ELEMENTS OF DIFFERENTIAL EQUATIONS
IN THE MATHEMATICS COURSE
FOR STUDENTS OF ECONOMICS

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Abstract. The article presents a proposal of one lecture with elements of differential equations included in a 30-hour course in mathematics for students of economics at the University of Economics in Wrocław. The author puts forward a presentation of some basic methods for solving first order differential equations exemplified by two macroeconomic growth models: the Domar model and the Solow model.

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1. Introduction

Is it possible that elements of differential equations are excluded from the course in mathematics for students of economics? Tools such as differential equations are indispensable for understanding dynamic economic models and emerge as a natural consequence of the syllabus in mathematics, especially of Analysis 1 and Analysis 2. The author teaches mathematics in lectures and tutorials to students of the Faculty of Economic Sciences at the University of Economics in Wrocław: a one-term course of 30-hour lecture with a 15-hour tutorial and two-term course of a 30-hour lecture with a 30-hour tutorial. The length of lectures and tutorials is evidently short, therefore one may ask whether it is reasonable to devote at least one lecture to elements of differential equations with applications in macroeconomics?!

It is worthwhile noticing that the textbooks on mathematics for economists provide the elements of differential equations in 30-hour lectures on Analysis [Smoluk 2007; Ostoja-Ostaszewski 1996]. Some academic teachers at colleges of economics believe that mathematics courses should not include any elements of micro or macroeconomics (due to time pressure), there-

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fore I would like to quote a few thoughts about teaching mathematics to students of economics shared by Professor T. Żylicz who teaches microeconomics, environmental economics and mathematical subjects (such as convex programming and differential equations) at the University of Warsaw.

"It can be observed that few economics students appreciate the value of rigorous reasoning in their analyses. Most try to grasp concepts quickly, relate them to everyday experience, and learn how to apply formulae or procedures in order to obtain quantitative results. They think that the task of proving things should be left to mathematicians. As a result, quite often the conclusions they arrive at are shallow and based on politically correct statements rather than established facts. Therefore teaching mathematics is more than simply providing routine solutions. If properly implemented, it can augment the analytical skills of students, raise the level of healthy scepticism, and nourish innovativeness. (...) Having claimed that the specific material studied is of minor importance, a serious qualification must be added. Even though the main purpose of teaching mathematics is training young minds, one cannot escape from the fact that a typical student expects to see a direct use of what he or she is learning. Hence mathematics courses have to take into account specific requirements and traditions of a given discipline. Otherwise the students will revolt" [Żylicz 2006, p. 1 and 3].

2. Elements of mathematics for undergraduate students of economics

Hours assigned to subjects on the Bachelor degree programme for students at the Faculty of Economic Sciences, specializing in economics, that require, at least supposedly, some mathematical tools are the following:

Mathematics: 30-hour lecture, 30-hour tutorial (semesters 1 and 2);

Introduction to Microeconomics: 15-hour lecture, 12-hour tutorial (semester 1);

Microeconomics: 15-hour lecture, 20-hour tutorial (semester 2);

Descriptive statistics: 15-hour lecture, 15-hour tutorial (semester 3)

Fundamentals of Macroeconomics: 30-hour lecture, 30-hour tutorial (semesters 3 and 4);

Economic Analyses: 15-hour lecture, 15-hour tutorial (semester 4);

Econometrics: 15-hour lecture, 15-hour tutorial (semester 4).

The titles of the 15 lectures in Mathematics taught by the author are presented below.
1.   Overview of elementary functions.
2.   Numerical sequences.

3. Limits, asymptotes and continuity of functions.
4. Differential calculus for functions of one variable.
5. Properties of functions (graph), applications of derivatives in economics.
6. Indefinite and definite integrals.
7. Definite integrals (with applications in economics), improper integrals.
8. Matrices and determinants.
9. Systems of linear equations.
10. Linear spaces (linear independence, basis, dimension).
11. Linear transformations.
12. Functions of two variables (contour curves, intersections, with examples in economics).
13. Differential calculus for functions of two variables (with applications in microeconomics, least-squares method).
14. Double integrals.
15. Elements of differential equations (with applications in macroeconomics).

Unfortunately, the amount of mathematical knowledge and skills of students decreases considerably with time – most students had passed only a basic level high-school exit exam in mathematics, typically scoring less than 50 per cent. Therefore, my lectures are given in a straightforward way: definitions of notions, theorems occasionally with general ideas of proofs, algorithms of solving the problems. As I try to maintain contact with my listeners, I often realize that elementary material from their primary education must be recalled during the lecture (for example, the order of operations, and the like).

3. Example of a lecture with elements of differential equations

As an example of applications of differential equations, we will consider problems of growth. First, the Malthusian model of population growth, next, a simple Domar model of economic growth, and its generalization, i.e. the Solow model.

The Malthusian model of population growth (1798)

Let $N(t)$ be the number of individuals in the given population. Then $dN(t)/dt$ is the speed of the change of the population which can occur for three reasons: births (B), deaths (D) or migrations (M). In the language of mathematics we can represent this relation in the form of the equation:

$$dN(t)/dt = B(t) - D(t) \pm M(t). \tag{1}$$

If we assume that births and deaths are linear functions, as well as there is no migration, then equation (1) can be rewritten in the form

$$d\mathrm{N}(t)/dt = a\mathrm{N}(t) - b\mathrm{N}(t) = (a - b)\mathrm{N}(t) = k\mathrm{N}(t), \tag{2}$$

where $k = a - b$, the rate a ($a > 0$) is called the population growth rate (natality), whereas the rate b ($b > 0$) is called the population decline rate (mortality).

We obtained the first order linear differential equation, which can be solved using separation of variables: $d\mathrm{N}(t)/\mathrm{N}(t) = kd(t)$, thus integrating yields: $\ln(N(t)) = kt$, that is

$$\mathrm{N}(t) = \mathrm{N}_0 e^{kt}. \tag{3}$$

It follows from (3) that when $k = 0$, which means that the birth rate equals the death rate, the population is in equilibrium and its number is N_0; when $k > 0$, which means that the birth rate is greater than the death rate, then the population size increases: $\mathrm{N}(t) > \mathrm{N}_0$; and when $k < 0$, which means that the birth rate is smaller than the death rate, then the population declines: $\mathrm{N}(t) < \mathrm{N}_0$. From this model we obtain either exponential growth (when $a > b$) or exponential decay (when $a < b$).

Assignment

In 1974, the global population amounted to 4 billion inhabitants, in 1999 – 6 billion people. Assuming that the number of population behaves according to the above Malthusian growth model, estimate the number of population in 2015. Compare the result obtained from the model with the actual number.

The Domar model of economic growth

The model is based on the following assumptions:

a) income $Y(t)$ at time t is proportional to capital invested at time t: $K(t)$, $Y(t) = kK(t)$;

b) a constant fraction s of income at time t is saved to finance investment $I(t)$, that is: $I(t) = sY(t)$, where constant s is called the marginal propensity to save;

c) investment at time t results in the increase of capital at time t, that is: $I(t) = dK(t)/dt$.

Conditions a) and b) yield: $I(t) = ksY(t)$. So $dI(t)/dt = ksdY(t)/dt$ and using c) we have: $dI(t)/dt = ksI(t)$.

In order to find a solution, we separate variables I and t so that they occur on different sides of the equation, and then integrating both sides yields:

$$\int \frac{dI(t)}{I(t)} = \int ksdt \, .$$

Hence, $lnI(t) = kst$, that is, $I(t) = Ae^{kst}$. Since $I(0) = Ae^0 = A$, then finally we obtain

$$I(t) = I(0)e^{kst}. \tag{4}$$

One can show [cf. Ostoja-Ostaszewski 1996], that if economic growth is exponential, at the growth rate $p \neq ks$, $I(t) = I(0)e^{pt}$, then, if $p > ks$, deficient productive capacity will appear in the economy after some time, and excess productive capacity in the other case. Solving this paradox (under some modified assumptions) leads to the Solow model of economic growth.

The Solow model of economic growth

We extend the Domar model by adding labour $L(t)$ to capital $K(t)$ as factors of production. We still assume that a constant fraction s of national income Q is saved and invested: $dK/dt = I(t) = sQ$, where $Q = F(K, L)$. We also assume that labour L grows at a constant rate λ: $dL/dt = \lambda L$. Solow assumed that the function F was linearly homogeneous, that is, $F(\beta K, \beta L) = \beta F(K, L)$, where $\beta > 0$. For $\beta = \dfrac{1}{L}$ we have: $F(K,L) = \dfrac{1}{L} F\left(\dfrac{K}{L}, 1\right)$.

Denote $k = K/L$ and define the function of one variable $\varphi(k) = F(k, 1)$, where k is capital-labour ratio, i.e. the value of capital input per unit of labour (one person-hour), called capital equipment of labour. In that case we have:

$K = kL$ and $I = \dfrac{dK}{dt} = L\dfrac{dk}{dt} + k\dfrac{dL}{dt}$. Hence, $sQ = sF(K,L) = L\dfrac{dk}{dt} + k\lambda L = sL\varphi(k)$.

Dividing this equality by $L \neq 0$ yields the Solow's equation:

$$\frac{dk}{dt} + k\lambda = s\varphi(k) \ . \tag{5}$$

Let the production function be of a Cobb-Douglas type: $Q(K, L) = K^{\alpha}L^{1-\alpha} = L(K/L)^{\alpha} = Lk^{\alpha}$, then the Solow equation gets the form of the Bernoulli equation: $\dfrac{dk}{dt} + \lambda k = sk^{\alpha}$. Multiplying its sides by $k^{-\alpha}$ and substituting $z = k^{1-\alpha}$ yields: $\dfrac{dz}{dt} = (1-\alpha)k^{-\alpha}\dfrac{dk}{dt}$ and the equation gets the form: $\dfrac{1}{1-\alpha}\dfrac{dz}{dt} + \lambda z = s$, i.e.

$$dz/dt + (1-\alpha)\lambda z = (1-\alpha)s. \tag{6}$$

By factoring out $e^{(1-\alpha)\lambda t}$, we get $e^{(1-\alpha)\lambda t}\dfrac{dz}{dt} + e^{(1-\alpha)\lambda t}(1-\alpha)\lambda z = e^{(1-\alpha)\lambda t}(1-\alpha)s$,

which is equivalent to the equation $\dfrac{d}{dt}\left(ze^{(1-\alpha)\lambda t}\right) = s(1-\alpha)e^{(1-\alpha)\lambda t}$, thus

$e^{(1-\alpha)\lambda t} = s\left(1-\alpha\right)\dfrac{e^{(1-\alpha)\lambda t}}{\left(1-\alpha\right)\lambda} + C$, therefore $k^{1-\alpha} = \dfrac{s}{\lambda} + Ce^{-(1-\alpha)\lambda t}$ and $k(0)^{1-\alpha} =$

$\dfrac{s}{\lambda} + C$; and finally, $k(t)^{1-\alpha} = \dfrac{s}{\lambda} + (k\left(0\right)^{1-\alpha} - \dfrac{s}{\lambda})e^{-(1-\alpha)\lambda t}$, where $0 < \alpha < 1$,

$\lambda > 0$. When $t \to \infty$, then $\left(\dfrac{K}{L}\right)^{1-\alpha} \to \dfrac{s}{\lambda}$, that is

$$\frac{K}{L} \to \left(\frac{s}{\lambda}\right)^{\frac{1}{1-\alpha}}. \qquad (7)$$

Notice that the rate of increase of capital is $s\lambda/s = \lambda$, so it is equal to the rate of increase of labour resources. This also holds for investment growth rate: $I = sQ = sLk^{\alpha}$, then: $lnI = lns + lnL + \alpha lnk$ and differentiating yields:

$$\frac{1}{I}\frac{dI}{dt} = \frac{1}{L}\frac{dL}{dt} + \alpha\frac{1}{k}\frac{dk}{dt} = \lambda + \frac{\alpha}{k}\left(sk^{\alpha} - \lambda k\right) = \lambda + \alpha s(k^{\alpha-1} - \frac{\lambda}{s}) \to \lambda.$$

Formula (7) shows that the economy converges to the static equilibrium without the need of investment adjustment characteristic of the Domar model.

4. Conclusions

Firstly, the presented lecture with elements of differential equations illustrates a nice application of differential and integral calculus for functions of one variable to solving macroeconomic problems. Secondly, it introduces some simple methods of differential calculus: separation of variables and integrating factor. Perhaps such a lecture will encourage students of economics to become more interested in mathematics and they will be motivated to deepen their knowledge in this field.

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