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EXTENDED CHOICE FUNCTIONALS – A CARDINAL FRAMEWORK FOR THE ANALYSIS OF CHOICE UNDER RISK

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Abstract

We propose a framework that extends the one developed by Professor Amartya Sen (with Arrowian roots), for the analysis of choice under risk by an individual, hereafter referred to as a decision maker. The framework is based on the decision maker's state dependent numerical evaluations − referred to as utility, worth, or pay-off − of the alternatives. We provide several examples to illustrate meaningful possibilities in the model proposed here. The expected utility choice functional assigns to each given state-dependent data profile (i.e., a pair consisting of a profile of state-dependent evaluation functions and a probability distribution over states of nature) the non-empty set of alternatives obtained by maximizing expected utility. A significant result in this paper, which illustrates the workability of our frameworks of analysis, is an axiomatic characterization of the expected utility choice functional using purely combinatorial techniques.

Aim/Purpose: To use a minor extension of the Arrow-Sen model of social choice theory to study individual decision making/aiding under risk and with state dependent evaluation functions.

Methodology: Combinatorics (theory of finite sets).

Findings: Plausible decision-aids for decision making under uncertainty with state dependent evaluation functions.

Research Implications: Exactly same model and results apply for the study of "weighted" multi-criteria decision making/aiding with state dependent evaluation functions.

Contribution: Apart from useful decision-aids for managerial decision making under risk and operations research, we provide an axiomatic characterization of the expected utility choice functional.

Keywords: risk, state-dependent evaluation, extended choice functionals.

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1 Introduction

Here we propose a framework for the analysis of choice under risk by an individual, hereafter referred to as a decision maker. The framework is based on the decision maker's state dependent numerical evaluations − referred to as utility, worth, or pay-off − of the alternatives. This framework is an extension of a model described in Sen (1970). A framework for the analysis of choice under risk, when the state-dependent preferences of the decision maker are expressed through rankings of alternatives, is motivated in Lahiri (2019b) and the framework in its entirety is discussed in Lahiri (2020a). Related axiomatic analysis, when the decision maker believes that all states of nature are equiprobable, is available (2019c) and a concrete analysis concerning the existence of "preferred with probability at least half" winners and when beliefs can be represented by any probability distribution is described in Lahiri (2020b). The problem of choosing one or more alternatives from a given set of alternatives was raised and rigorously formulated for the first time in a seminal contribution on majority voting by Pattanaik (1970). For the classical theory of decision making under uncertainty in the state dependent case − which is the other and major motivation behind this paper − one may refer to Karni (1985). Karni (1985) and Sen (1970) comfortably surpass the prerequisites related to decision making that is required to understand the frameworks of analyses developed here. An informative overall perspective of decision theory can be found in Resnik (1987). In the concluding section of this paper, we discuss a representation of uncertain prospects as ordered pairs of evaluation functions and probability distributions on the set of states of nature, motivated by a similar attempt in chapter 2 of Resnik (1987).

The reasons for our interest in state-dependent preferences are precisely the same as the ones discussed in Karni (1985), i.e., it is so obviously true that it does not need justification beyond citing trivial day-to-day examples as Karni has done in his book. Hence we can comfortably move ahead with our understanding of state-dependent preferences as in Karni (1985).

The major justification for the framework and the investigation presented in this paper is that the classical theory of decision making under uncertainty that rests on the assumption of maximization of expected utility (state-dependent or not) has significant limitations. It has often failed to be consistent with observed human behaviour in situations involving risk (i.e., uncertainty with probabilistic information about all states of nature available to or plausibly attributable by the decision maker) as was shown in the seminal work of Maurice Allais, also known as the Allais paradox (see Allais, 1953).

After defining the extended choice functional, we provide several examples of choice functionals. However, in order to show that our framework of analysis is very general and a “workable” model for the purpose of axiomatic characterizations, we provide here an axiomatic characterization of the expected utility choice functional. Related results are available in Lahiri (2019a). Going beyond that is the agenda for future research.

2 The model and some examples of extended choice functionals

The concept of an extended choice functional that is developed here, is a direct consequence of the concept of a social welfare functional introduced in Sen (1970) or its choice theoretic equivalent − choice functional − discussed in Lahiri (2019c), Sen’s framework has been the subject of extensive as well as intensive research, that lead to a comprehensive survey by d’Aspremont and Gevers (2002).

Consider a decision maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X. Let $\Psi(X)$ denote the set of all non-empty subsets of X. For a positive integer $n \geq 3$, let $N = \{1, 2, \ldots, n\}$ denote the set of states of nature. The satisfaction from the chosen alternative is realized only after the state of nature reveals itself.

We assume the satisfaction derived from the chosen alternative is represented by a numerically measurable worth or pay-off referred to as the **evaluation of the chosen alternative**.

An **evaluation function** is a function $u: X \times N \to \mathbb{R}$ such that for each alternative $x \in X$ and state of nature $i \in N$, $u(x,i)$ is the evaluation of x, in state of nature i. Let $\mathcal{U}$ denote the set of all evaluation functions.

Given $u \in \mathcal{U}$ and $x \in X$, we will often use $u(x)$ to denote the point $(u(x,1), \ldots, u(x,n))$ in $\mathbb{R}^N$ (the n-dimensional Euclidean space).

It is easy to see that $\{u(x)| \, u \in \mathcal{U} \text{ and } x \in X\} = \mathbb{R}^N$.

An **admissible set of evaluation functions** is any non-empty subset $\mathcal{D}$ of $\mathcal{U}$.

We denote vectors in $\mathbb{R}^N$ by letters a, b, c, d, etc. and when there is need for us to be explicit about (for instance) vector a, we write it as $(a_1, \ldots, a_n)$. $\mathbb{R}^N_+$ denotes the set $\{a \in \mathbb{R}^N | \, a_i \geq 0 \text{ for all } i \in N\}$.

The DM’s beliefs about the possibility of the various states of nature being realized is summarized by a probability distribution, i.e., $p \in \mathbb{R}^N_+$ such that $\sum_{i=}^{n} p_i = 1$. Let P^N denote the set of all probability distributions on N. Let π denote the **equi-probability** distribution, i.e. $\pi \in P^N$ such that $\pi_i = \frac{1}{n}$ for all $i \in N$.

Given a probability distribution p, the **set of most likely states of nature at** p is denoted by $ML(p) = \{i \in N | p_i \geq p_j \text{ for all } j \in N\} = \underset{i \in N}{\operatorname{argmax}}\, p_i$.

A **feasible set of probability distributions** (about the future states of nature being realized) is a non-empty subset of P^N denoted by Q. For whatever reasons, the DM's beliefs are restricted to belong to Q.

An **extended choice functional** (ECFL) on $\mathcal{D}\times Q$ is a function F: $\mathcal{D}\times Q \rightarrow \Psi(X)$, such that for each $(u,p) \in \mathcal{D}\times Q$, the decision maker chooses an alternative from F(u,p).

Before we proceed to examples, let us introduce the concept of regret which we shall require subsequently.

Given $u \in \mathcal{D}$, $x \in X$ and $i \in N$, **the regret from choosing** x **given** u **in state of nature** i, is: $\text{regret}(x,u,i) = \max_{y \in X} u(y,i) - u(x,i)$.

Note: If $\mathcal{D} = \{u \in \mathcal{U} | \text{ for all } i \in N, u(.,i): X \rightarrow \{1,2, \ldots, \#X\}$ is a one-to-one function$\}$, where #X denotes the cardinality of X, then each u could be considered to be an assignment of state-dependent rank-score of an alternative, with a higher rank-score corresponding to a better ranking.

Example 1 (Min-max Regret Choice Functional): An ECFL on $\mathcal{D}\times Q$ is said to be the **Min-max Regret Choice Functional**, denoted F^{mMR}, if for all $(u,p) \in \mathcal{D}\times Q$: $F^{mMR}(u,p) = \operatorname{argmin}_{x \in X}[\max_{i \in ML(p)} \text{regret}(x,u,i)]$.

Research on issues related to Example 1, but in an entirely different framework and from an entirely different perspective, is available in Puppe and Schlag (2009).

Example 2 (Max-min or Pessimistic rule): An ECFL on $\mathcal{D}\times Q$ is said to be the **Max-min rule**, denoted Mm, if for all $(u,p) \in \mathcal{D}\times Q$: $Mm(u,p) = \operatorname{argmax}_{x \in X}[\min_{i \in ML(p)} u(x,i)]$.

Example 3 (Max-max or Optimistic rule): An ECFL on $\mathcal{D}\times Q$ is said to be the **Max-max rule**, denoted MM, if for all $(u,p) \in \mathcal{D}\times Q$: $MM(u,p) = \operatorname{argmax}_{x \in X}[\max_{i \in ML(p)} u(x,i)]$.

Example 4 (Hurwicz's pessimism-optimism criterion): Let $\alpha \in [0,1]$. α is called the pessimism index. An ECFL on $\mathcal{D}\times Q$ is said to be the **Hurwicz** α **rule**, denoted H^α, if for all $(u,p) \in \mathcal{D}\times Q$: $H^\alpha(u,p) = \operatorname{argmax}_{x \in X}[\alpha \min_{i \in ML(p)} u(x,i) + (1-\alpha) \max_{i \in ML(p)} u(x,i)]$.

Example 5 (Pessimism-optimism regret criterion): Let $\alpha \in [0,1]$. α is called the pessimism index. An ECFL on $\mathcal{D}\times Q$ is said to be the **Regret** α **rule**, denoted Regret^α, if for all $(u,p) \in \mathcal{D}\times Q$: $\text{Regret}^\alpha(u,p) = \operatorname{argmax}_{x \in X}[(1-\alpha) \min_{i \in ML(p)} \text{regret}(x,u,i) + \alpha \max_{i \in ML(p)} \text{regret}(x,u,i)]$.

Example 6 (Expected Utility Choice Functional): An ECFL on $\mathcal{D}\times Q$ is said to be the **Expected Utility Choice Functional**, denoted F^e, if for all $(u,p)\in \mathcal{D}\times Q$: $F^e(u,p) = \text{argmax}_{x\in X} \sum_{i=1}^{n} p_i u(x,i)$.

In subsequent sections we are concerned with the expected utility choice functional which given state-dependent data chooses alternatives by maximizing expected utility. The properties we invoke for our axiomatic characterization are not very unusual and seem plausible in the context of our analysis.

3 Some important properties of extended choice functionals

In this section we introduce some important axioms for extended choice functionals.

We shall be **assuming** in what follows that $\pi\in Q$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy the **Weak Domination criterion** (WD), if for all $u\in\mathcal{D}$, $p\in Q$ and $x,y\in X$, $u(x,i) > u(y,i)$ for all $i\in N$ implies $y\notin F(u,p)$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy **Independence of Irrelevant Alternatives** (IIA), if for all $u,v\in\mathcal{D}$, $p\in Q$ and $x,y\in X$ with $x \neq y$, the following holds: $[u(x,i) = v(x,i)$, $u(y,i) = v(y,i)$ for all $i\in N$, $x\in F(u,p)$, $y\notin F(u,p)]$ implies $[y\notin F(v,p)]$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy **Equi-Probability Identical Evaluation** (E-PIE), if for all $u\in\mathcal{D}$, and $x,y\in X$ with $x \neq y$, the following holds: $[u(x,i) = u(y,i)$ for all $i\in N$ and $x\in F(u,\pi)]$ implies $[y\in F(u,\pi)]$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy **Equi-Probability Anonymity** (E-PAnon), if for all $u,v\in\mathcal{D}$, $i,j\in N$ and $x\in X$: $[v(x,k) = u(x,k)$ for all $k\in N\setminus\{i,j\}$, $v(x,i) = u(x,j)$, $v(x,j) = u(x,i)]$ implies $[F(v,\pi) = F(u,\pi)$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy **Equi-Probability Additivity** (E-PAdditivity), if for all $u,v\in\mathcal{D}$, and $a\in\mathbb{R}^N$:$[v(x) = u(x) + a$ for all $x\in X]$ implies $[F(v,\pi) = F(u,\pi)]$.

An ECFL F on $\mathcal{D}\times Q$ is said to satisfy **Evaluation Probability Conjunction** (EvPC), if for all $u,v\in\mathcal{D}$ and $p\in Q$ satisfying $v(x,i) = p_i u(x,i)$ for all $(x,i)\in X\times N$, it is the case that $F(u,p) = F(v,\pi)$.

4 The significance of Evaluation Probability Conjunction

EvPC is a fairly strong assumption, which is summarized in the following proposition whose proof is quite straightforward.

Proposition 1: An ECFL F on $\mathcal{U}$×Q is the expected utility choice functional on $\mathcal{U}$×Q if and only if the following two properties are satisfied.

(i) F(u,π) = F^e(u,π) for all u∈$\mathcal{U}$;

(ii) F satisfies EvPC on $\mathcal{U}$.

Proof: It is easy to see that the expected utility choice functional on $\mathcal{U}$ satisfies (i) and (ii). Hence, suppose an ECFL F on $\mathcal{U}$ satisfies (i) and (ii) and let (u,p)∈$\mathcal{U}$×Q.

By EvPC, F(u,p) = F(v,π), where for all (x,i)∈X×N: v(x,i) = p_iu(x,i).

By (i) F(v,π) = F^e(v,π) = $\text{argmax}_{x\epsilon X}\sum_{i=1}^{n} v_i(x,i)$ = $\text{argmax}_{x\epsilon X}\sum_{i=1}^{n} p_i u(x,i)$ = = F^e(u,p).

This proves the proposition. Q.E.D.

Using Proposition 1 and the main axiomatic characterization in Lahiri (2019a), we can easily obtain an axiomatic characterization of the EUCFL on $\mathcal{U}$×Q.

5 An axiomatic characterization of expected utility choice functional

The first lemma of this section leads to the starting point of the discussion of subjective expected utility theory due to Leonard Savage in lecture 7 of Rubinstein (2019).

A binary relation R on $\mathbb{R}^N$ whose asymmetric part is denoted P(R) and symmetric part is denoted I(R) is said to satisfy:

(i) **reflexivity (or be reflexive)** if for all a∈$\mathbb{R}^N$ it is the case that aRa holds;

(ii) **completeness (or be complete)** if for all a, b∈$\mathbb{R}^N$ it is the case that either aRb or bRa holds;

(iii) **transitivity (or be transitive)** if for all a, b, c∈$\mathbb{R}^N$: [aRb & bRc] implies [aRc];

(iv) **anonymity (or be anonymous)** if for a, b∈$\mathbb{R}^N$ and one-to-one functions (permutations) ρ:N→N on N: [$b_{\rho(i)}$ = a_i for all i∈N] implies [aI(R)b];

(v) **additivity** if for a, b, c∈$\mathbb{R}^N$: [aRb] implies [(a+c)R(b+c)].

Give a binary relation R on $\mathbb{R}^N$and any non-empty finite subset A of $\mathbb{R}^N$, let Best (A,R) = {a∈A| aRb for all b∈A}.

Suppose F is an ECFL on $\mathcal{U}$×Q. Define a binary relation R on $\mathbb{R}^N$ as follows: for a, b∈$\mathbb{R}^N$, aRb if and only if for some u∈$\mathcal{U}$ there exist x, y∈X such that u(x,i) = = a_i, u(y,i) = b_i for all i∈N and x∈F(u,π).

Claim 1: Suppose F is an ECFL on $\mathcal{U}$×Q satisfying WD, IIA and E-PIE. Let u∈$\mathcal{U}$, x∈F(u,π) and y∈X. Then y∈F(u,π) if and only if u(y)I(R)u(x).

Proof: $u \in \mathcal{U}$, $x \in F(u,\pi)$ and $y \in X$ implies $u(x)Ru(y)$. Hence we have to show that for $u \in \mathcal{U}$, $x \in F(u,\pi)$ and $y \in X$, [$y \in F(u,\pi)$ if and only if $u(y)Ru(x)$].

If $y \in F(u,\pi)$, then by definition of R, we have $u(y)Ru(x)$. Hence suppose $u(y)Ru(x)$ and, towards a contradiction, suppose $y \notin F(u,\pi)$.

$u(y)Ru(x)$ implies there exist $v \in \mathcal{U}$ and $z, w \in X$ with $v(z) = u(y)$, $v(w) = u(x)$ and $z \in F(v,\pi)$.

Let $v^* \in \mathcal{U}$ with $v^*(y) = v^*(z) = u(y)$, $v^*(x) = v^*(w) = u(x)$ and for all $x' \in X \backslash \{x,y,z,w\}$ and $i \in N$, $v^*(x',i) < \min\{u(i,x), u(i,y)\}$.

By WD, $F(v^*,\pi) \subset \{x,y,z,w\}$.

If $z \notin F(v^*,\pi)$, then by E-PIE, $y \notin F(v^*,\pi)$.

Thus, $F(v^*,\pi) \subset \{x,w\}$ and by E-PIE, $F(v^*,\pi) = \{x,w\}$.

Since $v^*(z) = v(z)$, $v^*(w) = v(w)$, $w \in F(v^*,\pi)$, $z \notin F(v^*,\pi)$ and $z \in F(v,\pi)$ contradicts IIA. Thus, $z \in F(v^*,\pi)$ and by E-PIE, $y \in F(v^*,\pi)$.

Since $v^*(y) = u(y)$, $v^*(x) = u(x)$, $x \in F(u,\pi)$, $y \notin F(u,\pi)$ and $y \in F(v^*,\pi)$ contradicts IIA. Thus, $y \in F(u,\pi)$.

This proves the claim. Q.E.D.

Lemma 1: Suppose F is an ECFL on $\mathcal{U} \times Q$ satisfying WD and IIA. Then R is a weak order on $\mathbb{R}^N$, i.e., R is reflexive, complete and transitive. If, in addition, F satisfies E-PIE, then for all $u \in \mathcal{U}$: $F(u,\pi) = \{y \in X |\ u(y) \in Best(\{u(x)|\ x \in X\},R)\}$.

Proof: Given $a, b \in \mathbb{R}^N$ and $x, y \in X$, let $u \in \mathcal{U}$ such that $u(x,i) = a_i$, $u(y, i) = b_i$ and $u(z,i) < \min\{a_i,b_i\}$ for all $i \in N$ and $z \in X \backslash \{x,y\}$.

Since F satisfies WD, $F(u,\pi)$ is a non-empty subset of $\{x,y\}$. Thus either aRb or bRa. Hence R is reflexive and complete.

To show that R is transitive, suppose aRb and bRc for some $a, b, c \in \mathbb{R}^N$ with $a \neq b \neq c \neq a$. Thus, there exist $u, v \in \mathcal{U}$ and $x, y, z \in X$ such that $u(x,i) = a_i$, $u(y,i) = b_i = v(y,i), c_i = v(z,i)$ for all $i \in N$, $x \in F(u,\pi)$ and $y \in F(v,\pi)$.

Let $u^* \in \mathcal{U}$ such that for all $i \in N$, $u^*(x,i) = u(x,i) = a_i$, $u^*(y,i) = u(y,i) = v(y,i) = b_i$, $u^*(z,i) = v(z,i) = c_i$ and $u^*(w,i) < \min\{a_i, b_i, c_i\}$ for all $w \in X \backslash \{x,y,z\}$.

By WD, $F(u^*,\pi)$ is a non-empty subset of $\{x,y,z\}$. Towards a contradiction suppose that $x \notin F(u^*,\pi)$.

Then by WD, $F(u^*,\pi)$ is a nonempty subset of $\{y,z\}$.

If $y \in F(u^*,\pi)$, then along with $x \notin F(u^*,\pi)$, $x \in F(u,\pi)$ and [for all $i \in N$ $u^*(x,i) = u(x,i) = a_i$, $u^*(y,i) = u(y,i) = b_i$], we get a violation of IIA. Thus, $y \notin F(u^*,\pi)$. Thus, $F(u^*,\pi) = \{z\}$ implying $z \in F(u^*,\pi)$ and $y \notin F(u^*,\pi)$.

However, $z \in F(u^*,\pi)$ and $y \notin F(u^*,\pi)$ along with $y \in F(v, \pi)$ and [for all $i \in N$, $u^*(y,i) = v(y,i) = b_i$, $u^*(z,i) = v(z,i) = c_i$] leads to a violation of IIA.

Thus, $z \notin F(u^*,\pi)$ and so $F(u^*,\pi) = \phi$, which contradicts the definition of an ECFL.

Thus $x \in F(u^*,\pi)$ and so xRz.

Thus, R is transitive.

That $F(u,\pi) \subset \{y \in X| u(y) \in Best(\{u(x)|x \in X\},R)\}$ follows immediately from the definition of R. Now suppose that in addition to WD and IIA, F satisfies E-PIE.

Let us show that $\{y \in X| u(y) \in Best(\{u(x)|x \in X\},R)\} \subset F(u,\pi)$.

Let $y \in X$ be such that $u(y) \in Best(\{u(x)|x \in X\},R)$ and let $z \in F(u,\pi)$. Since $F(u,\pi) \subset \{y \in X| u(y) \in Best(\{u(x)|x \in X\},R)\}$, $u(z) \in Best(\{u(x)|x \in X\},R)$. Thus, $u(y)I(R)u(z)$ and since $z \in F(u,\pi)$ it follows from claim 1 that $y \in F(u,\pi)$.

Thus, $\{y \in X| u(y) \in Best(\{u(x)|x \in X\},R)\} \subset F(u,\pi)$ and hence $F(u,\pi) = \{y \in X| u(y) \in Best(\{u(x)|x \in X\},R)\}$. Q.E.D.

It is possible to follow the discussion in lecture 7 of Rubinstein (2019) with the lemma 1 as given and arrive at an axiomatic characterization of EUCFL on $\mathcal{U} \times Q$. However, then we would require using either the separating or the supporting hyperplane theorem, which we do not want to do, since we want our axiomatic characterization to be based entirely on combinatorial techniques. We do not want to use any continuity assumption and/or topological properties of finite dimensional Euclidean space to prove our axiomatic characterization. Thus, we follow the route provided below.

Lemma 2: Suppose F is an ECFL on $\mathcal{U} \times Q$ that satisfies E-PAnon. Then R satisfies anonymity.

Proof: Since any permutation can be obtained as a succession of pair-wise interchanges it is enough to establish the result for the case of a permutation ρ such that for some i, $j \in N$ with $i \neq j$, $\rho(i) = j$, $\rho(j) = i$ and $\rho(k) = k$ for all $k \in N \setminus \{i,j\}$.

Thus, let a, $b \in \mathbb{R}^N$ with $a_i = b_j$, $a_j = b_i$ and $a_k = b_k$ for all $k \in N \setminus \{i,j\}$.

Let $u \in \mathcal{U}$ and x, $y \in X$ with $u(x) = a$, $u(y) = b$ and for all $z \in X \setminus \{x,y\}$ and $k \in N$, $u(z,k) = \beta$, where β $\min\{\min\{a_k,b_k\}|k \in N\} - 1$.

By WD, $F(u,\pi) \subset \{x,y\}$.

Without loss of generality suppose $x \in F(u,\pi)$. By the definition of R, u(x)Ru(y), i.e., aRb.

Now let $v \in \mathcal{U}$ with $v(z,i) = u(z,j)$, $v(z,j) = u(z,i)$ for all $z \in X$ and $v(z,k) = u(z,k)$ for all $z \in X$ and $k \in N \setminus \{i,j\}$.

By E-PAnon, $F(v,\pi) = F(u,\pi)$ and so $x \in F(v,\pi)$.

By definition of R, v(x)Rv(y), i.e., bRa.

Hence aI(R)b. Q.E.D.

Lemma 3: Suppose F is an ECFL on $\mathcal{U}\times Q$ that satisfies E-PAdditivity. Then R satisfies additivity.

Proof: Let a, b, $c\in\mathbb{R}^N$ and suppose aRb. By the definition of R, there exist $u\in\mathcal{U}$, x, $y\in X$ with u(x) = a, u(y) = b and $x\in F(u,\pi)$. Let $v\in\mathcal{U}$ be such that v(z) = = u(z) + c for all $z\in X$.

By E-PAdditivity, $F(v,\pi) = F(u,\pi)$ and so $x\in F(v,\pi)$.

By the definition of R, v(x)Rv(y), i.e. (a+c)R(b+c).

Thus, R is additive. Q.E.D.

Lemma 4: Suppose F is an ECFL on $\mathcal{U}\times Q$ that satisfies WD, IIA and E-PAdditivity. Let $<a^{(0)}, a^{(1)}, \ldots, a^{(n-1)}>$ be a sequence in $\mathbb{R}^N$, such that $a^{(k)}I(R)a^{(m)}$ for all k, $m\in\{0,1, \ldots, n-1\}$. Then $na^{(0)}I(R) \sum_{k=0}^{n-1} a^{(k)}$.

Proof: By lemma 1, R is a weak order on $\mathbb{R}^N$.

Suppose $ma^{(0)}I(R) \sum_{k=0}^{m-1} a^{(k)}$ for all $1 \leq m \leq K$ for some K < n.

Now $Ka^{(0)}I(R) \sum_{k=0}^{K-1} a^{(k)}$ and additivity of R implies $(K+1)a^{(0)}I(R)(a^{(0)} + + \sum_{k=0}^{K-1} a^{(k)})$.

But $a^{(0)}I(R)a^{(K+1)}$ and additivity of R implies $(a^{(0)} + \sum_{k=0}^{K-1} a^{(k)})I(R) (\sum_{k=0}^{K-1} a^{(k)}+ a^{(K)})$.

By transitivity of R, we get $(K+1)a^{(0)}I(R) \sum_{k=0}^{K} a^{(k)}$.

By a standard induction argument we now get $na^{(0)}I(R) \sum_{k=0}^{n} a^{(k)}$. Q.E.D

Lemma 5: Suppose F is an ECFL on $\mathcal{U}\times Q$ that satisfies WD, IIA, E-PAnon and E-PAdditivity. Let $a\in\mathbb{R}^N$, ρ be the permutation on N such that ρ(j) = j+1 for all $j\in\{1, \ldots, n-1\}$, ρ(n) = 1, $a^{(0)} = a$ and for $k\in\{1, \ldots, n-1\}$, let $a_j^{(k)} = a_{\rho(j)}^{(k-1)}$ for all $j \in 1, \ldots, n$. Then $aI(R)\frac{1}{n}\sum_{k=0}^{n} a^{(k)}$, where every coordinate of $\sum_{k=0}^{n} a^{(k)} = \sum_{j=1}^{n} a_j$.

Proof: By lemma 2, $aI(R)a^{(k)}$, for all k = 0,1, …, n-1 and $(\frac{1}{n}a)I(R)(\frac{1}{n} a^{(k)})$, for all k = 0,1, …, n – 1.

The lemma now follows from lemma 4. Q.E.D.

The following proposition is the stepping stone to our main result.

Proposition 2: Suppose F is an ECFL on $\mathcal{U}\times Q$ that satisfies WD, IIA, E-PIE, E-PAnon and E-PAdditivity. Then for all $u\in\mathcal{U}$, $F(u,\pi) = F^e(u,\pi)$.

Proof: Suppose F is an ECFL on $\mathcal{U}\times Q$ that satisfies WD, IIA, E-PIE, E-PAnon and E-PAdditivity and let $u\in\mathcal{U}$. Let $x\in F(u,\pi)$ and towards a contradiction suppose there exists $y\in X$ with $\sum_{i=1}^{n} u(y,i) > \sum_{i=1}^{n} u(x,i)$. Thus $\frac{1}{n}\sum_{i=1}^{n} u(y,i) > \frac{1}{n}\sum_{i=1}^{n} u(x,i)$.

By lemma 5, u(x)I(R)a and u(y)I(R)b, where $a_k = \frac{1}{n}\sum_{i=1}^{n} u(x,i)$ and $b_k =$ $= \frac{1}{n}\sum_{i=1}^{n} u(y,i)$ for all k∈N.

By lemma 1, F(u,π) = {z∈X|u(z)Ru(w) for all w∈X}.

Thus, u(x)Ru(z) for all z∈X and by transitivity of R, aRu(z) for all z∈X. Further, by transitivity of R, aRb.

Let v∈$\mathcal{U}$ with v(x) = a, v(y) = b, and v(z) = u(z) for all z∈X\{x,y}.

Since v(y,k) = $b_k > a_k$ = v(x,k) for all k∈N, by WD, x∉F(v,π) = = {{z∈X|u(z)Ru(w) for all w∈X}.

Hence it is not the case that aRb and aRv(z) for all z∈X\{x,y}.

Since v(z) = u(z) for all z∈X\{x,y}, it is not the case that aRb and aRu(z) for all z∈X\{x,y}, leading to a contradiction.

Thus, F(u,π) ⊂ F^e(u,π). Let x∈ F(u,π) ⊂ F^e(u,π) and y∈F^e(u,π).

Since x,y∈ F^e(u,π) implies $\frac{1}{n}\sum_{i=1}^{n} u(y,i) = \frac{1}{n}\sum_{i=1}^{n} u(x,i)$, by lemma 5, u(x)I(R)u(y).

Since F(u,π) = {z∈X| u(z)Ru(w) for all w∈X}, x∈F(u,π) and u(x)I(R)u(y), by transitivity of R, y∈{z∈X| u(z)Ru(w) for all w∈X}= F(u,π).

Thus, F(u,π) = F^e(u,π). Q.E.D.

With propositions 1 and 2 in place, we can prove the main theorem of this paper.

Theorem 1: An ECFL F on $\mathcal{U}$×Q is the expected utility choice functional if and only if it satisfies WD, IIA, E-PIE, E-PAnon, E-PAdditivity.and EvPC.

Proof: It is easy to verify that the EUCFL on $\mathcal{U}$×Q satisfies the six properties. Hence let us suppose that F is an ECFL on $\mathcal{U}$×Q that satisfies the six properties and let u∈$\mathcal{U}$. Let us show that F(u,π) = F^e(u,π).

By proposition 3, F(u,π) = F^e(u,π).

The theorem now follows from Proposition 1. Q.E.D.

An examination of the procedure by which we arrived at theorem 1, suggests that in order to axiomatically characterize the expected utility choice functional, weaker assumptions would suffice.

An ECFL F on $\mathcal{D}$×Q is said to satisfy the **Equi-Probability Weak Dominatio criterion** (E-PWD), if for all u∈$\mathcal{D}$ and x,y∈X, u(x,i) > u(y,i) for all i∈N implies y∉F(u,π).

An ECFL F on $\mathcal{D}$×Q is said to satisfy **Equi-Probability Independence of Irrelevant Alternatives** (E-PIIA), if for all u,v∈$\mathcal{D}$ and x,y∈X with x ≠ y, the following holds: [u(x,i) = v(x,i), u(y,i) = v(y,i) for all i∈N, x∈F(u,π), y∉F(u,π)] implies [y∉F(v,π)].

The alternative axiomatic characterization of EUCFL on $\mathcal{U}$ based on the using the above four properties instead of their analogues used in theorem 2 is the following.

Theorem 2: An ECFL F on $\mathcal{U}\times$Q is the expected utility choice functional if and only if it satisfies E-PWD, E-PIIA, E-PIE, E-PAnon, E-PAdditivity and EvPC.

6 Representation of uncertain prospects as an element in the domain of a choice functional

The following is based on Chapter 2 of Resnik (1987), where "states of nature" are related to "consequences".

Given a non-empty set $\mathcal{X}$, an uncertain prospect on $\mathcal{X}$ is a probability distribution p on $\mathcal{X}$ with finite support, i.e., support (p) = $\{x \in \mathcal{X} \mid p(x) > 0\}$ is a non-empty finite set. The elements of $\mathcal{X}$ are called prizes or consequences. If $\mathcal{X} = \mathbb{R}$, then the prizes are interpreted as monetary gains and losses, depending on whether the real number is positive or negative.

Let $\{p^{(1)}, \ldots, p^{(K)}\}$ for some positive integer K be a non-empty finite set of uncertain prospects.

Let X = $\{p^{(1)}, \ldots, p^{(K)}\}$ denote the set of alternatives from which the decision maker is required to choose. Note that $\bigcup_{j=1}^{K} \mathrm{support}(p^{(j)})$ is a non-empty finite subset of $\mathcal{X}$.

Let N = $\{1,2, \ldots, K\}\times\bigcup_{j=1}^{K} \mathrm{support}(p^{(j)})$ denote the set of states of nature. Clearly N is non-empty and finite.

Let $v:\{0\}\cup\mathcal{X}\rightarrow\mathbb{R}$ satisfying v(0) = 0 denote the utility function of the decision maker. The utility function is defined on a set consisting of consequences and the real number 0, which could belong to $\mathcal{X}$. Intuitively, v(x) > v(0) means that x is a gain and v(x) < v(0) means that x is a loss.

The corresponding evaluation function $u:X\times N\rightarrow\mathbb{R}$ of the decision maker is defined as follows: for all $p^{(k)}\in X$ and $(j,x)\in N$, $u(p^{(k)}, (j,x)) = v(x)$ if k = j and $u(p^{(k)}, (j,x)) = v(0)$ if $k \neq j$.

The decision maker's beliefs about the occurrence of the states of nature in N is given by a probability distribution q on N such that for all $(j,x)\in N$, $q_{(j,x)}$ = $= \frac{1}{\sum_{k=1}^{K} \#(\mathrm{support}(p^{(k)}))}$, where for each $k\in\{1, \ldots, K\}$, $\#(\mathrm{support}(p^{(k)})$ is the cardinality of support $(p^{(k)})$.

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