SOFTWARE TOOLS
IN DIDACTICS OF MATHEMATICS

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A picture is worth a thousand words
Chinese proverb

Abstract. The use of software tools in the teaching process allows us to enrich the traditional methods. Graphics and animation complement the text and create positive associations related to the presented content. Recent research shows that using the visual methods in teaching leads to better scores obtained by students. The aim of the paper is to present the dynamic visualisations of selected concepts taught in mathematics and other "quantitative" subjects at university. Our dynamic visualisations can be used during lectures to help students to better understand difficult ideas and dependencies, for example the derivate of a function, the concept of fuzzy sets and the operations on these sets as well as the concept of the best solution in multi-objective programming problems. Visualisations of these issues are prepared in GeoGebra, which combines algebra and geometry and allows for the dynamic visualisation of concepts with a mathematical background.

Keywords: visualisations, GeoGebra, multi-objective programming, fuzzy sets, derivative.

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1. Introduction

Computers are now an integral part of our lives, accompanying us at home, school and work. From an early age, children participate in computer classes, and the use of IT tools in the teaching process becomes quite natural for them. Common access to a variety of educational software and websites helps in gaining knowledge, and dynamic visualisations created by means of IT tools facilitate the transfer of knowledge as well as the explanation of concepts and various relationships. Visualization is a tool for analysis, selection and perception of the phenomena, facts and trends presented using graphical methods. Visualization can also be understood as the processing

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of complex relationships into a form which is readable for the customer (using the mechanism of perception) with the use of graphic elements [Dudycz 1998]. Empirical studies revealed that people perceive up to 87% of the information by sight, 10% of the information by hearing, and only 3% of the information by other senses [Niemann et al. 1994]. Graphics and animation complement the text and create positive associations related to the presented content. Verbal information enriched with appropriate graphics is better understood and remembered than the same message presented only in spoken form [Kozak, Łaguna 2009].

The aim of the paper is to present the dynamic visualisations of selected concepts taught in mathematics and other "quantitative" subjects at university. Our visualisations can be used during lectures to help students to better understand difficult ideas and dependencies, for example the derivate of a function, the concept of fuzzy sets and the operations on these sets as well as the concept of the best solution in multi-objective problems. Visualisations of these issues are prepared in GeoGebra, which is "a multi-platform mathematics software that gives everyone the chance to experience the extraordinary insights that math makes possible" [GeoGebra]. It combines algebra and geometry and allows for the dynamic visualisation of concepts with a mathematical background.

2. Visualisation of multi-objective programming problem

Decision-making practice abounds in problems involving several, often conflicting, objectives, e.g. to achieve maximum profit while minimizing costs. Each objective corresponds to the criterion that allows us to measure the level of achievement of the objective. In further discussions, without a loss of generality, we assume that the decision maker, making the feasible decision $\mathbf{x} = [x_1, x_2, ..., x_n]^T \in S$ (where S is a set of feasible solutions in the decision space $\mathbf{R}^n$), wants to maximise[1] the achievement of all objectives $f_1, f_2, ..., f_m$. We also assume that the objectives are measurable, which means that they can be expressed by numerical functions (in particular linear functions). The evaluation of a decision $\mathbf{x}$ is given by the vector $f(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_m(\mathbf{x})]^T \in Z$ where Z is a set of outcomes (which is a subset of criterion space). Multi-objective programming problem can be formulated as follows [Sikora (ed.) 2008]

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[1] If some objective function is to be minimised, it is equivalent to maximising its negative.

$$f(\mathbf{x}) = \mathbf{Cx} \to \max$$

$$\mathbf{Ax} \le \mathbf{b} \qquad , \qquad (1)$$

$$\mathbf{x} \ge 0$$

where $\mathbf{C}$ is the matrix of coefficients of the criteria. Each row of this matrix corresponds to one criterion.

The solution of the problem (1) involves the determination of the best decision in the feasible set S, considering the given set of criteria. The existence of one optimal solution, i.e. the solution which is better than other feasible solutions considering all the objectives, is very rare. The objectives in a multi-objective problem are usually conflicting, which means that improvement in one criterion is possible only as a result of the deterioration in the value of another criterion. If one optimal solution does not exist, it is sufficient to find Pareto optimal (efficient) solutions, thereby foregoing the simultaneous maximisation of all the criteria. An efficient solution is a solution for which it is not possible to improve the value of any of the criteria without the deterioration of at least one other criterion [Steuer 2001]. The Pareto optimal solutions in the decision space correspond to the Pareto front (Pareto curve) in the criterion space. A Pareto front is formed by the so called non-dominated points.

The following part of this section presents an example of the visualisation of the solution of the multi-objective programming problem[2].

Decision problem: A company produces two products P1 and P2. The unit revenues are \$3 and \$1, respectively. Two raw materials M1 and M2, used in production have respective daily availabilities of 10 units and 19 units. The preparation of one unit of product P1 requires 2 units of M1 and 1 unit of M2 and one unit of P2 requires 1 unit of M1 and 4 units of M2. Moreover, demand for P1 is not greater than 6 units, and for P2 is not greater than 4.5 units. Determine a plan of production P1 and P2 which will ensure maximal total revenue and maximal total volume of production.

Visualisation of solution: The model of the considered decision problem is written in the following form:

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[2] An example of the use of GeoGebra in solving the one-objective decision problem is presented in [Dudzińska-Baryła, Michalska 2011].

$$f(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} \rightarrow \max$$

$$2x_1 + x_2 \leq 10$$

$$x_1 + 4x_2 \leq 19 \qquad . \qquad (2)$$

$$x_1 \leq 6, \; x_2 \leq 4.5$$

$$x_1 \geq 0, \; x_2 \geq 0$$

Figure 1 presents the visualisation of the solution: the feasible set in the decision space and the set of outcomes in the criterion space. Moreover, the Pareto front and corresponding Pareto optimal (efficient) solutions are marked in appropriate spaces. The decision problem of the form (2) does not have one optimal solution. Non-dominated points are located on segment $\overline{F'M'}$ where $F' = (13, 7)$ and $M' = (15, 5)$, and the corresponding Pareto optimal solutions are located on segment $\overline{FM}$ where $F = (3, 4)$ and $M = (5, 0)$. Moving from F' to M', the value of the first criterion increases while at the same time the value of the second criterion decreases. Thus this decision problem does not have one best production plan (taking into consideration both criteria). In practice one of the Pareto optimal solutions is taken as a solution of such decision problems.
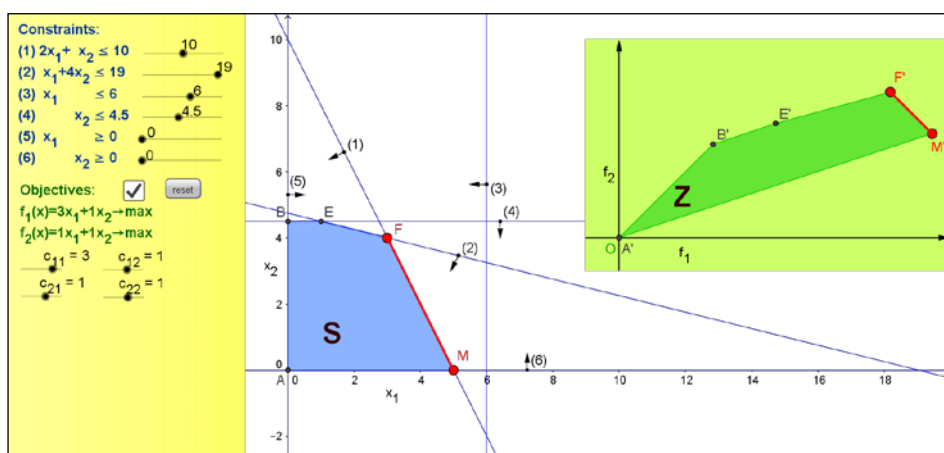

Fig. 1. Pareto optimal solutions in decision space and Pareto front in criterion space

Source: own elaboration.

Our visualisation in GeoGebra is prepared in such a way that it allows to observe the influence of the change of some parameters (constants in constraints and coefficients in objectives) on the feasible set and Pareto front and further on the optimal or efficient solutions. In our production problem, let us consider the change in the unit revenue of product P1 from \$3 to \$2. The visualisation of the modified problem is shown in Figure 2. Such a change does not affect the feasible set but only affects its image in the criterion space. Now our problem has one optimal solution in point $F = (3, 4)$ and dominant point (values of both criteria) are represented by point $F' = (10, 7)$. Point $M = (5, 0)$ is no longer the Pareto optimal solution.
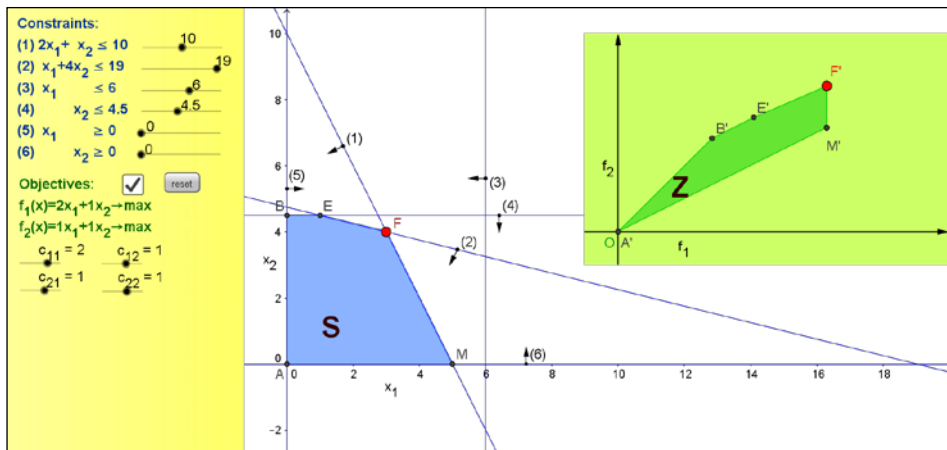

Fig. 2. Optimal solution in decision space and dominant point in criterion spaces

Source: own elaboration.

The use of sliders and properly defined dependencies in our visualisation in GeoGebra allows us to make the sensitivity analysis and to present the influence of some changes on the feasible set, outcome set and optimal or Pareto optimal solutions.

3. Operations on fuzzy sets

Fuzzy sets are used in, among others, decision-making under uncertainty, multistage decision-making and group decision-making [Ostasiewicz 1986] as well as financial analysis [Sojda 2012], control theory and artificial intelligence [Rutkowski 2006]. The universality of the practical applications of fuzzy sets results in the growing interest in this subject. We have devel-

oped teaching materials related to fuzzy sets. Our worksheets are a valuable aid for students of economic and information technology studies.

The aim of the following part is to propose the graphical presentation of the concept of fuzzy sets and some operations defined in fuzzy sets.

A fuzzy set A in non-empty space X, written as $A \subset X$, is a set of pairs (x, μ_A(x)) where μ_A is a membership function which associates with each point in X a real number in the interval [0,1] [Zadeh 1965]. μ_A represents the "grade of membership". $\mu_A = 1$ means that x is fully included, $\mu_A = 0$ – element x is not included. The values between 0 and 1 characterize fuzzy members, which belong to the fuzzy set only partially.

The most common classes of the membership function are: class t, class γ, class L and Gaussian [Sherer 2012].

Membership function of class t is defined as follows

$$\mu_A(x;a,b,c) = \begin{cases} 0 & \text{for } x \le a \\ \dfrac{x-a}{b-a} & \text{for } a < x \le b \\ \dfrac{c-x}{c-b} & \text{for } b < x \le c \\ 0 & \text{for } x > c \end{cases}, \tag{3}$$

where *a*, *b* and *c* are parameters of the function.

Membership function of class γ is given by the formula

$$\mu_A(x;a,b) = \begin{cases} 0 & \text{for } x \le a \\ \dfrac{x-a}{b-a} & \text{for } a < x \le b \\ 1 & \text{for } x > b \end{cases}, \tag{4}$$

where *a* and *b* are parameters of the function.

Membership function of class L is defined as follows

$$\mu_A(x;a,b) = \begin{cases} 1 & \text{for } x \le a \\ \dfrac{b-x}{b-a} & \text{for } a < x \le b \\ 0 & \text{for } x > b \end{cases}, \tag{5}$$

where *a* and *b* are parameters of the function.

Gaussian membership function is described by the formula

$$\mu_A(x;a,b) = \exp\left(-\left(\frac{x-a}{b}\right)^2\right), \tag{6}$$

where a (mean) and b (standard deviation) are parameters of the Gaussian function.

Basic operations defined for fuzzy sets include: union, intersection, algebraic product and Cartesian product of two fuzzy sets.

The union of two fuzzy sets A1 and A2 with respective membership functions μ_{A1} and μ_{A2} is a fuzzy set whose membership function is related to those of A1 and A2 by [Zadeh 1965]

$$\mu_{A1 \cup A2}(x) = \max\{\mu_{A1}(x), \mu_{A2}(x)\}. \tag{7}$$

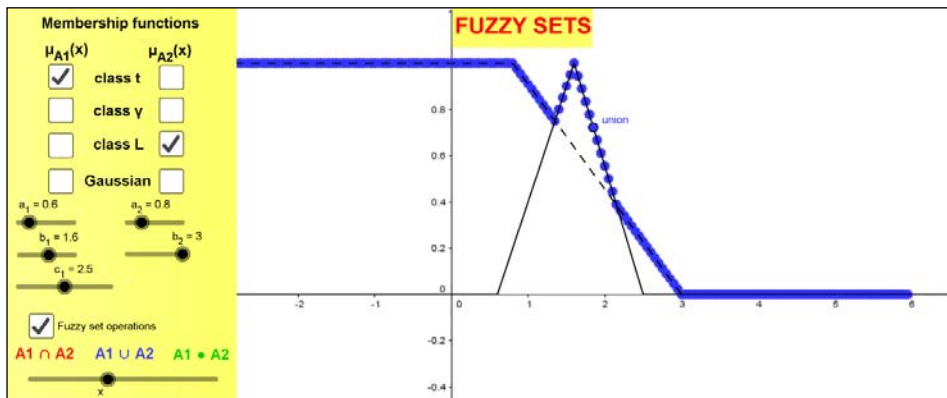Figure 3 shows the visualisation of the union of fuzzy sets A1 and A2 with the membership function of class t and L, respectively.

Fig. 3. Union of fuzzy sets with membership function
of class t (for A1) and class L (for A2)

Source: own elaboration.

The intersection of two fuzzy sets A1 and A2 with respective membership functions μ_{A1} and μ_{A2} is a fuzzy set whose membership function is related to those of A1 and A2 by [Zadeh 1965]

$$\mu_{A1 \cap A2}(x) = \min\{\mu_{A1}(x), \mu_{A2}(x)\}. \tag{8}$$

Figure 4 shows the visualisation of the intersection of fuzzy sets A1 and A2 with the membership function of class t and γ, respectively.
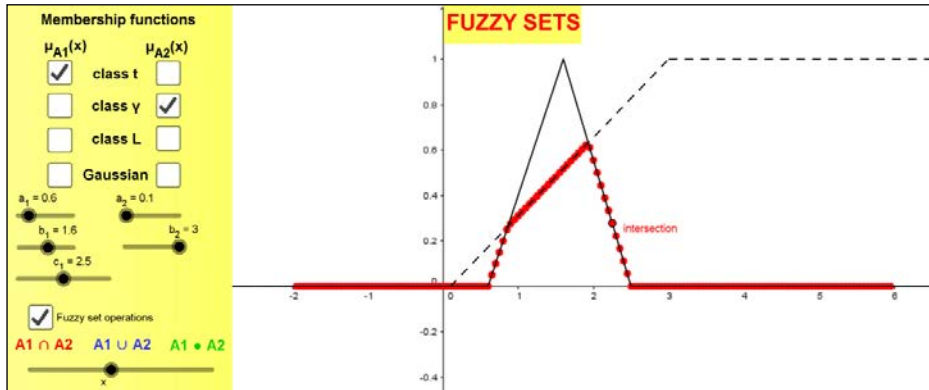
Fig. 4. Intersection of fuzzy sets with membership function
of class t (for A1) and class γ (for A2)

Source: own elaboration.

The algebraic product of A1 and A2 is defined in terms of the membership functions of A1 and A2 by the relation [Zadeh 1965]

$$\mu_{A1\cdot A2}(x) = \mu_{A1}(x)\cdot\mu_{A2}(x). \qquad (9)$$

Figure 5 shows the visualisation of the algebraic product of fuzzy sets A1 and A2 with the membership function of class t and Gaussian, respectively.
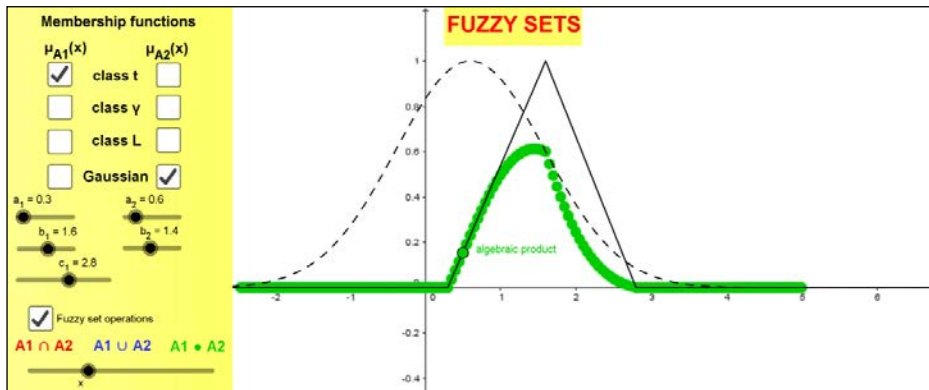

Fig. 5. Algebraic product of fuzzy sets with membership function
of class t (for A1) and Gaussian (for A2)

Source: own elaboration.

The Cartesian product of A1 and A2, where $A1 \subset X$ and $A2 \subset Y$, can be defined in terms of the membership functions as follows

$$\mu_{A1\times A2}(x, y) = \mu_{A1}(x)\cdot\mu_{A2}(y). \qquad (10)$$

Figure 6 shows the visualisation of the Cartesian product of fuzzy sets A1 and A2 with both Gaussian membership functions in a three-dimensional plane.
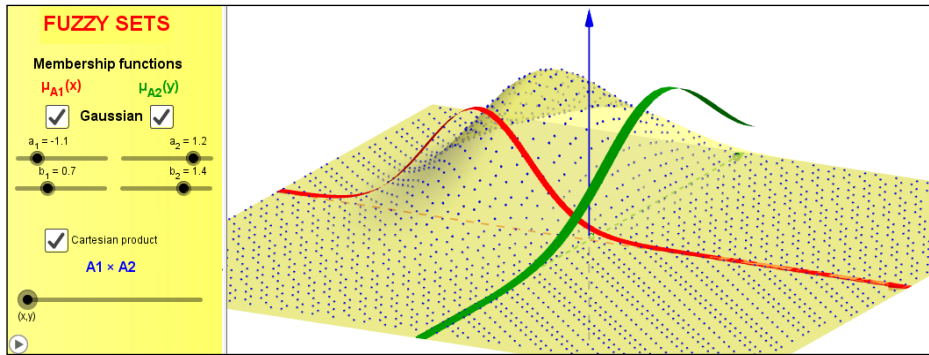

Fig. 6. Cartesian product of fuzzy sets with both Gaussian membership functions in 3D

Source: own elaboration.

All the figures above are our proposition of the visualisation of fuzzy sets and some operations on them. All the parameters of classes of membership functions can be changed by the lecturer and the influence of this change on membership functions of input fuzzy sets as well as the membership function of the output fuzzy set can be observed immediately. Such dynamic visualisations are very useful and can enrich his or her lecture and can lead to a better understanding of the presented issues.

4. Geometric interpretation of the derivative of a function

One of the more important issues in economics is a marginal analysis which uses differential calculus. Therefore students of economic studies should have a thorough knowledge and understanding of the concept of a derivative and its interpretation [Samuelson, Marks 2003; Bednarski 2004].

The geometric interpretation of a derivative of a function can be presented using a simple example of the quadratic function of the form $f(x) = ax^2 + bx + c$. In Figure 7 we assume $f(x) = 0.7x^2 - 0.8x + 1$. Let us take two points of that function $P_0 = (x_0, f(x_0))$ and $P = (x_0 + \Delta x, f(x_0 + \Delta x))$ (here $x_0 = 1.3$ and $\Delta x = 0.9$). The difference quotient of function $f(x)$ at point x_0 is defined as the quotient of the difference of values of the function by the increase in its arguments:

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{11}$$

This quotient is a slope of secant line P_0P as well as the tangent of an angle β which is the angle between the secant line and positive *x*-axis (see Figure 8).
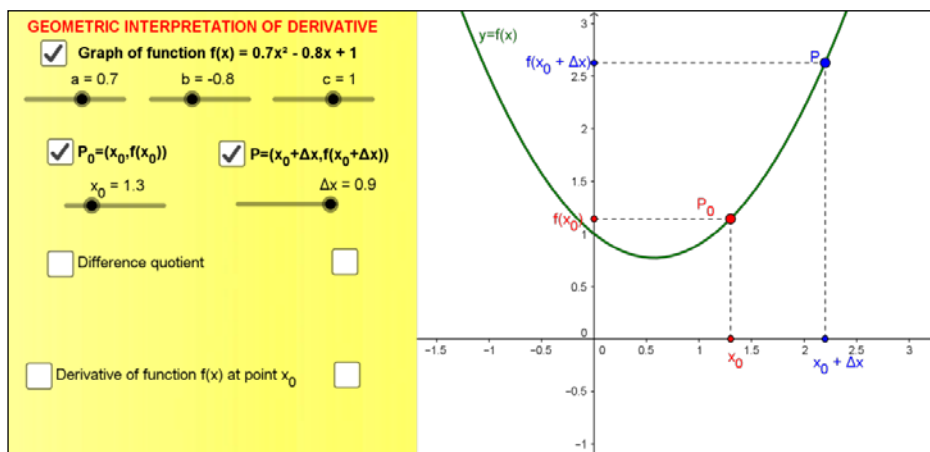

Fig. 7. Quadratic function *f(x)* with two points P_0 and P
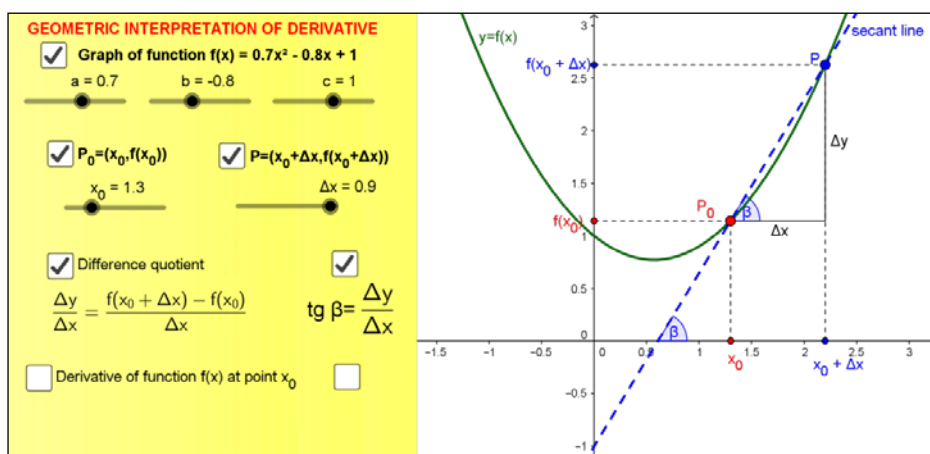
Source: own elaboration.

Fig. 8. Difference quotient

Source: own elaboration.

The derivative of the function is defined as the limit of the difference quotient as $\Delta x \to 0$. If Δx approaches 0, which means that P approaches P_0, the geometric equivalent of the existence of this limit is the existence of the tangent line to the graph of the function $f(x)$ in point P_0 (Figure 9).
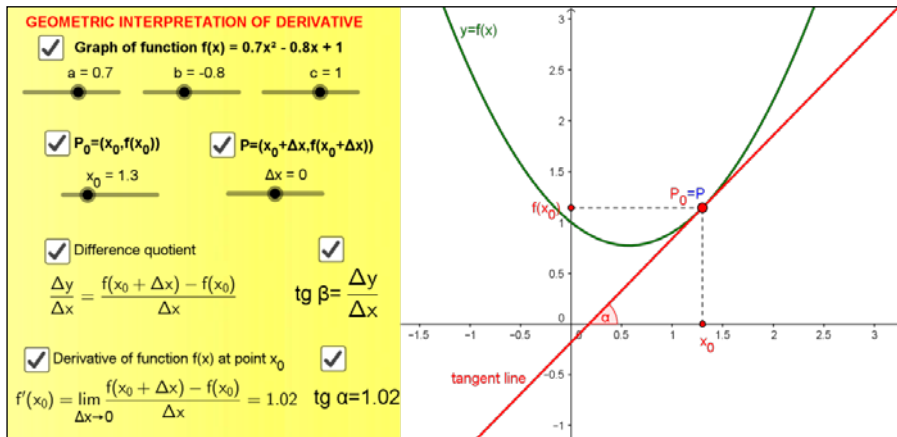
Fig. 9. Tangent line to the graph of function $f(x)$ when P approaches P_0

Source: own elaboration.

The derivative $f'(x_0)$ equals the tangent of the angle α which the positive x-axis forms with the line tangent to the curve $f(x)$ at a point x_0 (Figure 9) [Dubnicki et al. 1996].

The figures presented above are screenshots of our applet prepared in GeoGebra. This applet allows for changing parameters of the quadratic function, coordinates of points P_0 and P as well as to illustrate the process of moving point P along the curve towards the point P_0 and thus to dynamically illustrate the derivative of the function at point x_0 as a limit of the difference quotient as Δx approaches 0.

5. Conclusions

The use of IT in teaching allows for the enrichment of the traditional teaching process, but it should not replace the understanding of mathematical concepts, calculation skills or lead to a complete rejection of any traditional teaching methods. A study by MIND Research Institute [Wendt et al. 2014] shows that when teaching methods are extended by visual methods, more students gain better scores in the competence tests.

The teaching materials prepared by us and presented in this paper relate to selected issues which are taught on "quantitative" subject. We have also prepared other examples of visualisations of issues of Logistics (model of inventory control with continuous review, model of inventory control with periodic review), Mathematics (numerical integration), Operations Research (linear programming model and its graphical solving), Portfolio Analysis (Sharpe

single index model, Markowitz model) [Dudzińska-Baryła, Michalska 2011; 2015]. Anyone who is interested in our materials is very welcome to e-mail us.

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