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The method of the fast Radon transform calculation based on the usage of scheme calculation symmetry properties

Introduction

The Radon transformation (RT) [Radon 1917], like its particular form Hough transformation (HT) [Hough 1962], is a well-known tool in image processing, tomography, astronomy, microscopy etc [Toft 1996; Deans 1993]. The (ρ,θ) RT defined as integral of image $g(x,y)$ alone the integration line s. The matrix presentation of RT is

$$\begin{array}{lll} \breve{g}(\rho,\theta) & = \Re & g(x,y) \\ \downarrow & \quad\downarrow & \downarrow \\ \mathbf{b} & = \mathbf{W} & \mathbf{g} \end{array}, \tag{1}$$

where $\mathbf{W}\in\Re^{I\times J}$ is the system matrix (SM) with weight factors $\varpi_{i,j}$ between j-th image pixels and each orientation i of integration line s; I – dimensional vector $\mathbf{b}$ ($I = RT$) describes the parameter domain(PD); J – dimensional vector $\mathbf{g}$ ($J = MN$) describes the image; R and T are appropriately number of samples of ρ (the shortest distance from the source of the coordinate system to the integration line s) and θ (the angle between integration line s and axis of abscise); M and N is appropriately width and height of the image.

The discrete form of eq.1 is named beam sum. The full system of beam sums is a projection. The set of projections for $\theta\in[0;\pi]$ forms the PD.

The elements of SM $\varpi_{i,j}$ are calculated using one of these approaches: 1) the weighted, when $\varpi_{i,j}$ is calculated precisely; 2) the non-weighted, when $\varpi_{i,j}$ in eq.1 is equal to 1 if the integration line s cross the image element and $\varpi_{i,j}$ is equal to 0 in other cases. Usually the non-weighted approach is used. It requires the extra approximation, and usually applicable in case of line segment detection [Toft 1996].

The SM consists of $RTMN \approx N^2 \times N^2$ elements. It's very large. Also it's not structured. The matrix is sparse and has around $O(N)$ non-zero elements for each integration line. It's easy to determine that the number of non-zero values for line is less than $d(N-1)+1$, where d - dimension of the object (for images $d=2$).

For instance, the image of 100×100 pixels requires calculation and storage $\sim 10^8$ ($\sim 3.6 \cdot 10^6$ non-zero) elements of SM, the image of 512×512 pixels requires $\sim 6.8 \cdot 10^{10}$ ($\sim 5.3 \cdot 10^8$ non-zero) elements of SM. This requires essential resources even for modern computers.

1. The Radon transformation scheme calculation symmetry properties

Due to the method of fast calculation of RT uses scheme calculation symmetry properties let's prove the existence of these properties.

For square image $N \times N$, where N is even, the source of the coordinate system coincides with the image center (fig.1). The samples of angels $\Delta\theta$ and offsets of projection sums in these projections $\Delta\rho$ are arbitrary.

1.1. The central symmetry

Let's consider the central symmetry of SM elements for arbitrary angle α, where $\alpha \in (0;45^0)$. The projection sums are calculated with the same value (modulo) of positive and negative displacement relatively to the source of the coordinate system.

Due to fig.1, a $\rho = |-\rho| = OM = |-OM'|$, where $\rho \in \left(0; N\sqrt{2}\right)$.

For positive offsets the integration line forms the rectangular triangle ΔABC. For negative offsets it forms the rectangular triangle $\Delta A'B'C'$.

It should be proven that

- $\Delta ABC = \Delta A'B'C'$;
- the appropriate lengths of the line segments of the integration line in the image elements are equal;
- the coordinates of the line segments of the integration line in the image elements are determined using the same incidence matrixes for x and y coordinates.

1. ΔABC is a rectangular rectangle with $\angle ABC = 90^0$. Let's find the rest of angles.

$\angle E'OK' = \alpha$. The integration uses the same parallel displacement (offset), in other words $KK' \| AC \| C'A'$, whence $\angle OLS = \angle E'OK' = \angle E'L'K' = \alpha$. The line segments LO and AP are perpendiculars to axis Oy. So $LO \| AP$, whence

$\angle PAS = \alpha$. ΔPAS and ΔOLS are rectangular triangles. Since $\angle ASP = \angle LSO$ (the joint angle) and $\angle PAS = \angle OLS = \alpha$, then ΔPAS and ΔOLS are similar due to theorem about similarity of the triangles about three angles with the same measurements, in other words $\Delta PAS \infty \Delta OLS$.

Since $\angle PAS + \angle CAB = 90^0$, then $\angle CAB = 90^0 - \alpha$. The sum of angles of arbitrary triangle equals 180^0 , whence

$\angle BCA = 180^0 - \angle ABC - \angle CAB = = 180^0 - 90^0 - (90^0 - \alpha) = \alpha$.

2. Similarly the $\Delta A'B'C'$ is rectangular triangle with $\angle A'B'C' = 90^0$. Let's find the rest angles.

Since $KK' \| C'A'$, then $\angle E'OK' = \angle E'L'A' = \alpha$. For rectangular triangle $\Delta A'E'L'$ $\angle L'A'E' = 180^0 - \angle A'E'L' - \angle E'L'A'$, whence $\angle L'A'E' = 90^0 - \alpha$.

Since $E'L' \| B'C'$, then rectangular triangles $\Delta A'B'C'$ and $\Delta A'E'L'$ with joined angle $\angle L'A'E = 90^0 - \alpha$ are similar (due to the upper mentioned theorem of similarity) and $\angle E'L'A' = \angle B'C'A' = \alpha$.

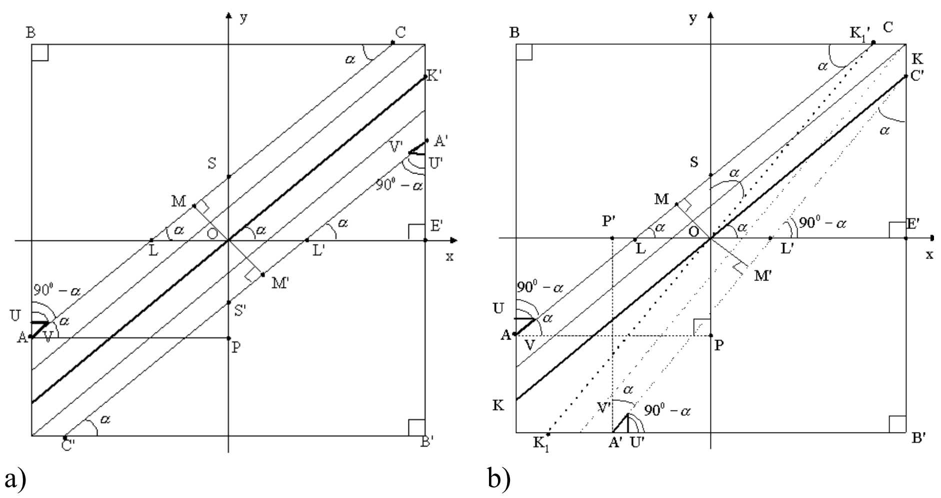

Fig. 1. The illustration for the proves of existence of central (a), rotational and rotational-mirrored (b) symmetries of SM

3. So $\Delta ABC \infty \Delta A'B'C'$ due to theorem of similarity of rectangular triangle with acute angle (in rectangular $\Delta S'OL'$ $\angle OL'S' = \angle EL'A' = \alpha$ as joint-vertical angles). Since $OM = |OM'|$ and the image is square, then $\Delta S'OL' = \Delta SOL$ and respectively $\Delta ABC = \Delta A'B'C'$.

Let's consider any point V , which is located on line segment AC between two adjacent image columns. The point U is the point on left image boundary of the perpendicular through point V to the axis Oy . The point U' is the point on right image boundary of the perpendicular through point V' to the axis Oy. Let's consider ΔAUV and $\Delta A'U'V'$. $\angle AVU = \angle A'V'U'$ are similar due to theorem about similarity of the triangles about three angles with the same measurements, in other words $\Delta AUV \infty \Delta A'U'V'$. Due to drawing $UV = U'V'$, in other words $\Delta AUV = \Delta A'U'V'$. The last formula is correct for any parallel offset of point V to the axis Oy on segment AC.

The same way we prove the existence of the central symmetry of the SM elements of the integration lines with $\alpha \in (45^0;90^0)$, $\alpha \in (90^0;135^0)$ and $\alpha \in (135^0;180^0)$, which are located on the same distances from the image source.

1.2. The rotational and rotational-mirrored symmetries

It should be proven that the appropriate line segments in image elements of integration line are equal for any angle $\alpha \in (0;45^0)$ and they will equal for in $\frac{\pi}{4}$. Also the incidence matrixes of appropriate coordinates will be equal too.

Again the projection sums are calculated with the same value (modulo) of positive and negative displacement relatively to the source of the coordinate system. Due to fig.1,b $\rho = |-\rho| = OM = |-OM'|$, where $\rho \in \left(0; N\sqrt{2}\right)$.

For any positive offset the integration line with angle α forms rectangular triangle ΔABC (fig.1, b). Another line with the same positive offset (relative to the source of the coordinate system) with angle $\frac{\pi}{2} - \alpha$ forms rectangular triangle $\Delta A'B'C'$. It should be proven that $\Delta ABC = \Delta A'B'C'$, the appropriate lengths of the integration line in the image elements are equal and the line segment coordinates are determined using the same incidence matrixes for x and y coordinates.

1. ΔABC is a rectangular triangle (as it was proven upper).

2. The same way $\Delta A'B'C'$ is a rectangular triangle with $\angle C'B'A' = 90^0$. Let's find the rest angles.

Since $K_1K_1' \| C'A'$, then $\angle E'OK_1' = \angle E'L'C' = 90^0 - \alpha$. For rectangular triangle $\Delta C'E'L'$ $\angle L'C'E' = 180^0 - \angle C'E'L' - \angle E'L'C'$, whence $\angle L'C'E' = \alpha$.

Since $E'L' \| B'A'$, then the rectangular triangles $\Delta A'B'C' \infty \Delta L'E'K'$ are similar due to theorem about similarity of the triangles about three angles. So, $\angle E'L'K' = \angle B'A'C' = 90^0 - \alpha$.

3. $\Delta ABC \infty \Delta A'B'C'$ are similar due to theorem of similarity of rectangular triangle with acute angle. Since $OM = |OM'|$ and the image is square, then $\Delta ABC = \Delta A'B'C'$.

The same way as for central symmetry it could be proven 1) $\Delta AUV = \Delta A'U'V'$; 2) the existence of the rotational and rotational-mirrored symmetries of the SM elements of the integration lines with $\alpha \in (45^0;90^0)$, $\alpha \in (90^0;135^0)$ and $\alpha \in (135^0;180^0)$ to the SM elements of the integration lines with $\alpha \in (0^0;45^0)$, which are located on the same distances from the image source.

2. The method of the fast calculation of the weighted Radon transformation

The method of the fast calculation of the weighted Radon transformation (WRT) was proposed. In this method the SM elements are calculated as a length of the integration line in image elements.

From the analysis of the geometry of integration it can be seen that for the arbitrary samples of offset and angles the line segments of integration lines s, limited by boundaries of the image elements, the coordinates of the beginnings and ends of these line segments, the lengths of line segments of integration lines s, limited by image sizes, and coordinates of the beginnings and ends of these line segments have properties of the central, rotational and rotational-mirrored symmetries. It should be mentioned again, that these symmetries are applicable for the image $N \times N$, where N is even and center of the coordinate system matches the image center. This allows to calculate and store less number of characteristics related to the geometry of integration. These characteristics are used by appropriate indexes.

The integration lines of the same angle with the same displacement (offset) relative the image center form the SM elements. These elements have central symmetry, which allows to calculate geometrical characteristics only for positive and zero offsets. This allows to reduce number of necessary characteristics in 2 times. The usage of the rotational and rotational-mirrored symmetries for different angle samples $\theta \in [0^0;180^0)$ allows to calculate the geometrical characteristics of the integration lines s only for $\theta \in [0^0;45^0]$, which decrease the number of necessary characteristics in 4 times.

The carried out researches have shown (fig. 2), that the usage of the symmetry properties in case of 2D interpolation of RT decreases the time of calculation in 2.8–5.7 times, in case of 1D interpolation it decreases the time of calculation in 2.4–3.6 times.

To compare the existing methods of computation of RT and HT some researches were carried out, in particular the dependence of time of calculation of PD as a function of image sizes for different interpolations of (ρ, θ) RT and HT (this interpolations are described in [Toft 1996]) was measured. The results of researches are presented on fig. 3. The charts 1–4 match to different optimization techniques of the nearest neighborhood interpolation of (ρ, θ) HT. The charts 5, 6 match to nearest neighborhood and linear interpolations of (ρ, θ) RT.

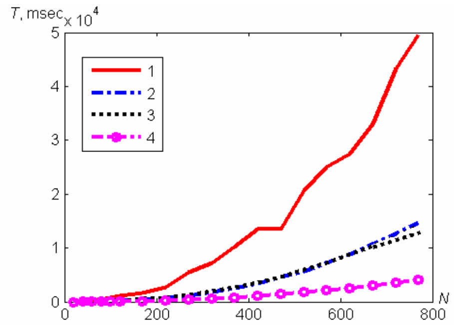

Fig. 2. The dependence of time of calculation of RT as function of the image sizes for linear 2D (1,2) and 1D (3,4) interpolations of coordinates for existing (1,3) and proposed (2,4) approaches.

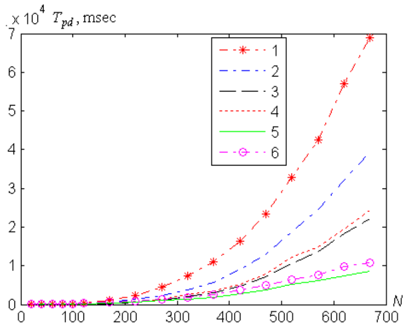

Fig. 3. The dependence of time of calculation of RT as function of the image sizes for different methods of interpolations of (ρ,θ) HT (1-4) and (ρ,θ) RT (5-6).

Conclusions

The new method of the fast calculation of the Radon transform with the usage of scheme calculation symmetry properties was proposed. It allowed to calculate and store 1/8 characteristics necessary for calculation of the full systems matrix. The speed of calculation of RT was increased in 2.4–3.6 times for 1D interpolation of coordinates and in 2.8–5.7 times for 2D interpolation. The non-weighted and weighted Radon and Hough transforms were implemented. The proposed method with the usage of the symmetry properties provides ability of the parallel calculation of 8 integrals. The implementation of this method in parallel systems will require minimal changes. It should be expected the increase of speed of calculation in 7.8 times compare to classic approaches. The theoretical increase of the speed in 8 times is not possible due to usage of several extra operations for determination of coordinates of current image elements.

Literature

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Abstract

The method of the fast Radon transform calculation whith uses properties of symmetry has been proposed. The scheme calculation symmetry properties were investigated. It allows to calculate less number of necessary characteristics. The carried out researches have shoun affectivity of proposed method, with essentially alloved to decrease time of Radon transform calculation.

Key words: transformation, integration line, rotational-mirrored symmetries, system matrix.

Metoda szybkich przekształceń Radona bazująca w obliczeniach na użyciu własności symetrii

Streszczenie

Zaproponowano metodę szybkiego przetwarzania Radona, która wykorzystuje własności symetrii. Zbadano schemat własności symetrii. Pozwala to obliczyć mniejszą ilość potrzebnych charakterystyk. Prowadzone badania pokazują efektywność zaproponowanej metody, która pozwala zmniejszyć czas obliczenia szybkiego przetwarzania Radona.

Słowa kluczowe: transformacja, linia integracji, obrotowa symetria obrazu, system macierzy.