,\ \frac{pq-s}{p}\right\}.$$

Then the interval (a,b) *is notempty if and only if*

1. *For* $0<s\leq\dfrac{1}{2}$ *we have* $\dfrac{1-p-s}{p}<q<\dfrac{3-p}{p-2}$. *Moreover*

$$a=\begin{cases}-1 & \text{if}\quad q<\dfrac{2-p}{p-1}\\[2ex] \dfrac{q(p-1)-2}{p} & \text{if}\quad q\geq\dfrac{2-p}{p-1}\end{cases}.$$

and

$$b=\begin{cases}\dfrac{q+1-p}{p} & \text{if}\quad q>\dfrac{2-p-s}{p-1}\\[2ex] \dfrac{pq+s-1}{p} & \text{if}\quad q\leq\dfrac{2-p-s}{p-1}\end{cases}.$$

or

2. *For* $\dfrac{1}{2}<s<1$ *we have* $\dfrac{s-p}{p}<q<\dfrac{3-p}{p-2}$. *Moreover*

$$a=\begin{cases}-1 & \text{if}\quad q<\dfrac{2-p}{p-1}\\[2ex] \dfrac{q(p-1)-2}{p} & \text{if}\quad q\geq\dfrac{2-p}{p-1}\end{cases}.$$

and

$$b=\begin{cases}\dfrac{q+1-p}{p} & \text{if}\quad q>\dfrac{1-p+s}{p-1}\\[2ex] \dfrac{pq-s}{p} & \text{if}\quad q\leq\dfrac{1-p+s}{p-1}\end{cases}.$$

Theorem 4.7. *Let* $2 < p < \infty$, $0 < s < 1$, $\max\left\{\frac{1-p-s}{p}\ ,\ \frac{s-p}{p}\right\} < q < \frac{3-p}{p-2}$ *and* $a < \beta < b$, *with* a *and* b *as in the Lemma 4.5. Then*

$$P_\beta : L^p(\mathbb{D}, dA_q) \mapsto {}_s\mathcal{A}^p_q$$

is a bounded operator. Moreover

$$I(a) \preceq (1-|a|^2)^s \|f\|^p_{p,q}\ .$$

Proof. The proof is similar to the made in the Theorem 4.5. □

We describe explicitly the intervals (a, b) of the previous theorem.

Proposition 4.3. *I. For* $2 < p < \infty$, $0 < s \leq \frac{1}{2}$, $\frac{1-p-s}{p} < q < \frac{3-p}{p-2}$ *and* a, b *as in 1 from Lemma 4.5 we have that*

i. $(a, b) = \left(-1, \frac{q+1-p}{p}\right)$ *if and only if* $\frac{2-p-s}{p-1} < q < \frac{2-p}{p-1}$.

ii. $(a, b) = \left(-1, \frac{pq+s-1}{p}\right)$ *if and only if* $\frac{1-p-s}{p} < q < \frac{2-p-s}{p-1}$.

iii. $(a, b) = \left(\frac{q(p-1)-2}{p}, \frac{q+1-p}{p}\right)$ *if and only if* $\frac{2-p}{p-1} < q < \frac{3-p}{p-2}$.

iv. $(a, b) = \left(\frac{q(p-1)-2}{p}, \frac{pq+s-1}{p}\right) = \emptyset\ .$

II. For $2 < p < \infty$, $\frac{1}{2} < s < 1$, $\frac{s-p}{p} < q < \frac{3-p}{p-2}$ *and* a, b *as in 2 from Lemma 4.5 we have that*

i. $(a, b) = \left(-1, \frac{q+1-p}{p}\right)$ *if and only if* $\frac{1-p+s}{p-1} < q < \frac{2-p}{p-1}$.

ii. $(a, b) = \left(-1, \frac{pq-s}{p}\right)$ *if and only if* $\frac{s-p}{p} < q < \frac{1-p+s}{p-1}$.

iii. $(a, b) = \left(\frac{q(p-1)-2}{p}, \frac{q+1-p}{p}\right)$ *if only if* $\frac{2-p}{p-1} < q < \frac{3-p}{p-2}$.

iv. $(a, b) = \left(\frac{q(p-1)-2}{p}, \frac{pq-s}{p}\right) = \emptyset.$

We now consider the case $L = 0$. The result is formulated in the following theorem, but we need the following straightforward result.

Lemma 4.6. *Let* $1 < p < \infty$, $-1 < q < \infty$ *and* $\frac{1}{2} < s < 1$. *Let* $\beta = \frac{pq-s}{p}$

$$a = \max\left\{-1\ ,\ \frac{q(p-1)-2}{p}\right\}, \quad \textit{and} \quad b = \min\left\{\frac{q+1-p}{p}\ ,\ \frac{pq+s-1}{p}\right\}.$$

Then $a<\beta<b$ *if and only if*

$$\frac{s-p}{p}<q<\frac{1-p+s}{p-1}\ .$$

Moreover

$$a=\begin{cases}-1 & \text{if}\quad q<\dfrac{2-p}{p-1}\\[2ex] \dfrac{q(p-1)-2}{p} & \text{if}\quad q\geq\dfrac{2-p}{p-1}\ .\end{cases}$$

and

$$b=\begin{cases}\dfrac{q+1-p}{p} & \text{if}\quad q<\dfrac{2-p-s}{p-1}\\[2ex] \dfrac{pq+s-1}{p} & \text{if}\quad q\geq\dfrac{2-p-s}{p-1}\ .\end{cases}$$

Theorem 4.8. *Let* $1<p<\infty$, $\frac{1}{2}<s<1$, $\frac{s-p}{p}<q<\frac{1-p+s}{p-1}$ *and* $a<\beta<b$, *with* a, b *and* $\beta=\frac{pq-s}{p}$ *as in the Lemma 4.6. Then*

$$P_\beta : L^p(\mathbb{D}, dA_q)\mapsto {}_s\mathcal{A}^p_q$$

is a bounded operator. Moreover

$$I(a)\preceq C(1-|a|^2)^s\|f\|^p_{p,q}\ .$$

Proof. The proof is similar to the made in the Theorem 4.6. □

Again, we can describe explicitly the intervals (a,b) of the previous theorem.

Proposition 4.4. *Let* $1<p<\infty$, $\frac{1}{2}<s<1$, $\frac{s-p}{p}<q<\frac{1-p+s}{p-1}$ *and* a, b, β *as in Lemma 4.6. Then*

i. $(a,b)=\left(-1,\dfrac{q+1-p}{p}\right)$ *if and only if* $\dfrac{s-p}{p}<q<\dfrac{2-p-s}{p-1}$.

ii. $(a,b)=\left(-1,\dfrac{pq+s-1}{p}\right)$ *if and only if*

ii.1 $1<p<\dfrac{s}{2s-1}$ *and* $\dfrac{2-p-s}{p-1}<q<\dfrac{1-p+s}{p-1}$ *or*

ii.2 $\dfrac{s}{2s-1}<p<\infty$ *and* $\dfrac{s-p}{p}<q<\dfrac{1-p+s}{p-1}$.

iii. $(a,b)=\left(\dfrac{q(p-1)-2}{p},\dfrac{q+1-p}{p}\right)=\emptyset$

iv. $(a,b)=\left(\dfrac{q(p-1)-2}{p},\dfrac{pq+s-1}{p}\right)=\emptyset$.

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WAŻONE PRZESTRZENIE BERGMANA I PROJEKCJE BERGMANA

S t r e s z c z e n i e

Wiadomo, że gdy $-1 < q, \beta < \infty$, projekcja Bergmana P_β jest ograniczonym operatorem działającym z przestrzeni $L^p(\mathbb{D}, dA_q)$ na przestrzeń Bergmana $\mathcal{A}^p_q$ wtedy i tylko wtedy, gdy $q+1 < (\beta+1)p$. W pracy badany jest operator Bergmana P_β z przestrzeni $L^p(\mathbb{D}, dA_q)$ w przestrzeń Bergmana z wagą ${}_s\mathcal{A}^p_q$ i jest udowodnione, że P_β jest ograniczonym operatorem dla pewnych wartości $\beta,\ p,\ q$ oraz s, a w szczególności spełnia warunek $q+1 \geq (\beta+1)p$. Tak więc praca dotyczy klas funkcji na kole jednostkowym, stanowiących przestrzenie Banacha przy odpowiednich normach zadanych całkami, z pewnych potęg modułu z odpowiednimi gęstościami. Projekcje Bergmana to pewne uogólnienia transformaty Möbiusa na takich przestrzeniach.

Słowa kluczowe: przetrzeń Banacha, przestrzeń Bergmana $\mathcal{A}^p_q$, przestrzeń ważona