NICOLAS BERNOULLI AS A STATISTICIAN

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Nicolas Bernoulli belonged to the glorious Bernoulli dynasty, he was a nephew of both Jakob and Johann Bernoulli and a student of the latter. De Moivre [1733/1756, p. 243], called him a *great mathematician*, but he was unable to devote enough time to mathematics and did not live up to his capability [Hald 1990, p. 393].

In statistics, he is mostly remembered for his dissertation on the application of the art of conjecturing to jurisprudence (1709/1975). Its English translation www.cs.xu.edu/math/sources/NBernoulli/de_usu_artis.pdf is horrible (and translations in other languages are lacking). Latin constructions are rendered slavishly, without any consideration of English grammar. Many phrases are very long, up to 25 lines (in one case, 40 lines long) and in some cases the translation is either incomprehensible or certainly wrong but Kohli [1975], provided a useful commentary. The dissertation contained:

a) The calculation of the mean duration of life for persons of different ages. Nicolas issued from Graunt's life table. He, just like Jakob Bernoulli, certainly had not seen Graunt's classic and did not know that his table was largely erroneous, and just as certainly, he did not know about another classic, the 1693 paper of Halley (the Breslau life table).

b) A recommendation of its use for ascertaining the value of annuities and estimating the probability of death of absentees about whom nothing is known. If the probability of death of an absentee, calculated in accordance with a mortality table, was twice higher than the probability of the absentee being alive, he should be declared legally dead. For the first time ever, this recommendation introduced (at least theoretically) objectivity in this problem. The way of approaching this issue has really changed since the time when Kepler refused to consider such a problem.

Stochastic studies of judicial decisions, of the voting procedures adopted by assemblies and at general elections, began in the late 18th

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century but many later scientists denied any possibility of numerically examining these subjects. Thus, probability, misapplied to jurisprudence, had become *the real opprobrium of mathematics* [Mill 1843/1886, p. 353]; or, in law courts people act like the *moutons de Panurge* [Poincaré 1912, p. 20]. Nevertheless, Gauss (*Werke*, Bd. 12, pp. 401–404), as described by W.W. Weber in a letter of 1841, stated that the theory of probability provided a clue for establishing the proper number of witnesses and jurors. True, he also argued that applications of the theory of probability can be greatly mistaken if the essence of the studied object is not taken into account.

c) Methodical calculations of expected losses in marine insurance.

d) The calculation of expected gains (more precisely, of expected losses) in the classical Genoese lottery.

e) Calculation of the probability of truth of testimonies. Bernoulli considered this problem too formally, he had not duly allowed for its complexities. The same is true even with regard to Poisson's celebrated study [1837], but in spite of Poincaré's verdict and only having stochastic arguments at his disposal, Poisson was able to study the consequences of some changes in the administration of justice in France.

f) The determination of the life expectancy of the last survivor of a group of men [Todhunter 1865, pp. 195–196]. Assuming a continuous uniform law of mortality, he calculated the expectation of the appropriate [order statistic]. He was the first to use, in a published work, both this distribution and an order statistic.

In 1669, Huygens corresponded with his brother [Huygens 1888–1950/1895, vol. 6] and they discussed stochastic problems in mortality and life insurance. Huygens was the first to apply that same law of mortality (but N. Bernoulli was the first to apply it in a published work). In one case Huygens wrongly assumed that the number of dying people from a given group decreases in time. Actually, order statistics will separate the studied interval of time into roughly equal periods. N. Bernoulli did not make that mistake.

g) A comment on the introduction of expectation by Huygens [Kohli 1975, p. 542]. Bernoulli interpreted it as a generalized arithmetic mean and the centre of gravity *of all probabilities* (this is rather loose).

Apparently in accordance with his subject he had not discussed the treatment of observations. Bernoulli's work undoubtedly fostered the spread of stochastic notions in society, but I ought to add that not only did he pick up some hints included in the manuscript of the *Ars conjectandi*, he borrowed separate passages both from it and even

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from the *Meditationes* [Kohli 1975, p. 541], which were never intended for publication. His numerous general references to Jakob Bernoulli do not excuse his plagiarism. As a smokescreen he even stated that, when the *Ars Conjectandi* of Jakob appears, *we will see whether I have found an approximation as good as his own*, see his letter to Montmort of 23 January 1713 [Montmort 1708/1713, p. 394].

I am now discussing Nicolas' achievements of 1713 contained in his letters to Montmort and published by him [Montmort 1708/1713].

The strategic game Her [Hald 1990, pp. 314–322]. The modern theory of games studies it by means of the minimax principle. Nevertheless, already Bernoulli indicated that the gamblers ought to keep to mixed strategies.

The gambler's ruin. Montmort wrote out the results of his calculations for some definite initial conditions whereas Bernoulli indicated, without derivation, the appropriate formula (an infinite series). Hald believes that he obtained it by means of the method of inclusion and exclusion.

A study of contemporary games of chance [Todhunter 1865, pp. 90 and 105–126].

The sex ratio at birth [Montmort 1708/1713, pp. 280–285]. I only dwell on *Bernoulli's indirect derivation of the normal distribution* [Sheynin 1968, only in its reprint of 1970, p. 232; 1970, pp. 201–203]. Let the sex ratio be m/f, n, the total yearly number of births, and μ and $(n - \mu)$, the numbers of male and female births in a year. Denote

$$n/(m + f) = r,\ m/(m + f) = p,\ f/(m + f) = q,\ p + q = 1,$$

and let s be of the order of $\sqrt{n}$. Then Bernoulli's derivation [Montmort 1708/1713, pp. 388– 394] can be presented as follows:

$$P(|\mu - rm| \le s) \approx (t - 1)/t,$$
$$t \approx [1 + s\,(m + f)/mfr]^{s/2} \approx \exp[s^2(m + f)^2/2mfn],$$
$$P\,(|\mu - rm| \le s) \approx 1 - \exp(s^2/2pqn),$$
$$P[|\mu - np|/\sqrt{npq} \le s] \approx 1 - \exp(-s^2/2).$$

This result does not however lead to an integral theorem since s is restricted (see above) and neither is it a local theorem; for one thing, it lacks the factor $\sqrt{2/\pi}$.

Youshkevich [1986] reported that at his request three mathematicians (!), issuing from the description offered by Hald, had concluded that Bernoulli had come close to the local theorem. Neither he, nor Hald [1998, p. 17] mentioned that lacking factor.

The Petersburg game. In a letter to Montmort, Bernoulli [Montmort 1708/1713, p. 402] described his invented game. *B* throws a die; if a six arrives at once, he receives an *écu* from *A*, and he obtains 2, 4, 8, … *écus* if a six only occurs at the second, the third, the fourth, … throw. Determine the expectation of *B*'s gain. Gabriel Cramer insignificantly changed the conditions of the game: a coin appeared instead of the die, and the occurrence of heads (or tails) has been discussed ever since. The expectation of gain became

$$\mathrm{E}\xi = 1\cdot 1/2 + 2\cdot 1/4 + 4\cdot 1/8 + \ldots = \infty,$$

although a reasonable man would never pay any considerable sum in exchange for it. In 1738, Daniel Bernoulli discussed this game in a Petersburg journal, hence its name.

This paradox is still being examined. Additional conditions were being introduced, for example, suggestions were made to neglect unlikely gains, i.e. to truncate this series, to restrict beforehand the possible payoff and, the most interesting, to replace expectation by *moral expectation*.

Suppose that the observations of a random variable are $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, then its *usual* expectation is

$$\frac{p_1x_1 + p_2x_2 + \ldots + p_nx_n}{p_1 + p_2 + \ldots + p_n},$$

where as the moral expectation is equal to that fraction with x_i replaced by $\ln x_i$. In most cases *usual* infinite becomes moral finite, expectation.

In addition, Condorcet [1784, p. 714], noted that the possibly infinite game nevertheless provided only one trial and that only some mean indicators describing many such games could lead to an expedient solution. Actually issuing from the same idea, Freudenthal [1951], proposed to consider a number of games with the role of the gamblers in each of them to be decided by lot. Finally, the Petersburg game caused Buffon [1777, § 18] to carry out what was apparently the first statistical experiment. He conducted a series of 2048 games; the mean payoff was 4.9 units, and the longest duration of play (in six cases), nine throws.

From a theoretical point of view, the game was interesting because it introduced a random variable with an infinite expectation. Also, conforming to common sense, its study implied that the expectation of even a large gain ought to be disregarded if the probability of obtaining it is low.

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Spieß [1975], dwelt on the early history of the Petersburg game and Jorland [1987], and Dutka [1988], described later developments. Dutka also adduced the results of its examination by means of statistical simulation.

Moral expectation had become popular and Laplace [1812/1886, p. 189], therefore proposed a new term for the previous *usual* expectation calling it *mathematical*; his expression persists at least in the French and Russian literature, regrettably since moral expectation is not anymore applied in statistics. At the end of the 19th century, issuing from Bernoulli's idea, economists began to develop the theory of marginal utility thus refuting Bertrand's opinion [1888, p. 66], that moral expectation was useless:

The theory of moral expectation became classic, and never was a word used more appropriately. It was studied and taught, it was developed in books and really celebrated. With all that, the success came to a stop - no application was made, or could be made, of it.

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