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PATTANAIK'S AXIOMS AND THE EXISTENCE OF WINNERS PREFERRED WITH PROBABILITY AT LEAST HALF

SOMDEB LAHIRI*

School of Petroleum Management, Pandit Deendayal Energy University, Gujarat 382426, India

We show that three conditions due to Pattanaik, when satisfied by a given profile of state-dependent preferences (linear orders) on a given and fixed set of alternatives and a probability distribution with which the various states of nature occur, are individually sufficient, for the non-emptiness of the set of alternative(s) which are individually preferred to all alternatives other than itself with probability at least half. Before this, we show that since each axiom individually implies Sen-coherence, then, as a consequence of a result obtained earlier, each axiom along with asymmetry of the preferred with at probability at least half relation implies the transitivity of the relation. All the sufficient conditions discussed here are required to apply at least to all those otherwise relevant events that have positive probability. This observation also applies to a sufficient condition for the non-emptiness of the set of alternative(s) which are individually preferred to all alternatives other than itself with probability at least half, called generalised Sen coherence introduced and discussed in earlier research.

Keywords: *state-dependent preferences, preferred with probability at least half, fixed set of alternatives*

1. Introduction

In this paper, we are concerned with choosing an alternative from a given non-empty finite set of alternatives, when the material consequence of the chosen alternative is realised only after the uncertain future state of nature has been revealed. The preferences of the decision-maker are given by state-dependent rankings of alternatives, and information about the future that is known to the decision-maker is contained in a probability distribution over the non-empty finite set of states of nature that is revealed only after the choice has been made. This framework of analysis is discussed in [3] and [4]. This framework extends the Arrowian model of multi-criteria decision theory which along with significant and original theoretical contributions is available in [9]. For the

*Email address: somdeb.lahiri@gmail.com
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classical theory of decision making under uncertainty in the state-dependent case which is the other and major motivation behind this paper, one may refer to [2]. Other authors [2, 9] comfortably surpass the prerequisites related to decision making that is required to be able to understand the framework of analysis and results presented here. An informative overall perspective of decision theory can be found in [6]. State-dependent utility functions from a more advanced perspective are discussed in chapters 11 and 12 of [1]. However, the classical theory of decision-making/aiding under uncertainty that rests on the assumption of maximisation of expected utility (state-dependent or not) has an important limitation, namely that it fails to apply if the decision maker's preferences are not available in the form of cardinal utility functions, but only as rankings. That leads to a departure from the classical theory of decision aiding and opens up the possibility of decision-makers using other algorithms (decision aids) for decision-making under risk. One such procedure is what we are concerned about within the paper. The reasons for our interest in state-dependent preferences are precisely the same as the ones discussed in [2], i.e., it is so obviously true that it does not need a justification beyond citing trivial day-to-day examples as Karni did in his book.

A very simple and mundane example from [3] could be used to illustrate what we are concerned with here. Consider a person who, before going out to work in the morning, has to choose between a cap and a raincoat to carry with himself. Before going out to work, the sky appears to be clear but the weather could change when he has to return from work in the evening. There are three different possible states of nature that he may have to reckon with while returning from work: sunshine, rain, or cloudy skies without rain. In the first and third states of nature, he prefers a cap to a raincoat, and in the second state of nature, his preferences are opposite.

The first thing to notice regarding this example is that since the only available information is rankings, expected utility as a "decision aid" is completely useless, regardless of whether the person has probabilistic information about the weather in the evening or not. In case the person does have probabilistic information about the weather that he may have to confront in the evening, a reasonable rule for him to apply would be the "probabilistic plurality" or what may also be called "the most likely best alternative" rule, i.e., choose the alternative that is ranked first with the highest probability. This and the credibility problems associated with this rule are discussed in detail in [3], because of which we introduce and discuss thread-bare an alternative and a more credible decision aid, the preferred with probability at least half rule, i.e., the rule that selects one or more alternatives each of which is at least as likely to be preferred to an alternative than the latter is to it, for all other alternatives, i.e., to any other alternative. Choosing an alternative that is ranked above another alternative with a probability of at least half where the latter can be any other alternative appears to be a plausible algorithm that the decision-maker may use. However, as discussed in [3], it may happen that under certain circumstances such a "winner" may not exist. An example (using the probabilities and configuration used in Condorcet's paradox) whose context was suggested by an anonymous referee may be used to illustrate the example. The characters in this example are chosen so that while being realistic, the situation being discussed is impossible and hence completely uncontroversial.

Consider the election of a president for Japan (they are a monarchy with an elected government headed by a Prime Minister but they do not have a president. The three candidates for the presidency are Jimmy Carter (JC), Winston Churchill (WC), and Sun Yat-sen (Doc). Suppose that the relevant states of nature are: 1) status quo continues (i.e., the power relationships, trade and political constellations of major powers remain the same), 2) major conflict ensues between Japan and (PR) China, 3) a major change occurs in the non-political environment (e.g., a pandemic emerges). Further, the situation at election time is like the perfect; lull before a storm, so that all three states of nature appear to be equally likely. In the popular imagination, Jimmy Carter is viewed as a low profile and humane individual who is best suited to continue with the peacetime status quo, and Winston Churchill is the exact opposite. If there is a major conflict between Japan and (PR) China the best known Commander-in-Chief that Japan could imagine is Winston Churchill with the liberator (as opposed to warrior) SunYat-sen being at the other extreme. In the final case, where a pandemic (such as COVID-19) breaks out the best person to steer Japan through the crisis would be the formally qualified medical practitioner Sun Yat-sen and the next most qualified one for leading during an emergency would be Winston Churchill. Thus, there are there states of nature and three alternatives, the latter being JC, WC, and Doc. In the first state of nature a typical Japanese citizen would rank JC first, Doc second, and WC last. In the second state of nature the typical ranking would be WC first, JC second, and Doc last. In the third state of nature, the typical ranking would be Doc first, WC second, and JC third. Hence, if all three states of nature are perceived to be equally likely, the representative Japanese prefers JC to Doc with probability two-thirds, Doc to WC with probability two-thirds, and WC to JC with probability two thirds. Hence, no candidate is preferred in one-to-one comparison with a probability greater than half to all other candidates.

As in [3], the main result here provides sufficient (though not necessary) conditions under which for a given profile of linear orders (strict rankings, one for each state of nature) on a given (and fixed) set of alternatives, and a probability distribution with which various states of nature occur, there exists a preferred with probability at least half winner. Unlike [3], all the new sufficient conditions discussed here are available in the seminal work on majority rule on a fixed set alternative by Pattanaik in [5]. So far all research on this topic (including that in [5]) has been restricted to the situation where all states of nature are equally likely (equiprobable states of nature). We extend the investigation to situations where the probabilities with which the states of nature are realised may differ across states of nature. However, it ought to be emphasised that the discussion in [5] allows for state-dependent weak rankings, which is more general and includes the state-dependent strict rankings that we confine ourselves to, except in section 5 to our main proposition, where we indicate a possible proof to show that the corresponding results hold if we allow individual rankings with the possibility of ties. A detailed literature survey of related research for decision making under uncertainty with equiprobable states of nature can be found in [3]. The paper bt Pattanaik [5] is a noticeable and unforgivable omission in that literature survey. Whatever the reasons for such an unintentional omission, none can be considered as a justification for the error on the part of the author.

Elsewhere, the author shows [3] that Sen coherence – an axiom in [8] – along with asymmetry of the preferred with probability at least half relation implies its transitivity. Here we show that in the previous statement Sen coherence can be replaced by the three axioms due to Pattanaik – considered individually. Each of the three properties in [5] in Sen coherence, and hence the transitivity results implied by Pattanaik's axioms, can be considered to be a corollary of the corresponding result in [3].

Contrary to issues related to the transitivity or for that matter acyclicity of the preferred with probability at least half relation, in [3] it is shown that the satisfaction of a property, called generalised Sen coherence, guarantees the existence of a preferred with probability at least half winner. Here, we show that each of the three axioms in [5] achieves the same result and as in the case of generalised Sen coherence, and none of the three axioms is necessary for the existence of a preferred with probability at least half winner.

It should be admitted and taken note of by all readers of this and related research that without the lucid and transparent statement that Sen coherence is, the "mental gymnastics" requires to conceptualise sufficient conditions like "single-peakedness" and "single-dippedness" that were in existence till then is way beyond the intellectual abilities of this author and none of this research would be possible.

It may be of some interest to relate our framework of analysis to multi-criteria decision analysis. Yes, indeed, this is an example of multi-criteria decision analysis, where each criterion is a state of nature. The importance of a criterion, in this case, is the probability that one assigns to a particular state of nature that the criterion represents.

Our contribution here, as well as in related research, should be viewed in its proper perspective – as contributions to the theory of decision making/aiding under uncertainty and not as any contribution to decision theory which is discussed in detail in [1]. It may or may not be interesting to investigate whether and which rules of decision making/aiding are reasonable from the standpoint of decision theory. To give an example, suppose that rational behaviour in consumer demand theory is defined as budget-constrained utility maximisation. Hence, choosing the most preferred alternative from a budget set may not be rational unless the preferences have a numerical representation. Now, consider a consumer whose preferences are lexicographic, and this consumer's behaviour in the market is dictated by choosing the most (lexicographically) preferred alternative from his/her budget set. It is well-known that lexicographic preferences on the non-

-negative orthant of any Euclidean space are not representable by utility functions. Hence, the dictates of this consumer's market behaviour are not rational in the sense we have defined it in this paragraph. However, the consumer's market behaviour would consist of spending all of his/her wealth on the good which has "first priority" for him/her and this is also the behaviour of a budget-constrained utility maximising consumer whose utility of a basket of commodities is equal to the amount consumed of the first priority well. Thus, our lexicographic consumer's behaviour may be considered to be rational although what dictates the consumer's market behaviour is not. It is in this sense that the observed behaviour dictated by a decision aid may or may not be reasonable (acceptable?) from the point of view of decision theory.

2. Model

The motivation for the following discussion and the proposition thereafter comes from [8]. There is a lucid discussion of the same in [10].

Consider a decision-maker (DM) faced with the problem of choosing one or more alternatives from a non-empty finite set of alternatives X containing at least three alternatives.

If R is a binary relation on X (i.e., $R \subset X \times X$), then whenever $(x, y) \in R$ where $x, y \in X$, we write this simply as xRy.

Given a binary relation R on X, let P denote the asymmetric part of R (i.e., xPy if and only if xRy and not [yRx]), and I denote the symmetric part of R (i.e., xIy if and only if xRy and yRx). Further, let $G(X, R) = \{x \in X | xRy \text{ for all } y \in X\}$. $G(X, R)$ is said to be the set of greatest alternatives in the set X concerning the binary relation R. It is possible that $G(X, R)$ is empty.

Let $B(X, R) = \{x \in X | xRy \text{ for all } y \in X \backslash \{x\}\}$. $B(X, R)$ is said to be the set of best alternatives in the set X with respect to the binary relation R. It is easy to see that $G(X, R) = B(X, R) \cup \{x \in B(X, R | xRx\}$. Hence if R is reflexive, then $G(X, R) = B(X, R)$.

For a positive integer, $n \geq 3$, let $N = \{1, 2, ..., n\}$ denote the set of states of nature. The satisfaction from the chosen alternative is realised only after the state of nature reveals itself.

A preference relation/(weak) ranking on X is a reflexive, complete/connected/total and transitive binary relation on X. If for $x, y \in X$, it is the case that xRy, then we will say that x is at least as good as y. Similarly, xPy is interpreted as x is strictly preferred to y, and xIy is interpreted as there is indifference between x and y.

Given a (weak) ranking R on X, a non-empty subset Y of X and an alternative x in Y the rank of (alternative) x at/by R among Y denoted $rk^Y(x, R)$ = cardinality of $\{y \in Y | yPx\}) + 1$.

Given $x, y \in Y$ where Y is a non-empty subset of X and a ranking R on X, x is said to be ranked above (below) y at/by R among Y if $rk^Y(x, R) < (>) \ rk^Y(y, R)$.

Given a (weak) ranking R on X and an alternative x in X, $rk^X(x, R)$ is (for the sake of simplicity) denoted by $rk(x, R)$.

A strict preference relation/strict ranking on X is a reflexive, complete/connected/total, transitive and anti-symmetric binary relation (linear order) on X. A strict preference relation is, therefore, an antisymmetric preference relation/(weak) ranking. Since by definition a strict preference is a linear order, indifference between two alternatives is possible if and only if the two alternatives are identical.

Let $\mathcal{L}$ denote the set of all strict preference relations on X.

Note. Trivial but important observation that can be made at this point is that if R is a strict preference relation/strict ranking, then (i) for all $x \in X$, $\{y \in X|yRx\} = \{x\} \cup \{y \in X|yP(R)x\} = \{x\} \cup \{y \in X\backslash\{x\}|yRx\}$; (ii) for all $x \in X$, $\{y \in X|xRy\} = \{x\} \cup \{y \in X|xP(R)y\} = \{x\} \cup \{y \in X\backslash\{x\}|xRy\}$.

A preference profile denoted R_N is a function from N to $\mathcal{L}$. R_N is represented as the array $\langle R_i|i \in N\rangle$, where R_i is the strict preference relation/strict ranking of the DM in state of nature i. The set of all preference profiles is denoted $\mathcal{L}^N$.

The DM's beliefs or assessments about the possibility of the various states of nature being realised are summarised by a probability distribution, i.e., $p \in \mathbb{R}_+^N$ such that $\sum_{i=1}^{n} p_i = 1$. Let P^N denote the set of all probability distributions on N.

The following two concepts will prove important in what follows.

• Given a pair (R_N, p) consisting of a preference profile R_N and a probability distribution p, a property concerning states of nature for the pair is said to hold/be satisfied/be valid almost always if the property is satisfied for all $i \in N$ with $p_i > 0$.

• Given a pair (R_N, p) consisting of a preference profile R_N and a probability distribution p, a property concerning states of nature for the pair is said to hold/be satisfied/be valid almost never if the property is not satisfied when for $i \in N$ it is the case that $p_i > 0$.

Given a pair (R_N, p), define the preferred with probability at least half (PPALH) relation $R^\#(R_N, p)$ on X as follows: for all $x, y \in X$ with $x \neq y$, [$xR^\#(R_N, p)y$ if and only if $\sum_{\{i \in N|xR_iy\}} p_i \geq \sum_{\{i \in N|yR_ix\}} p_i$, i.e., $\sum_{\{i \in N|xR_iy\}} p_i \geq 1/2$]. It is clear from the definition of $R^\#(R_N, p)$, that this binary relation is irreflexive. However, it is complete, i.e., for all $x, y \in X$, with $x \neq y$, either $xR^\#(R_N, p)y$ or $yR^\#(R_N, p)x$ (and possibly sometimes both!).

From the note above (immediately) after the definition of strict rankings it is clear that for all $i \in N$ and $x, y \in X$ with $x \neq y$: xR_iy if and only if xP_iy.

If $xR^\#(R_N, p)y$, then we say that x is preferred with probability at least half to y.

Since n may be even with $p_i = 1/n$ for all $i \in N$, the inequality need not be strict.

When there is no scope for confusion, we will write $R^\#$ instead of $R^\#(R_N, p)$.

The asymmetric part of $R^{\#}$, denoted $P(R^{\#})$, is called the preferred with probability greater than half relation.

When there is no scope for confusion, we will write $P^{\#}$ instead of $P(R^{\#})$.

Note that $xP(R^{\#})y$ if and only if $\sum_{\{i \in N | xR_i y\}} p_i > \sum_{\{i \in N | yR_i x\}} p_i$. The latter naturally implies that $x \neq y$.

$xP^{\#}y$ then we say that x is preferred with probability greater than half to y.

We want to obtain condition(s) under which $B(X, R^{\#})$ is non-empty. An alternative in $B(X, R^{\#})$ is called a preferred with probability greater than half winner (PPALHW).

Note. If in the definition of $R^{\#}$ (R_N, p), we dropped the requirement that $x \neq y$, then by the reflexivity of strict rankings we would have got $xR^{\#}x$, and then, as we pointed out earlier, $B(X, R^{\#})$ would be equal to $G(X, R^{\#})$. However, "an alternative is preferred to itself with probability at least half" is an absurd statement and linguistically jarring. Hence, although our results would remain unaffected, we decided to introduce the caveat $x \neq y$.

3. Pattanaik's axioms and transitivity of PPALH relation

Transitivity of the PPALH relation clearly implies that there exists a PPALHW. In [10], there is a property on preference profiles referred to as Sen coherence, which, as shown in [8], implies the transitivity of the PPALH relation when the number of states of nature is odd and all are equally likely.

A pair (R_N, p) consisting of a preference profile R_N and a probability distribution p is said to satisfy Sen coherence if given any three distinct alternatives $x, y, z \in X$, there exists an alternative $w \in \{x, y, z\}$ such that either (i) w is almost never ranked first among $\{x, y, z\}$, i.e., $rk^{\{x,y,z\}}(w, R_i)$ is almost never equal to 1; or (ii) w is almost never ranked second among $\{x, y, z\}$, i.e., $rk^{\{x,y,z\}}(w, R_i)$ is almost never equal to 2; or (iii) w is almost never ranked third among $\{x, y, z\}$, i.e., $rk^{\{x,y,z\}}(w, R_i)$ is almost never equal to 3.

The proof of the following proposition is available in [3].

Proposition 1. Let (R_N, p) satisfy Sen coherence. Suppose $R^{\#}(R_N, p)$ is asymmetric. Then, $R^{\#}(R_N, p)$ is transitive.

Following the work by Sen in [8], the following two axioms were proposed by Pattanaik in [5].

A pair (R_N, p) consisting of a preference profile R_N and a probability distribution p is said to satisfy Pattanaik's first axiom if given any three distinct alternatives $x, y, z \in X$, there exists an alternative $w \in \{x, y, z\}$ such that w is almost always ranked first among $\{x, y, z\}$.

A pair (R_N, p) consisting of a preference profile R_N and a probability distribution p is said to satisfy Pattanaik's second axiom if given any three distinct alternatives $x, y, z \in X$, there exists an alternative $w \in \{x, y, z\}$ such that w is almost always ranked second among $\{x, y, z\}$.

A pair (R_N, p) consisting of a preference profile R_N and a probability distribution p is said to satisfy Pattanaik's third axiom if given any three distinct alternatives $x, y, z \in X$, there exists an alternative $w \in \{x, y, z\}$ such that w is almost always ranked last among $\{x, y, z\}$.

Since each axiom due to Pattanaik implies Sen coherence, the following result, which says that each axiom along with asymmetry of $R^\#$ is sufficient to guarantee the transitivity of $R^\#$, is an immediate Corollary of Proposition 1.

Corollary of Proposition 1. Let (R_N, p) satisfy at least one of the three Pattanaik's axioms. Suppose $R^\#(R_N, p)$ is asymmetric. Then, $R^\#(R_N, p)$ is transitive.

4. Pattanaik's axioms and the existence of a PPALHW

The purpose of this section is to show that in the absence of asymmetry Pattanaik's axioms considered individually are sufficient to ensure that $B(X, R^\#)$ is non-empty though not the transitivity of $R^\#$.

The following example illustrates that $B(X, R^\#)$ may be non-empty even though $R^\#$ is not transitive.

Example 1: $X = \{y_1, y_2, y_3, y_4\}$ and $n = 3$. Suppose $p_k = 1/3$ for all $k \in \{1, 2, 3\}$. Let $y_4P_1y_1P_1y_2P_1y_3$, $y_4P_2y_3P_2y_1P_2y_2$, $y_4P_3y_2P_3y_3P_3y_1$. Here we have $y_1P^\#y_2P^\#y_3P^\#y_1$ and hence $R^\#$ is not transitive. In fact, it is not even acyclic. However, $B(X, R^\#) = \{y_4\}$.

In [3] it is shown that a property called generalised Sen coherence is sufficient to ensure that $B(X, R^\#)$ is non-empty.

Here we show that Pattanaik's axiom considered individually ensures that $B(X, R^\#)$ is non-empty.

Recall the definition of a rank.

Given a (weak) ranking R on X the rank of (alternative) x at/by R denoted $rk(x, R)$ = cardinality of $\{y \in X|yPx\}) + 1$.

Proposition 2. Suppose (R_N, p) satisfies at least one of the three Pattanaik's axioms. Then $B(X, R^\#(R_N, p))$ is non-empty. However, the converse need not hold for any of the three Pattanaik's axioms, i.e., there exists (R_N, p) such that $B(X, R^\# (R_N, p))$ is non-empty and yet (R_N, p) does not satisfy any of the three Pattanaik's axioms.

Proof. First, let us show that the respective converses are not true. Let $X = \{x_1, x_2, x_3\}$, $n = 4$ and $p \in P^N$ with $p_i = 1/4$ for all $I \in N$. Let $x_1P_1x_2P_1x_3$, $x_3P_2x_1P_2x_2$, $x_2P_3x_3P_3x_1$, and

$x_1P_4x_2P_4x_3$. Clearly, all three axioms of Pattanaik are violated by (R_N, p). However, $B(X, R^{\#}) = \{x_1\}$.

Now, let us show that if (R_N, p) satisfies Pattanaik's first axiom, then $B(X, R^{\#})$ is non-empty.

Hence, suppose that some (R_N, p) satisfies Pattanaik's first axiom and towards a contradiction suppose $B(X, R^{\#})$ is empty.

Consider any state of nature which occurs with positive probability, say state of nature 1, and without loss of generality suppose x is ranked first in state of nature 1. Note $p_1 > 0$.

Since $B(X, R^{\#})$ is empty, for all $w \in X$ $\sum_{\{i\in N|xR_i w\}} p_i < 1/2$. Thus, $\{i \in N|p_i > 0$ and x is not ranked first in state of nature i$\} \neq \phi$ and for x not ranked first in state of nature i $\sum_{\{i\in N|p_i>0\}} p_i > 1/2$.

Suppose $y \in X\setminus\{x\}$ such that y is ranked first in some state of nature with positive probability. Without loss of generality this state of nature is 2. Thus, $p_2 > 0$.

If y is ranked first in every state of nature that occurs with positive probability and which does not rank x first (i.e., y is ranked first in every state of nature belonging to $\{i\in N \mid p_i > 0$ and x is not ranked first in state of nature $i\}$, then for $\{i \in N|y$ ranked first in state of nature $i\}$ $\sum_{i\hat{I}N|y} p_i > 1/2$ leading to $y \in B(X, R^{\#})$ and a contradiction.

Thus, there exists $z \in X\setminus\{x, y\}$ and a state of nature different from that which occurs with positive probability but is different from both 1 and 2, that ranks z first.

Thus, there does not exist $w \in \{x, y, z\}$ such that w is ranked first among $\{x, y, z\}$ almost always.

Clearly $\{x, y, z\}$ violates Pattanaik's first axiom and proves the first part of the proposition.

The above proof shows that if $B(X, R^{\#})$ is empty, then there exist three distinct alternatives x, y, z, and three states of nature, say 1, 2, 3, all of which occur with positive probability such that x is ranked first among $\{x, y, z\}$ in state of nature 1, y is ranked second among $\{x, y, z\}$ in state of nature 2, and z is ranked first among $\{x, y, z\}$ in state of nature 3.

Thus, there does not exist $w \in \{x, y, z\}$ such that w is ranked second among $\{x, y, z\}$ almost always. This violates Pattanaik's second axiom.

Further, there does not exist $w \in \{x, y, z\}$ such that w is ranked second. Thus, there does not exist $w \in \{x, y, z\}$ such that w is ranked last among $\{x, y, z\}$ almost always among $\{x, y, z\}$. This violates Pattanaik's Third Axiom.

This proves the proposition. Q.E.D.

5. State-dependent rankings with the possibility of ties and Pattanaik coherence

In the case of weak rankings (i.e., rankings with the possibility of ties), choose any alternative x which come first with highest probability, and let M be the set of states of nature where x comes first. Since x comes first with highest probability, the probability of the event $M = \{h \in N | p_h > 0$ and x is ranked first in state of nature $h\}$ is greater than zero. Since $B(X, R^{\#})$ is assumed to be empty, the probability of M is less than 1/2. Hence and since the probability of the event $N \backslash M$ is greater than 1/2, then there exists an alternative $y \in X \backslash \{x\}$ such that the event $\{h \in N \backslash M | p_h > 0, y$ is ranked first at $h\}$ has a positive probability that is greater than or equal to the probability of the event $\{h \in N \backslash M | p_h > 0, w$ is ranked first at $h\}$ for all $w \in X \backslash \{x\}$. Let $M_1 = \{h \in N \backslash M |$ y is ranked first at $h\}$. Since $B(X, R^{\#})$ is assumed to be empty, probability of M_1 is less than 1/2 which, in turn, is less than the probability of $\{h \in N \backslash M | p_h > 0\}$. Hence, M_1 is a proper subset of $\{h \in N \backslash M | p_h > 0\}$. Let $k \in N \backslash (M \cup M_1)$ and $z \in X \backslash \{x, y\}$ such that z is ranked first at k. Clearly, neither x nor y is ranked first at k. If z is ranked first at all $h \in M_1$, then the assumption that y is ranked first with highest probability, conditional on the state of nature belongs to $N \backslash M$, stands contradicted. Hence, let $j \in M_1$ such that y is ranked first at j, but z is not. Clearly, x is not ranked first at j either. Hence, there exist distinct states of nature j, $k \in N \backslash M$ with $p_j > 0$ and $p_k > 0$ such that y but not z or x is ranked first in state of nature j, and z but not y or x is ranked first in state of nature k. Since x is an alternative ranked first with highest probability it is not possible for y to be ranked first in all states of nature in M that occur with positive probability and it is not possible for z to be ranked first in all states of nature in M that occur with positive probability. Let $i \in M$ be such that $p_i > 0$ and y is not ranked first in state of nature i. Clearly, x is ranked first at i.

Then, there does not exist any $w \in \{x, y, z\}$ such that w is ranked first almost always among $\{x, y, z\}$ and each $w \in \{x, y, z\}$ is ranked first among $\{x, y, z\}$ at least once with positive probability. This should prove the corresponding version of Proposition 2, if we allowed for ties in individual rankings.

The following axiom which we may refer to as Pattanaik coherence is implied by the three axioms of Pattanaik that we have been discussing so far.

A pair (R_N, p) consisting of a preference profile R_N and a probability distribution p is said to satisfy Pattanaik coherence if there do not exist three distinct states of nature $i, j, k \in N$ with $p_i > 0$, $p_j > 0$, $p_k > 0$, and three distinct alternatives $x, y, z \in X$ such that $x \in G(X, R_i) \neq \{x, y, z\}$, $y \in G(X, R_j) \neq \{x, y, z\}$, $z \in G(X, R_k) \neq \{x, y, z\}$ and $G(X, R_i) \cap G(X, R_j) \cap G(X, R_k) \cap \{x, y, z\} = \phi$.

What Pattanaik coherence says is that there do not exist three different states of nature, each of which occurs with positive probability and three distinct alternatives satisfying the following three properties:

- All three alternatives are ranked first at least once (perhaps not simultaneously) by the three states of nature.
- At none of the three states of nature are all three alternatives ranked first simultaneously.
- None of the three is ranked first in all three states of nature.

While the above discussion and the counterexample at the beginning of the proof indicates that the following proposition holds even if we allow the possibility of state-dependent rankings with ties, we will restrict the formal statement of the proposition to the case of strict rankings so that Proposition 2 and its proof is applicable.

Proposition 3. Suppose (R_N, p) satisfies Pattanaik coherence. Then, $B(X, R^{\#}(R_N, p))$ is non-empty. However, the converse need not hold, i.e., there exists (R_N, p) such that $B(X, R^{\#}(R_N, p))$ is non-empty and yet (R_N, p) does not satisfy Pattanaik coherence.

In the case of state-dependent rankings, it is not possible for more than one alternative to be ranked first in any state of nature, and the assumption that each alternative is ranked first at least once rules out the possibility of any alternative being ranked first in all three states of nature.

6. Generalised Sen coherence and the existence of a PPALHW

A property called generalised Sen coherence is shown to be sufficient though not necessary for the existence of a PPALHW [3]. We propose here a slightly more restrictive version of the same property and refer to it by the same name as earlier, since the difference is marginal. An almost identical proof as the one for Proposition 2 in [3] shows that this new property is also sufficient, though not necessary, for the existence of a PPALHW.

Given a pair (R_N, p) consisting of a preference profile R_N and a probability distribution p, an (R_N, p)-cycle is a list of $K - 1 \geq 2$ distinct alternatives $<x_1, ..., x_{K-1}>$ $x_K = x_1$ (not necessa rily distinct) states of nature $i_1, ..., i_K \in N$ with $p_{j_k} > 0$ for all $k \in \{1, ..., K\}$ and $k \in \{1, ..., K - 2\}$ such that: a) $x_{j+2} P_{i_j} x_{j+1} P_{i_j} x_j$ for all $j \in \{1, ..., \; - 2\}$; b) $x_2 P_{i_{K-1}} x_K P_{i_{K-1}}) x_{K-1}$ and $x_3 P_{i_K} x_2 P_{i_K} x_K$.

More formally: Given a pair (R_N, p) consisting of a preference profile R_N and a probability distribution p, an (R_N, p)-cycle is a triplet $(<x_1, ..., x_K>, (i_1, ..., i_K), k)$ for some $K \geq 3$ satisfying the following properties:

(i) for $i, j \in \{1, ..., K - 1\}$: $x_i \neq x_j$, $x_K = x_1$;

(ii) $i_j \in N$ and $p_{i_j} > 0$ for all $j \in \{1, ..., K\}$;

(iii) $k \in \{1, ..., K-2\}$;

(iv) $x_{j+2} P_{i_j} x_{j+1} P_{i_j} x_j$ for all $j \in \{1, ..., K-2\}$;

(v) $x_2 P_{i_{K-1}} x_K P_{i_{K-1}} x_{K-1}$;

(vi) $x_3 P_{i_K} x_2 P_{i_K} x_K$.

The reason why we refer to the above as an (R_N, p)-cycle is because while depicting expressions like $x_{j+2} P_{i_j} x_{j+1}$ on a piece of paper, if one begins at $x_k P_{i_{K-1}} x_K$ and moves clock-wise through x_K, x_{K-1}, ..., x_{k-1} and returns to x_k, then the resulting figure will then look like the lower case Greek letter sigma (σ) which ends with $x_k P_{i_K} x_1$.

$x \in X$ is said to be a member of an (R_N, p)-cycle if there exists a (R_N, p)-cycle $\langle x_1, ..., x_K \rangle$ with $x_1 = x$.

A preference profile R_N is said to satisfy generalised Sen coherence (GSC) if $\{x \in X | x$ is not is not a member of any (R_N, p)-cycle$\}$ is non-empty.

Note. In example 1, $B(X, R^{\#}) = \{y_4\} = \{x \in X | \, x$ is not a member of any $(R_{\{1,2,3\}}, p)$--cycle$\}$ and so generalised Sen coherence is satisfied, although $R^{\#}$ is not transitive – in fact not even acyclic.

Proposition 4. Suppose (R_N, p) satisfies generalised Sen coherence. Then $B(X, R^{\#}(R_N, p))$ is non-empty. However, the converse need not hold, i.e., there exists (R_N, p) such that $B(X, R^{\#}(R_N, p))$ is non-empty and yet (R_N, p) does not satisfy generalised Sen coherence.

7. Conclusion

The theory and applications of decision aiding is a component (or, if I may dare say, the essence) of operations research. When preferences are available in the form of state--dependent utility functions, decision aiding adopts the criteria recommended by decision theory, i.e., maximisation of expected utility. The genesis of this subject area of study can be traced back to even before the seminal work of Howard Raiffa, i.e., [6] appeared, giving it a name of its own, namely decision analysis. Hence, decision analysis is a branch of operations research. In general, it “works” and is robust enough to survive the occasional and well-known counterexamples (like early morning starting problems that cars sometimes face, particularly during winter) that any well-established theory has to be prepared to face. This is the classical version of decision analysis based on state-dependent utility functions. Classical decision analysis has nothing to say if preferences are not available as utility functions. It is precisely for situations where instead of state-dependent utility we have a state-dependent ranking of alternatives, thus requiring much less information than in classical analysis, we propose an algorithm that individuals could use to make decisions under probabilistic uncertainty risk.

The starting point of subjective expected utility maximisation in the sense of John C. Harsanyi consists of preferences on state-dependent utility vectors based on which probability distributions over states of nature are obtained. In the absence of state-dependent utility vectors, Harsanyi's SEU has nothing to say about probabilities. In fact, closer to our line of investigation are the definitions of probability as "betting quotients" of risk-neutral agents proposed independently by Bruno de Finetti in 1937, and by Frank P. Ramsey in 1926. It is precisely for situations where instead of state-dependent utility we have a state-dependent ranking of alternatives, thus requiring much less information than in classical analysis, we propose an algorithm that individuals could use to make decisions under probabilistic uncertainty risk.

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