Mateusz Dzicher

https://orcid.org/0000-0003-0917-3111

Department of Applied Mathematics
College of Finance
University of Economics in Katowice, Poland
mateusz.dzicher@ue.katowice.pl

Sampling methods for investment portfolio formulation procedure at increased market volatility

Abstract

Aim/purpose – In this paper, a market volatility-robust portfolio composition framework under the modified Markowitz's approach with the use of sampling methods is developed in order to improve the allocation efficiency for a portfolio of financial instruments formulation procedure at an increased market volatility.

Design/methodology/approach – In order to overcome the risk of not receiving an optimal solution to the portfolio optimization (suboptimal outcomes of attribution of weights in allocation procedures) the developed model, first, implements the rationale that financial markets largely feature two states, i.e., quiescent (non-crisis; low market volatility) periods that are occasionally interspersed with stress (crisis; high market volatility) periods and, second, relies on many input samples of rates of return, either from an empirical distribution or a theoretical distribution (mitigating estimation risk). All computational results are reported for publicly available historical daily data sets on selected Polish blue-chip securities.

Findings – Not only did the presented method produce more diversified allocation, but also successfully minimized the unfavorable effects of increased market volatility by providing less risky portfolios in comparison to Newton's method, typically used for optimization under portfolio theory.

Research implications/limitations – The research emphasized that in order to get a more diversified investment portfolio it is crucial to outdo the limitations of a single sample approach (utilized in Markowitz's model) which may on some occasions be statistically biased. Thus it was proved that sampling methods allow to obtain a less concentrated and volatile allocation which contributes the investment decision-making.

However, the current research focused solely on publicly available input data of particular securities. In this manner, an additional analysis can be prepared for other jurisdictions and asset classes. There can also be considered a use of other than variance risk measures.

Originality/value/contribution – The suggested framework contributes to existing methods a wide array of quantitative data analysis and simulation tools for composing an unique approach that directly addresses the task of minimizing the adverse implications of increased market volatility that, in consequence, pertains to knowledgeable attributing of investment portfolio proportions of either individual or institutional investors. The prepared method is also proved to hold demanded computational quality and, importantly, the capacity for further development.

Keywords: investment decisions, optimization techniques, portfolio selection, statistical simulation methods.
JEL Classification: C150, C610, G110.

1. Introduction

Markowitz (1952) introduced a quantitative framework that intends to define a portfolio formulation process, widely known as the foundation of the portfolio theory. As part of the concept, the financial market investors commit their capital resources to selected investment opportunities and anticipate a satisfactory benefit in reward which is, typically, an adequate rate of return from their individual asset allocation. In that manner, in relation to portfolio characteristics, investment participants are assumed to favor higher rates of return whilst exposed to a certain risk level on all occasions, or conversely for a particular expected rate of return, they always prefer lower extent of the volatility. Therefore, the optimal portfolio is assumed to hold a combination of assets characterized with a possibly best risk-return trade-off. Importantly, it was distinctly confirmed that a statistically significant positive trade-off between expected rates of return on financial instruments and associated risk exists (Lundblad, 2007). Consequently, the more volatile a financial instrument is, or otherwise the higher its potential loss, the higher its expected rate of return is. Thus, a reduction of the investment volatility should be of vital concern. The cited rationale will be considered as the fundamental notion of the portfolio theory (e.g., Elton, Gruber, Brown, & Goetzmann, 2014).

Essentially, in periods of market turbulence, as a result of either systemic or idiosyncratic events, an increase in the volatility of market parameters and accordingly in the prices of investment assets are observed (Aliber, 2011). For instance, the sudden wave of economic losses resulted from global failures of credit institutions that began in 2007 and 2008 or, currently experienced, after-

math of a lockdown related to SARS-CoV-2 pandemic remind the *raison d'être* of the portfolio risk management. In specific, during a financial crisis, correlations among the rates of return on securities use to appreciate in excess of that implied by the economic fundamentals, what may, e.g., influence the efficiency of the idiosyncratic risk diversification effect, emphasized by Goetzmann & Kumar (2001). Loretan & English (2000), Hartmann, Straetmans, & de Vries (2004) and Bekaert, Campbell, & Ng (2005) have highlighted this issue for financial market collapses since 1980s and identified it as the contagion phenomena. Accordingly, more robust security selection strategies are sought at present. Therefore, in order to formulate an investment portfolio, it is essential to apply a competent optimization technique for rational investment decision-making and proper selection of financial instruments, especially in times of increased volatility. This should be understood that a rational asset allocation assures a desirable trade-off between an expected rate of return on a portfolio and its volatility.

In this paper, a market volatility-robust portfolio composition framework is elaborated to enhance the efficiency of attributing investment proportions to selected securities in a composition problem of a portfolio of financial instruments under the modified Markowitz's approach with the use of sampling methods at an increased market volatility. The presented model implements the idea, described by Litzenberger & Modest (2008), that financial markets do feature quiescent (non-crisis; low market volatility) periods that are occasionally interspersed with stress (crisis; high market volatility) periods. Accordingly, the suggested framework aims at minimizing the adverse effects of an identified market volatility relation between both instances which remains not extensively explored in the literature. Thus, alike constructed investment portfolios may be considered an addition to commonly utilized alternatives to mean-variance optimization techniques, e.g., a regime-switching asset allocation (or Markov--switching asset allocation), discussed, e.g., by Ang & Bekaert (2002; 2004), which requires early recognition and pre-stress period strategy adjustments. Importantly, by and large, volatility-robust portfolios reduce the necessity of investment position rebalancing, which, ceteris paribus, diminishes the related transaction costs. Additionally, if this technique is employed to markets that exhibit, an opposite to a contagion phenomenon, a flight-to-quality effect, i.e., volatility increases while correlation between selected assets decreases (Baur & Lucey, 2006; Beber, Brandt, & Kavajecz, 2006; Berben & Jansen, 2005), it might provide an opportunity to compose as stress-insensitive portfolios as possible. Finally, the suggested methodology also addresses the estimation risk is-

sue, related to potentially biased historical input data, described by Orwat-
-Acedańska & Acedański (2013).

In view of the above, this paper aims at answering the research question if by
use of the developed framework and based on the rationale that financial markets
are characterized by quiescent and stress states, an individual or institutional inves-
tor can improve the allocation efficiency of a portfolio of securities in presence of
an elevated market volatility, e.g., resulting from an exogenous shock.

The remainder of this paper was organized as follows. Chapter 2 presents
a formulation of the market volatility-robust portfolio optimization model, derived
from the Markowitz's portfolio theory and with the use of sampling methods (to
overcome the limitations of, e.g., Newton's method), was defined. Chapter 3 ad-
dressed a specified portfolio optimization problem using the developed framework
and the obtained computational results were presented for the selected input data
sets. Next, Chapter 4 presented a dedicated discussion and optimized backtesting
of portfolios was included. Finally, conclusions were formulated.

2. Theoretical background in the modified Markowitz's approach with the use of sampling methods

2.1. Portfolio composition background

Markowitz assumed that rates of return on assets follow the multivariate
normal distribution. This entails that an investment portfolio is comprehensively
characterized by its expected rate of return, $E(r_p)$, given as:

$$E(r_p) = \sum_{i=1}^{n}(w_i E(r_i)) = W^T R \tag{1}$$

and its variance (volatility, or risk), σ_p^2, such as:

$$\sigma_p^2 = \sum_{i=1}^{n}(w_i^2 \sigma_i^2) + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j \neq i}}^{n}(w_i w_j Cov(r_i, r_j)) = W^T C W \tag{2}$$

where W is a matrix of proportions, w_i, invested in i-th security, R is a matrix of
expected rates of return of selected financial instruments, and C is a symmetric
variance-covariance matrix (Markowitz, 1959). Based on the above, the rate of
return of a constructed portfolio is defined as a weighted average of rates of re-
turn of its comprising securities and is the basic measure of the average receipts
of the investment opportunities. The mentioned weight is regarded simply as
a proportion of resources allocated in particular assets. In this place, it is im-

portant to notice that the variance of an investment portfolio is not the weighted average of the separate asset variances, e.g., the portfolio built upon two securities bears the variance:

$$\sigma_p^2 = E(w_i(r_i - E(r_i)) + w_j(r_j - E(r_j)))^2 \tag{3}$$

where σ_p^2 is the two asset portfolio variance, w_i and w_j are the weights attributed to securities i and j, r_i and r_j are the rates of return of the securities i and j, whereas $E(r_i)$ and $E(r_j)$ are their expected values. Further reduction and application of the binomial theorem, i.e., $(x + y)^2 = x^2 + 2xy + y^2$, leads the variance of the two securities portfolio to the form of:

$$\sigma_p^2 = E\left[w_i^2\left(r_i - E(r_i)\right)^2 + 2w_iw_j\left(r_i - E(r_i)\right)\left(r_j - E(r_j)\right) + w_j^2\left(r_j - E(r_j)\right)^2\right] \tag{4}$$

from where it is useful to recall the properties of expected value statistic. One should apply that the mean of a sum of two rates of return is equivalent to the sum of mean of each rates of return (i.e., $E(r_i + r_j) = E(r_i) + E(r_j)$) and that a mean of constant multiplied by the rate of return is equivalent to the product of the constant and the mean of the rate of return (i.e., $E(c(r_i)) = cE(r_i)$) to obtain:

$$\sigma_p^2 = w_i^2 E\left[\left(r_i - E(r_i)\right)^2\right] + 2w_iw_j E\left[\left(r_i - E(r_i)\right)\left(r_j - E(r_j)\right)\right]$$
$$+ w_j^2 E\left[\left(r_j - E(r_j)\right)^2\right] \tag{5}$$

where the expression $E\left[\left(r_i - E(r_i)\right)\left(r_j - E(r_j)\right)\right]$ is the covariance, $Cov(r_i, r_j)$, of assets i and j. Therefore, it is true that formula (2) stands for the variance of an investment portfolio, which is summarized, e.g., by Alexander (2008).

In this manner, an investor with a single period portfolio construction problem considers two feasible approaches. First, (a) to minimize the volatility of an investment portfolio for a given expected rate of return. Second, (b) to maximize the expected rate of return of an investment portfolio for a particular volatility level. Approach (a) involves finding the solution to an optimization issue with continuous random variables, quadratic objective and linear constraints, whereas approach (b) implies solving an optimization issue with continuous random variables, however, with linear objective and all linear constraints but one quadratic limitation. Bodie, Kane, & Marcus (2010) noticed that despite the fact that the methods proposed logically resemble each other, approach (a) is computationally more efficient. For that reason, formally, the portfolio optimization problem is widely defined as:

$$min \ \sigma_p^2 = W^T C W \tag{6}$$

subjected to: $\sum_{i=1}^{n} w_i = 1$ and $0 \leq w_i \leq 1$ for $i = 1, 2, ..., n$ where proportions, w_i, always sum up to one (i.e., short selling is not allowed) and for a desired minimal benchmark rate of return, $E(r_b)$.

In addition, the suggested in this paper market volatility-robust investment portfolio composition procedure incorporates the principle that financial markets are characterized, in most of the times, by quiescent periods that are infrequently interspersed with exogenous stress periods. Therefore, investment participants take into account the market turbulence possibility, as stated in Kole, Koedijk, & Verbeek (2006). Typically, the crisis intervals are specified by acute decline in the rates of return on financial instruments and increased volatility. Thus, in consequence, the asset allocation is done in the process of minimizing the ratio of variances in both non-crisis (quiescent) period, σ_q^2, written as:

$$\sigma_q^2 = W^T C_q W \tag{7}$$

and in crisis (stress) period, σ_c^2, formulated as:

$$\sigma_c^2 = W^T C_c W \tag{8}$$

where C_q and C_c are symmetric variance-covariance matrices for quiescent and stress intervals respectively. Thus, if w_i ($i = 1, 2, \ldots, n$) are the proportions of resources invested in i-th asset under either non-crisis or crisis regime and $\sum_{i=1}^{n} w_i = 1$, then the optimization issue is defined, in general, as follows:

$$\widehat{w} = argmin \frac{\sigma_c^2}{\sigma_q^2} \quad \begin{cases} 0 \leq w_i \leq 1 \text{ where } i = 1, 2, \ldots, n \\ \sum_{i=1}^{n} w_i = 1 \end{cases} \tag{9}$$

The market volatility-robust portfolios reduce the necessity of investment position rebalancing, which, ceteris paribus, curbs the associated transaction costs (e.g., fees but also operational costs). Passive investment strategies enthusiasts, who wish to limit detrimental effects of heightened volatility, may consider it especially useful to adapt the model in their asset allocation. The proposed framework seems suitable for all asset classes, provided that the non-crisis/crisis regime is implemented and data are segregated accordingly.

2.2. Market volatility-robust portfolio composed of two assets

In order to determine the market volatility-robust portfolio analytically, let us consider a two asset market spectrum ($i = 1, 2$). Volatility is measured by variance in both quiescent period, σ_q^2, and in time of market turmoil, σ_c^2. Fur-

thermore, the correlation coefficient of securities between rates of return is ρ_q, during non-crisis interval and ρ_c, while crisis. If short selling is not allowed, or $w_i \in <0, 1>$ and $\sum_{i=1}^{n} w_i = 1$, then the portfolio composed of two assets consists of weights w_A for asset A and $(1 - w_A)$ for asset B. As a result, it is true that:

$$\sigma_q^2 = w_A^2 \sigma_{Aq}^2 + (1 - w_A)^2 \sigma_{Bq}^2 + 2w_A(1 - w_A)\sigma_{Aq}\sigma_{Bq}\rho_q \tag{10}$$

$$\sigma_c^2 = w_A^2 \sigma_{Ac}^2 + (1 - w_A)^2 \sigma_{Bc}^2 + 2w_A(1 - w_A)\sigma_{Ac}\sigma_{Bc}\rho_c \tag{11}$$

With the purpose of finding the value of the proportion w_A that minimizes the relation $\frac{\sigma_c^2}{\sigma_q^2}$, it is necessary to differentiate with respect to w_A, which leads to:

$$\left(\frac{\rho_c}{\rho_q} - 1\right)\left[4w_A(1 - w_A)\left(w_A\sigma_{Aq}^2 - (1 - w_A)\sigma_{Bq}^2\right) \\ - 2(1 - 2w_A)\left(w_A^2\sigma_{Aq}^2 + (1 - w_A)^2\sigma_{Bq}^2\right)\right] = 0 \tag{12}$$

If the correlation coefficient of rates of return on financial instruments in quiescent interval is a lower value then its equivalent in stress period, or $\rho_c > \rho_q$, then it reveals the contagion phenomena. In the opposite scenario, or $\rho_c < \rho_q$, flight-to-quality effect is observed. Thus, in case when the flight-to-quality effect is present the correlation spread between crisis and non-crisis periods is always negative ($\rho_c - \rho_q < 0$), which creates the anticipated volatility and correlation trade-off. In essence, if the relation $\frac{\rho_c}{\rho_q}$ varies from one ($\neq 1$), which is analogous to the statement that correlations of investment assets' rates of return differ in crisis periods, the condition may be simplified and be given by:

$$w_A^2 \sigma_{Aq}^2 - (1 - w_A)^2 \sigma_{Bq}^2 = 0 \tag{13}$$

Therefore, the distinctive asset allocation determined by proportions:

$$\begin{cases} w_A = \frac{\sigma_{Bq}}{\sigma_{Aq} + \sigma_{Bq}} \\ w_B = 1 - w_A = \frac{\sigma_{Aq}}{\sigma_{Aq} + \sigma_{Bq}} \end{cases} \tag{14}$$

for the market volatility-robust portfolio is being completed.

2.3. Sampling methods in portfolio formulation

In the Markowitz's approach, the proportions w_i are found based on observations of rates of return on particular securities obtained from a single sample. In consequence, the estimates may be biased, especially, if outliers are observed or rates of return are asymmetrically distributed which results in suboptimal

allocation due to estimation errors, i.e., materialization of estimation risk. The mentioned issue became a catalyst for the development of a variety of statistical methods, which are utilized to reduce the unfavorable outcomes of estimation risk in investment portfolio selection process. Such methods are notably: robust estimation methods, Bayesian estimation methods, robust optimization methods and, analyzed in this paper, sampling methods.

The sampling methods allow to formulate an optimal portfolio built on many samples either from an empirical distribution or a theoretical distribution. In order to overcome the risk of not receiving an optimal solution to the portfolio optimization (via, e.g., Newton's method), the results are an average of weights, w_i, attained from multiple scenarios. Michaud (1998) was first to introduce the technique to the field of asset allocation. Orwat-Acedańska & Acedański (2013) noted that, typically, the phases of portfolio construction procedure in accordance with the Michaud's sampling methods approach are as follows:

Phase 1. Based on an initial sample – (K x n) matrix of rates of return observations – a k-amount of subsamples of the same size as the initial sample is formed. Subsamples may be derived from an empirical distribution (bootstrapping methods) or from a theoretical distribution (Monte Carlo simulations).

Phase 2. For each subsample $j(j = 1, 2, \dots , n)$ an estimation of [optionally – $E(r_j)$ vector and] C_j matrix is done.

Phase 3. [Optionally – for an desired level of minimal benchmark rate of return, $E(r_b)$] Proportions, w_j, are obtained.

Phase 4. Subsample j portfolio proportions are averaged:

$$w_{p\_n} = \frac{1}{n}\sum_{j=1}^{n} w_j \qquad (15)$$

so that an optimal solution, or weights, w_{p_0}, are calculated. Scherer (2002) noticed that investment portfolios derived in the above manner hold a higher level of diversity than those achieved by means of a classic, i.e., Markowitz's mean-variance optimization. The received proportions are also less sensitive to significant changes due to accepted risk alterations. Moreover, sampling methods may be used for a wide variety of rates of return distribution classes and for risk measures other than variance, i.e., Conditional Value-at-Risk (i.e., *CVaR*).

3. Research findings on the market volatility-robust portfolio optimization with the use of sampling methods

3.1. Input data

In this section the computational results for allocation of the investment portfolio proportions, w_{p_n}, with the use of sampling methods (for subsamples derived from both an empirical distribution and a theoretical distribution, i.e., normal distribution) are presented in comparison to Markowitz's approach. For the purpose of a portfolio formulation process and said comparison, the model stated and described in Chapter 2 was implemented.

Therefore, in order to validate the usefulness of the suggested framework, an investments in 10, diversified-by-sectors WIG20 traded blue-chip securities, i.e., CCC (clothing), CDR (gaming), CPS (telecommunication), JSW (coal mining and production), KGH (metal extracting and production), MBK, PEO (both banking), PGN (gas extraction and energy), PKN (crude oil extraction and production) and PZU (insurance) was considered. Table 1 presents a summary of information on the percentage share of securities in WIG20.

Table 1. Share of selected blue-chips securities in WIG20 at June 1, 2020

%	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
%WIG20	1.4%	7.9%	4%	0.8%	12.6%	1.4%	5.5%	4.5%	8.3%	8.3%

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Based on historical daily prices (where all used prices were publicly available), logarithmic rates of return of selected financial instruments were calculated for the period from July 1, 2014 until June 20, 2020 (input data; i.e., 1,477 trading day observations). Next, importantly, for the purpose of the allocation, the reference, or risk benchmark was determined (i.e., WIG20 index volatility) and afterwards the input data were segregated and assigned to period subsamples q – quiescent and c – crisis. As a rule, it was presumed that the volatility (i.e., standard deviation) of the risk benchmark, σ_{WIG20}, should be calculated for at least three, moving time intervals (e.g., 1M, 3M, and 6M). Thus, it was assumed that if the reference index volatility for the shortest-term interval was a higher value than for the medium-term interval and longer-term interval, and if volatility for the medium-term interval was a higher value than longer-term interval (i.e., $\sigma_{WIG20\_1M} > \sigma_{WIG20\_3M} > \sigma_{WIG20\_6M}$) then a stress period, or simply a crisis was observed. Figure 1 is a graphical illustration of the above assumption for the risk benchmark volatility (i.e., 1M, 3M, and 6M) in the considered time period.

Figure 1. Risk benchmark volatility for 1M, 3M and 6M intervals
from July 1, 2014 until June 1, 2020
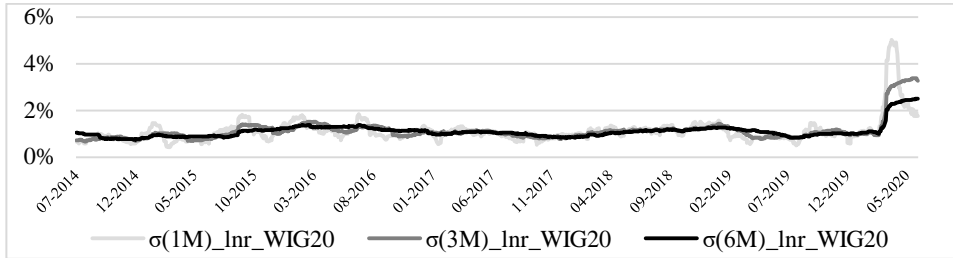
Source: Based on data from Warsaw Stock Exchange statistical data (2020).

The allocation per se was performed at four moments, i.e., at the end of 2018 and 2019 (December 28, 2018 and December 30, 2019, respectively), March 31, 2020 and June 1, 2020, where the last two dates specifically included conditions related directly to SARS-CoV-2 pandemic influence on financial markets (i.e., a significant increase in volatility was observed; Figure 1 and Table 2).

Table 2. Risk benchmark volatility at December 28, 2018, December 30, 2019, March 4, 2020, March 31, 2020 and June 1, 2020

σ_{WIG20_t}	1.06.2020	31.03.2020	4.03.2020*	30.12.2019	28.12.2018
σ_{WIG20_1M}	0.0268	0.0492	0.0212	0.009	0.0144
σ_{WIG20_3M}	0.0321	0.0308	0.0152	0.0095	0.0135
σ_{WIG20_6M}	0.0238	0.0229	0.013	0.01	0.0126

* Date of first official SARS-CoV-2 infection in Poland.

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

After the input data were declared for both period subsamples q and c, new data subsamples were derived by means of sampling methods either from an empirical distribution (bootstrapping methods) or a theoretical distribution (Monte Carlo simulations). Next, the symmetric variance-covariance matrices, C_q and C_c, were constructed. Ultimately, the asset allocation was done via minimization of the relation of the specific time interval data subsamples c and q volatilities. The final results, i.e., portfolios, were an average of proportions and statistics obtained from 1,000, drawn with replacement, iterations, as stated in Subchapter 2.3. In the end, the received allocations were compared to proportions and portfolio statistics received with the use of Markowitz's approach.

3.2. Empirical results and analysis

First set of portfolios was formulated for the end of 2018 (i.e., December 28, 2018), when the input data consisted of logarithmic rates of return of 1,125 trading day observations (starting from July 1, 2014). In accordance with the risk benchmark volatility criterion, $\sigma_{WIG20\_1M} > \sigma_{WIG20\_3M} > \sigma_{WIG20\_6M}$, the initial input data were segregated into two subsamples q and c. As such, data set q included 801 trading days, whereas data set c regarded 324 trading days. These data sets were described by volatilities presented in Table 3.

Table 3. Segregated data volatilities at December 28, 2018

σ_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
σ_1	0.0216	0.024	0.0176	0.0337	0.0229	0.02	0.0156	0.0192	0.0195	0.0154
σ_{c_1}	0.0236	0.0238	0.0174	0.0361	0.0247	0.0216	0.0163	0.0194	0.0218	0.0173
σ_{q_1}	0.0209	0.0241	0.0178	0.0325	0.0212	0.0195	0.0153	0.0186	0.0187	0.0148
$\sigma_{c\_1}/\sigma_{q\_1}$	1.1286	0.9872	0.9743	1.1096	1.1637	1.1073	1.0682	1.0407	1.1627	1.1666

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

It was anticipated that volatilities within the q data set would be lower than their counterparts in c. However, such relationship was not observed for CDR and CPS, which, in otherwise relatively stable market conditions, remained more unexpected, with increased rates of return and trading volume. Nonetheless, out of initially segregated input data, subsamples q and c were further generated with both bootstrapping and Monte Carlo methods. For newly obtained data sets corresponding C_q and C_c matrices were calculated. Next, 1000 above-defined optimizations were performed and afterwards results were averaged to finally obtain the minimal value of the relation of data subsamples c and q volatilities. Before not mentioned, assumed constraints included: no short selling allowed, maximal 20% commitment to one instrument and non-negative portfolio expected rate of return. Complementary, Markowitz allocation was done. A summary of the optimized portfolio proportions and volatility statistics are presented in Table 4.

Table 4. Optimized portfolio proportions and volatility statistics at December 28, 2018

w_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
w_{Boot_1}	0.1043	0.0452	0.0791	0.0782	0.137	0.1069	0.1269	0.0794	0.1088	0.1343
w_{MC_1}	0.1132	0.0382	0.0796	0.0609	0.1179	0.1101	0.1059	0.0774	0.1295	0.1673
w_{M_1}	0.1023	0.0559	0.1761	0.01	0.01	0.0275	0.192	0.1131	0.124	0.1891
Bootstrapping volatility	$\sigma_{q\_Boot\_1} = 0.0076$			$\sigma_{c\_Boot\_1} = 0.0095$			$\sigma_{c\_Boot\_1}/\sigma_{q\_Boot\_1} = 1.2455$			
Monte Carlo volatility	$\sigma_{q\_MC\_1} = 0.0075$			$\sigma_{c\_MC\_1} = 0.0092$			$\sigma_{c\_MC\_1}/\sigma_{q\_MC\_1} = 1.2272$			
Markowitz volatility	$\sigma_{q\_M\_1} = 0.0093$			$\sigma_{c\_M\_1} = 0.0117$			$\sigma_{c\_M\_1}/\sigma_{q\_M\_1} = 1.261$			

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Therefore, the presented results put emphasis on a few important issues. First, sampling methods in comparison to Markowitz's approach provide a more diversified allocation, i.e., lower concentration in particular instruments. Second, the averaged volatilities for both subsamples q and c are lower, which also results, as desired, in their lower quotient. Additionally, with the averaged correlation spreads between subsamples q and c considered, mostly a flight-to-quality effect was observed. A corresponding summary is included in Table 5.

Table 5. Correlation spreads for subsamples q and c at 28/12/2018

	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
CCC	–									
CDR	0.0603	–								
CPS	−0.0263	0.0717	–							
JSW	−0.0785	−0.1621	0.0188	–						
KGH	−0.0155	−0.0009	0.0032	−0.0124	–					
MBK	0.0202	0.0913	−0.123	−0.0278	−0.0631	–				
PEO	0.0535	0.0087	0.0804	−0.0156	−0.044	−0.0325	–			
PGN	−0.0249	−0.0541	−0.1359	0.0533	−0.041	0.0257	−0.0666	–		
PKN	−0.0801	0.1106	0.0018	0.1091	0.1328	0.0165	−0.1036	−0.0256	–	
PZU	0.0887	0.012	−0.0002	0.0674	0.0512	0.0073	0.113	−0.0476	−0.0912	–

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Next, second set of portfolios was prepared for the end of 2019 (i.e., December 30, 2019), where the input data consisted of logarithmic rates of return of 1,370 trading day observations (period extended by 245 trading days since the end of 2018). The input data were further segregated in line with the risk benchmark volatility criterion so that data set q included 1001 trading days, whereas data set c included 369 trading days. These data sets were described by volatilities presented in Table 6.

Table 6. Segregated data volatilities at December 30, 2019

σ_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
σ_2	0.0223	0.0235	0.0174	0.033	0.0225	0.0197	0.0153	0.0193	0.0192	0.0149
σ_{c_2}	0.0237	0.0234	0.0174	0.036	0.0247	0.0219	0.016	0.0198	0.0216	0.0168
σ_{q_2}	0.0221	0.0236	0.0176	0.032	0.0207	0.019	0.0149	0.0187	0.0184	0.0143
$\sigma_{c\_2}/\sigma_{q\_2}$	1.0761	0.9925	0.9927	1.125	1.192	1.1542	1.0784	1.0571	1.1780	1.1741

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Again, CDR and CPS volatilities at q and c had similar characteristics throughout 2019 as in 2018, i.e., featured increased rates of return and trading volumes. Since allocation procedure was identical as previously described, thus a corresponding summary of the optimized portfolio proportions and volatility statistics are shown in Table 7.

Table 7. Optimized portfolio proportions and volatility statistics at December 30, 2019

w_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
w_{Boot_2}	0.0703	0.0579	0.0706	0.0976	0.1164	0.1277	0.109	0.0797	0.1309	0.14
w_{MC_2}	0.0787	0.0587	0.0578	0.0706	0.1395	0.1383	0.0922	0.0892	0.1321	0.1428
w_{M_2}	0.0844	0.0664	0.1829	0.01	0.01	0.0336	0.1923	0.0947	0.1258	0.2
Bootstrapping volatility	$\sigma_{q\_Boot\_2} = 0.0078$			$\sigma_{c\_Boot\_2} = 0.0091$			$\sigma_{c\_Boot\_2}/\sigma_{q\_Boot\_2} = 1.1666$			
Monte Carlo volatility	$\sigma_{q\_MC\_2} = 0.0074$			$\sigma_{c\_MC\_2} = 0.0088$			$\sigma_{c\_MC\_2}/\sigma_{q\_MC\_2} = 1.1898$			
Markowitz volatility	$\sigma_{q\_M\_2} = 0.0098$			$\sigma_{c\_M\_2} = 0.0118$			$\sigma_{c\_M\_2}/\sigma_{q\_M\_2} = 1.1986$			

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Similarly to the previously analyzed example, sampling methods in comparison to Markowitz approach provided a more diversified allocation. Furthermore, the averaged volatilities for both subsamples q and c were lower, similar to their quotients. Interestingly, flight-to-quality effect was less present than at the end of 2018 which is presented in Table 8.

Table 8. Correlation spreads for subsamples q and c at 30/12/2019

	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
CCC	–									
CDR	0.0593	–								
CPS	−0.0241	−0.0026	–							
JSW	0.0228	−0.0156	0.1201	–						
KGH	0.0705	−0.0103	0.0115	−0.1356	–					
MBK	−0.0127	−0.092	−0.0105	−0.0085	−0.0343	–				
PEO	0.0055	0.008	0.1135	0.0551	−0.0618	−0.1723	–			
PGN	−0.0689	0.0648	0.0215	0.0246	0.0887	0.0846	0.0928	–		
PKN	0.0003	−0.0504	0.0754	−0.0423	−0.0586	0.0447	−0.0281	0.0332	–	
PZU	0.046	0.0077	0.0035	−0.0369	0.0655	0.0326	0.0423	0.0499	0.042	–

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Another, third set of portfolios was prepared for the end of March 2020 (i.e., March 31, 2020), as such the input data included logarithmic rates of return of 1, 433 trading day observations (period extended by 63 trading days since the end of 2019). Afterwards, due to the risk benchmark volatility criterion, the input

data were segregated so that data set q comprised of 1,037 trading days and data set c included 396 trading days. As such, the data sets were described by volatilities depicted in Table 9.

Table 9. Segregated data volatilities at 31/03/2020

σ_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
σ_3	0.026	0.0245	0.0178	0.0369	0.0258	0.0216	0.0173	0.0204	0.02	0.0159
σ_{c_3}	0.0347	0.0266	0.0192	0.0402	0.0286	0.0283	0.0226	0.0238	0.0242	0.0202
σ_{q_3}	0.0225	0.0234	0.0174	0.0324	0.0207	0.0189	0.0149	0.0188	0.0184	0.0143
$\sigma_{c\_3}/\sigma_{q\_3}$	1.5411	1.135	1.1029	1.2427	1.3852	1.4925	1.5195	1.2665	1.3132	1.4148

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

In March 2020, there was observed the deepest slump in the valuations of securities since the subprime crisis not only across Warsaw Stock Exchange indexes but also at major stocks around the world. As a rule, the sudden increase in volatility within the financial markets was a direct aftermath of the imposed lockdowns in real economy due to the SARS-CoV-2 pandemic. In regards to the selected portfolio components, the volatility shocks were clearly visible in the data set c, either in comparison to the end of 2018 or 2019. Thus, in all cases the relationship between the data sets c and q results in values higher than one. Under such circumstances, the allocation procedure, with consequently the same assumptions as previously, was executed and the obtained portfolio proportions and volatility statistics are included in Table 10.

Table 10. Optimized portfolio proportions and volatility statistics at March 31, 2020

w_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
w_{Boot_3}	0.1628	0.0461	0.0496	0.0336	0.0953	0.1324	0.1499	0.1026	0.0878	0.1398
w_{MC_3}	0.1754	0.0310	0.0468	0.0312	0.1316	0.1137	0.1536	0.0726	0.0886	0.1554
w_{M_3}	0.0517	0.0717	0.2	0.01	0.01	0.01	0.1816	0.1092	0.1558	0.2
Bootstrapping volatility	$\sigma_{q\_Boot\_3}=0.0073$			$\sigma_{c\_Boot\_3}=0.0113$			$\sigma_{c\_Boot\_3}/\sigma_{q\_Boot\_3}=1.5456$			
Monte Carlo volatility	$\sigma_{q\_MC\_3}=0.0073$			$\sigma_{c\_MC\_3}=0.0114$			$\sigma_{c\_MC\_3}/\sigma_{q\_MC\_3}=1.555$			
Markowitz volatility	$\sigma_{q\_M\_3}=0.0099$			$\sigma_{c\_M\_3}=0.0158$			$\sigma_{c\_M\_3}/\sigma_{q\_M\_3}=1.5971$			

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

In line with the results of the preceding examples, it was noted that higher level of diversification was observed with the use of sampling methods in comparison to Markowitz approach. Correspondingly, the averaged volatilities for both subsamples q and c were again lower, similarly as their quotient. In terms of the flight-to-quality effect, it was clearly noticed that correlation spreads between crisis and non-crisis periods were on their rise (Table 11).

Table 11. Correlation spreads for subsamples q and c at March 31, 2020

	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
CCC	–									
CDR	−0.0724	–								
CPS	−0.1406	0.0733	–							
JSW	0.0606	−0.1154	−0.0292	–						
KGH	−0.0205	0.0813	0.0229	−0.1063	–					
MBK	0.0259	−0.0008	−0.0151	0.0027	−0.0517	–				
PEO	−0.0551	0.1184	−0.109	−0.0253	0.0576	−0.026	–			
PGN	0.0362	−0.0217	−0.0305	−0.0589	0.0713	0.0191	−0.0007	–		
PKN	−0.0056	−0.025	−0.0569	−0.002	−0.0025	−0.0889	−0.0058	−0.0082	–	
PZU	−0.0105	0.1547	−0.0269	−0.1366	−0.0537	−0.0628	−0.0201	0.09	0.0012	–

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

Finally, fourth set of portfolios was composed for the beginning of June 2020 (i.e., June 1, 2020). Hence, at that moment, the input data consisted of logarithmic rates of return of 1, 474 trading day observations (period extended by 104 trading days since the end of March 2020). Next, the input data were segregated in accordance with the risk benchmark volatility criterion, therefore set q included logarithmic rates of return of 1,071 trading days, whereas data set c included logarithmic rates of return of 403 trading days. These data sets were described by volatilities shown in Table 12.

Table 12. Segregated data volatilities at June 1, 2020

σ_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
σ_4	0.0282	0.0246	0.0179	0.037	0.0262	0.022	0.0178	0.0204	0.0203	0.0162
σ_{q_4}	0.0354	0.027	0.0193	0.0402	0.029	0.0283	0.0236	0.0238	0.0244	0.0202
σ_{q_4}	0.0258	0.0235	0.0175	0.0328	0.0212	0.0196	0.0154	0.0188	0.0187	0.0148
$\sigma_{c\_4}/\sigma_{q\_4}$	1.3722	1.1484	1.1012	1.2267	1.3659	1.4429	1.5347	1.2644	1.3046	1.3706

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

As presented in Figure 1, at the beginning of June 2020 at Warsaw Stock Exchange an elevated short-term volatility (i.e., 1M) was accompanied by an increased medium-term volatilities (i.e., 3M and 6M). Thus, similarly as at the end of March 2020, the relationship between the data sets q and c was always a value higher than one. Taking into consideration the underlying conditions, the allocation was done again with the presented, predefined assumptions. Correspondingly, the portfolio proportions and volatility statistics are included in Table 13.

Table 13. Optimized portfolio proportions and volatility statistics at June 1, 2020

w_j	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
w_{Boot_4}	0.0934	0.0471	0.055	0.0511	0.1223	0.1329	0.1598	0.0858	0.1031	0.1494
w_{MC_4}	0.1116	0.0452	0.0515	0.0371	0.1214	0.1687	0.1489	0.082	0.0847	0.149
w_{M_4}	0.0346	0.0836	0.2	0.01	0.01	0.01	0.1781	0.1213	0.1525	0.2
Bootstrapping volatility	$\sigma_{q\_Boot\_4} = 0.0075$			$\sigma_{c\_Boot\_4} = 0.0114$			$\sigma_{c\_Boot\_4}/\sigma_{q\_Boot\_4} = 1.5234$			
Monte Carlo volatility	$\sigma_{q\_MC\_4} = 0.0075$			$\sigma_{c\_MC\_4} = 0.0113$			$\sigma_{c\_MC\_4}/\sigma_{q\_MC\_4} = 1.5158$			
Markowitz volatility	$\sigma_{q\_M\_4} = 0.0103$			$\sigma_{c\_M\_4} = 0.0158$			$\sigma_{c\_M\_4}/\sigma_{q\_M\_4} = 1.5313$			

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

In consistence with the results of the previous examples, allocation with the use of sampling methods in comparison to Markowitz's approach provided more diversified portfolios. Furthermore, the averaged volatilities for both subsamples *q* and *c* remained again lower, which also resulted in their lower quotients. Pertaining to the flight-to-quality effect, a relatively worrying relationship between correlations and volatilities was observed (correlations spreads were decreasing). Such observations suggest the presence of contagion phenomena. A relevant correlation summary is presented in Table 14.

Table 14. Correlation spreads for subsamples *q* and *c* at June 1, 2020

	CCC	CDR	CPS	JSW	KGH	MBK	PEO	PGN	PKN	PZU
CCC	–									
CDR	−0.0417	–								
CPS	0.0024	−0.0452	–							
JSW	0.0715	0.0525	−0.1112	–						
KGH	−0.0274	0.0396	0.0443	0.0162	–					
MBK	0.0682	−0.1184	0.0223	−0.0652	−0.1032	–				
PEO	−0.1045	0.0139	−0.0533	0.0817	−0.0295	−0.0788	–			
PGN	−0.0314	0.1184	0.027	−0.0087	0.0898	0.0727	−0.0499	–		
PKN	0.0813	−0.1282	0.028	−0.0498	0.0134	0.0709	0.0364	0.0441	–	
PZU	−0.0108	0.0221	−0.0414	−0.0609	−0.0307	0.1008	0.0416	0.0295	−0.0308	–

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

4. Discussion and backtesting

Consistent for all considered allocations, sampling methods produced more diversified portfolios which, simultaneously, provided lower averaged volatilities for both subsamples *q* and *c* in comparison to Markowitz's approach. However, in order to verify if historically obtained market volatility-robust portfolios

reduced the necessity of investment position rebalancing (which, inter alia, diminishes related transaction costs) in consecutive time periods an allocation backtesting was performed. In that manner, optimized proportions from previously analyzed time periods were substituted for their next time period counterparts. In that manner, a summary of backtesting volatility calculations is included in Table 15.

Table 15. Backtested portfolio proportions and volatility statistics at December 30, 2019, March 31, 2020 and June 1, 2020

December 30, 2019			
Bootstrapping volatility	$\sigma_{q\_Boot\_B1} = 0.0063$	$\sigma_{c\_Boot\_B1} = 0.0071$	$\sigma_{c\_Boot\_B1}/\sigma_{q\_Boot\_B1} = 1.1323$
Monte Carlo volatility	$\sigma_{q\_MC\_B1} = 0.0061$	$\sigma_{c\_MC\_B1} = 0.007$	$\sigma_{c\_MC\_B1}/\sigma_{q\_MC\_B1} = 1.1417$
Markowitz volatility	$\sigma_{q\_M\_B1} = 0.0099$	$\sigma_{c\_M\_B1} = 0.0117$	$\sigma_{c\_M\_B1}/\sigma_{q\_M\_B1} = 1.1909$
March 31, 2020			
Bootstrapping volatility	$\sigma_{q\_Boot\_B2} = 0.0063$	$\sigma_{c\_Boot\_B2} = 0.0091$	$\sigma_{c\_Boot\_B2}/\sigma_{q\_Boot\_B2} = 1.4314$
Monte Carlo volatility	$\sigma_{q\_MC\_B2} = 0.0063$	$\sigma_{c\_MC\_B2} = 0.0092$	$\sigma_{c\_MC\_B2}/\sigma_{q\_MC\_B2} = 1.4481$
Markowitz volatility	$\sigma_{q\_M\_B2} = 0.0098$	$\sigma_{c\_M\_B2} = 0.0161$	$\sigma_{c\_M\_B2}/\sigma_{q\_M\_B2} = 1.6427$
June 1, 2020			
Bootstrapping volatility	$\sigma_{q\_Boot\_B3} = 0.0067$	$\sigma_{c\_Boot\_B3} = 0.0102$	$\sigma_{c\_Boot\_B3}/\sigma_{q\_Boot\_B3} = 1.5059$
Monte Carlo volatility	$\sigma_{q\_MC\_B3} = 0.0072$	$\sigma_{c\_MC\_B3} = 0.0106$	$\sigma_{c\_MC\_B3}/\sigma_{q\_MC\_B3} = 1.4708$
Markowitz volatility	$\sigma_{q\_M\_B3} = 0.0103$	$\sigma_{c\_M\_B3} = 0.0158$	$\sigma_{c\_M\_B3}/\sigma_{q\_M\_B3} = 1.5369$

Source: Based on data from Warsaw Stock Exchange statistical data (2020).

In view of what has been said, it was observed that the suggested framework efficiently minimizes the unfavorable effects of increased market volatility in an investment portfolio allocation problem by providing less risky portfolios. Therefore, it seems that surprisingly little attention is paid in the literature to the use of multiple scenario analysis in solving the portfolio formulation problem.

5. Conclusions

5.1. Research contributions

This paper adopts a unique model under the modified Markowitz's approach with the use of sampling methods to improve the efficiency of allocation in a security portfolio composition procedure at an increased market volatility, thus contributes to the existing methods of overcoming the risk of not receiving an optimal solution in the attribution of investment proportions. The research also adds a rationale to implement the developed quantitative framework to the literature and to practical use either in individual or institutional investors' strategies.

5.2. Research implications

The computational results highlight that, first, in order to obtain a more diversified investment portfolio, it is important to surpass the limitations of a single sample analysis that may be biased, e.g., due to statistical properties of expectations of rates of return (especially, if outliers are observed or rates of return are asymmetrically distributed). In that manner, a portfolio formulation procedure with the use of sampling methods is proved to provide a less concentrated allocation in comparison to, e.g., Newton's method, typically used for optimization under Markowitz's approach in portfolio theory. Second, the analyzed averaged volatilities for both subsamples representing quiescent and stress periods are lower in value. As intended, this resulted in their lower quotient and further means that portfolios formulated with the use of multiple samples derived either from an empirical distribution or a theoretical distribution are less risky. In the end, all of the above enhance the investment decision-making process.

5.3. Research limitations and further development

Further research is required in order to determine if similar conclusions are directly applicable to other jurisdictions and, importantly, to different asset classes. Currently, this paper puts emphasis on model verification based on publicly available input data of WIG20 traded blue-chip securities, thus all influential market characteristics are mostly local. Additionally, since sampling methods may be used for a wide variety of rates of return distribution classes and for risk measures other than variance, it seems interesting to improve the analysis with the use of, e.g., *CVaR* (i.e., a coherent risk measure).

References

Alexander, C. (2008). *Market risk analysis, quantitative methods in finance*. New York: Wiley & Sons.

Aliber, R. (2011). Financial turbulence and international investment. In *Portfolio and risk management for central banks and sovereign wealth funds* (BIS Papers, No. 58, pp. 5-17). Basel: BIS, ECB, World Bank.

Ang, A., & Bekaert, G. (2002, July). International asset allocation with regime shifts. *Review of Financial Studies, 15*(4), 1137-1187. https://doi.org/10.1093/rfs/15.4.1137

Ang, A., & Bekaert, G. (2004). How regimes affect asset allocation. *Financial Analyst Journal*, *60*(2), 86-99. https://doi.org/10.2469/faj.v60.n2.2612

Baur, D., & Lucey, B. M. (2006). Flight-to-quality or contagion? An empirical analysis of stock-bond correlations (IIIS Discussion Paper, No. 122). Retrieved from https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.148.8440&rep=rep1&type=pdf

Beber, A., Brandt, M., & Kavajecz, K. (2007). *Flight-to-quality or flight-to-liquidity? Evidence from the euro-area bond market* (NBER Working Papers, No. 12376). Cambridge, MA: NBER. https://doi.org/10.3386/w12376

Bekaert, G., Campbell, H., & Ng, A. (2005, January). Market integration and contagion. *Journal of Business, 78*(1), 39-69. https://doi.org/10.1086/426519

Berben, R.-P., & Jansen, J. (2005, September). Co-movement in international equity markets: a sectoral view. *Journal of International Money and Finance, 24*(5), 832-857. https://doi.org/10.1016/j.jimonfin.2005.04.001

Bodie, Z., Kane, A., & Marcus, A. (2009). *Investments* (8th ed.). New York: McGraw-Hill Irwin.

Elton, E., Gruber, M., Brown, S., & Goetzmann, W. (2014). *Modern portfolio theory and investment analysis* (9th ed.). New York: John Wiley & Sons.

Goetzmann, W., & Kumar, A. (2001). *Equity portfolio diversification* (Working Papers, No. 8686). Cambridge, MA: NBER. https://doi.org/10.3386/w8686

Hartmann, P., Straetmans, S., & de Vries, C. (2004, February). Asset market linkages in crisis periods. *Review of Economics and Statistics, 86*(1), 313-326. https:/doi.org/10.1162/003465304323023831

Kole, E., Koedijk, K., & Verbeek, M. (2006, August). Portfolio implications of systemic crises. *Journal of Banking & Finance, 30*(8), 2347-2369. https://doi.org/10.1016/j.jbankfin.2005.08.006

Litzenberger, R., & Modest, D. (2008, July). Crisis and non-crisis risk in financial markets: An unified approach to risk management. *SSRN Electronic Journal*. https://doi.org/10.2139/ssrn.1160273

Loretan, M., & English, W. (2000). *Evaluating "correlation breakdowns" during periods of market volatility* (International Finance Discussion Paper, No. 658). Washington, DC: Board of Governors of the Federal Reserve. https://doi.org/10.2139/ssrn.231857

Lundblad, C. (2007, July). The risk return trade-off in the long run: 1836–2003. *Journal of Financial Economics, 85*(1), 123-150. https://doi.org/10.1016/j.jfineco.2006.06.003

Markowitz, H. (1952). Portfolio selection. *Journal of Finance, 7*(1), 77-91. https://doi.org/10.1111/j.1540-6261.1952.tb01525.x

Markowitz, H. (1959). *Portfolio selection: Efficient diversification of investments*. New York: Wiley & Sons (reprinted by Yale University Press, 1970).

Michaud, R. (1998). *Efficient asset management: A practical guide to stock portfolio optimization and asset allocation*. Boston: Harvard Business School Press.

Orwat-Acedańska, A., & Acedański, J. (2013). Zastosowanie programowania stochastycznego w konstrukcji odpornych portfeli inwestycyjnych [An application of the stochastic programming to build robust investment portfolios]. *Studia Ekonomiczne, 135*, 121-136.

Scherer, B. (2002, January). Portfolio resampling: review and critique. *Financial Analyst Journal, 58*(6), 98-109. https://doi.org/10.2469/faj.v58.n6.2489

Warsaw Stock Exchange statistical data on WIG20 traded blue-chip securities (CCC, CDR, CPS, JSW, KGH, MBK, PEO, PGN, PKN, PZU). (2020). Retrieved from https://stooq.pl