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THE ECONOMY WITH PRODUCTION AND CONSUMPTION SYSTEMS CHANGING IN TIME

1. INTRODUCTION

The Debreu private ownership economy (see Debreu, 1959) models the situation, where two groups of agents: producers and consumers operate in a market. Both of them tend to realize their aims: maximizing the profit on the one hand and maximizing the preferences on the budget sets on the other hand. If the total supply equals the total demand (so Walras Law is satisfied) at given prices, then the economy is in equilibrium, producers and consumers do not have the incentives to change their activities. In the initial model, we do not observe the change of agents' activities in time. In the spirit of comparative statics, the influence of the change of prices on the production and the consumption plans is usually elaborated.

In the real world, the production and the consumption plans are often changed in time. Consequently, the models of the market, elaborated the changes of production and consumption in time, were also constructed. Some of the results, the reader can find in Radner (1970), Chiang (1992), Aliprantis (1996), Panek (1997), Malawski (2001), Magill, Quinzii (2002), Malawski (2005), Malawski, Woerter (2006), Panek (2006).

In this paper, the changes, in the production and consumption systems observed in time, are also elaborated, but the dynamic structure of the economy has its origin in Lipieta (2010), where the influence of the specific properties of consumption sphere on the production plans was studied. Hence, the mathematical tools used in this paper, differ from that what have been used so far, in connection to the description of changes of production or consumption in time. In Lipieta (2010), the case of the consumption system in which all consumption plans are contained in a proper subspace of the commodity – price space was studied. According to the idea of the competitive mechanism, the producers, thinking about the losses' minimization, adjust the quantities of commodities in their production plans to the given relationship. The modification needs alterations in time. The geometric properties of consumption sets and modification of production sets make that the equilibrium in the economy will not be disturbed during the adjustment process. This change of the production sphere leads to the definition of the economy defined on the basis of the Debreu economy, in which the production and consumption plans are continuous functions defined on the (time) interval $[0, 1]$.

The paper is organized as follows: in the next section, the construction of the Debreu private ownership economy is presented. In the third part, the description of the change of production sphere in a Debreu economy, where consumption sets are contained in the proper subspace of the commodity – price space, is presented; the forth part focuses on the definition and the structure of action of the economy with the commodity – price space dependent on time.

2. MODEL

A model of the private ownership Debreu economy described in Debreu (1959) is defined in the form of a relational system consisted of the combination of the production and the consumption systems (see for instance Malawski, 2005; Lipieta, 2010). The linear space $\mathbb{R}^{\ell}$, $\ell \in \{1, 2, \ldots\}$ with the scalar product

$$(x \circ y) = (x_1, \ldots, x_\ell) \circ (y_1, \ldots, y_\ell) = \sum_{k=1}^{\ell} x_k \cdot y_k,$$

is interpreted as the $\ell-$ dimensional space of commodities and prices. Let $n \in \{1, 2, \ldots\}$. Then

- $J = \{1, \ldots, n\}$ is a finite set of producers,
- $y : J \ni j \to Y^j \subset \mathbb{R}^{\ell}$ is the correspondence of production sets, which to every producer j assigns a production set $Y^j \subset \mathbb{R}^{\ell}$ of the producer's feasible production plans,
- $p \in \mathbb{R}^{\ell}$ denotes a price vector.

Now we can assume the following definitions:

Definition 1. If for the given price vector $p \in \mathbb{R}^{\ell}$

$$\forall\, j \in J\; \eta^j(p) \overset{\text{def}}{=} \{y^{j*} \in Y^j : p \circ y^{j*} = \max\{p \circ y^j : y^j \in Y^j\}\} \neq \emptyset, \tag{1}$$

then

- $\eta : J \ni j \to \eta^j(p) \subset \mathbb{R}^{\ell}$ is called the correspondence of supply, which to every producer $j \in J$ assigns the set $\eta^j(p)$ of production plans maximizing their profit at given price system p; the plans belonging to set $\eta^j(p)$ will be called optimal plans of producer j,
- $\pi : J \ni j \to \pi^j(p) \in \mathbb{R}$ is the maximal profit function at given price system p, where for every $j \in J$, $\pi^j(p) = p \circ y^{j*}$ for $y^{j*} \in \eta^j(p)$.

Definition 2. The two – range relational system

$$P = (J, \mathbb{R}^{\ell}; y, p, \eta, \pi)$$

is called the production system.

Similarly, let $m \in \{1, 2, \ldots\}$. Then

- $I = \{1, \ldots, m\}$ is a finite set of consumers,
- $\Xi \subset \mathbb{R}^\ell \times \mathbb{R}^\ell$ is the family of all preference relations defined on commodity space $\mathbb{R}^\ell$,
- $\chi : I \ni i \to X^i \subset \mathbb{R}^\ell$ is the correspondence of consumptions sets which to every consumer $i \in I$ assigns a consumption set $\chi(i) = X^i$ being a subset of the commodity space and representing the consumer's feasible consumption plans,
- $e : I \ni i \to e^i \in X^i$ is an initial endowment mapping which to every consumer $i \in I$ assigns an initial endowment vector $e^i \in X^i$,
- $\varepsilon \subset I \times (\mathbb{R}^\ell \times \mathbb{R}^\ell)$ is a correspondence of preference relations, which to every consumer $i \in I$ assigns a preference relation $\preccurlyeq^i$ from the set Ξ restricted to the consumption set X^i,
- $p \in \mathbb{R}^\ell-$ is a price vector.

Notice that the expenditures of every consumer $i \in I$ are not greater than the value

$$w^i = p \circ e^i. \tag{2}$$

The following definitions may be assumed on the basis of the above:

Definition 3. If for the given price vector $p \in \mathbb{R}^\ell$

$$\forall\, i \in I \quad \beta^i(p, w^i) \overset{\text{def}}{=} \{x \in \chi(i) : p \circ x \leqslant w^i\} \neq \emptyset \tag{3}$$

$$\forall i \in I \ \varphi^i(p, w^i) \overset{\text{def}}{=} \{x^{i*} \in \beta^i(p, w^i) : \ \forall x^i \in \beta^i\left(p, w^i\right) \ x^i \preccurlyeq^i x^{i*}, \preccurlyeq^i \ \in \Xi\} \neq \emptyset, \tag{4}$$

then

- $\beta : I \ni i \to \beta^i\left(p, w^i\right) \subset \mathbb{R}^\ell$ is the correspondence of budget sets, which to every consumer assigns his set of budget constrains $\beta^i\left(p, w^i\right) \subset \chi(i)$ at the price system p and the initial endowment $e(i)$,
- $\varphi : I \ni i \to \varphi^i(p, w^i) \subset \mathbb{R}^\ell$ is the demand correspondence at given price system p, which to every consumer $i \in I$ assigns his consumption plans maximizing his preference on the budget set $\beta^i\left(p, w^i\right)$; the plans belonging to set $\varphi^i(p, w^i)$ will be called optimal plans of consumer i.

Definition 4. The three – range relational system

$$C = (I, \mathbb{R}^\ell, \Xi; \chi, e, \varepsilon, p, \beta, \varphi)$$

is called the consumption system.

Let $p \in \mathbb{R}^\ell$ be a price vector, $P-$ a production system and $C-$ a consumption system in the same space $\mathbb{R}^\ell$. Suppose that the mapping $\theta : I \times J \to [0, 1]$ satisfying

$$\forall j \in J \ \sum_{i \in I} \theta(i, j) = 1 \tag{5}$$

is given. Let's assume that every consumer shares in the producers' profits. Number $\theta(i, j)$ indicates that part of the profit of producer j which is owned by consumer i. Then the value w^i $(i \in I)$ in (2) is changed by the rule

$$w^i = p \circ e^i + \sum\nolimits_{j \in J} \theta(i, j) \cdot \pi^j(p). \tag{6}$$

Let

$$\omega = \sum\nolimits_{i \in I} e(i) \in \mathbb{R}^\ell. \tag{7}$$

If, at the given $p \in \mathbb{R}^\ell$, conditions (1), (3) and (4) are satisfied with w^i for every consumer i calculated by (6), then the following definition may be formulated:

Definition 5. The relational system

$$E_p = (P, C, \theta, \omega)$$

is called the Debreu private ownership economy (shortly called the Debreu economy).

The vector (7) is called the total endowment of the economy E_p.

It is well known that a Debreu private ownership economy E_p operates as follows. Let a price vector $p \in \mathbb{R}^\ell$ be given. Every producer j chooses a production optimal plan $y^{j*} \in \eta^j(p) \subset Y^j$ maximizing his profit at the price system p. The vector

$$y^* = y^{1*} + \ldots + y^{n*} \tag{8}$$

is called the optimal total production plan. Maximal profit of each producer is divided among all consumers according to function θ (see (5)). Now the expenditures of every consumer i cannot be greater than value w^i (see (6)). In this situation, every consumer i chooses his optimal consumptions plan $x^{i*} \in \varphi^i(p, w^i) \subset X^i$ maximizing his preference on the budget set $\beta^i\left(p, w^i\right)$. The vector

$$x^* = x^{1*} + \ldots + x^{m*} \tag{9}$$

is called the optimal total consumption plan. If

$$x^* - y^* = \omega, \tag{10}$$

which means the total supply equals the total demand (so Walras Law is satisfied), then vector p is called the equilibrium price vector and it is denoted by p^*, vector (8) is called the equilibrium total production plan and vector (9) – the equilibrium total consumption plan. Consequently, the sequence

$$((x^{i*})_{i \in I}, (y^{j*})_{j \in J}, p^*) \stackrel{\text{def}}{=} (x^{1*}, \ldots, x^{m*}, y^{1*}, \ldots, y^{n*}, p^*) \in (\mathbb{R}^\ell)^{m+n+1}$$

is called the state of Walras equilibrium in the private ownership economy E_p (see Malawski, 2005).

3. DEBREU ECONOMY WITH REDUCED CONSUMPTION OR PRODUCTION SPHERE

At the beginning of this part of the paper, some properties of subspaces of space $\mathbb{R}^{\ell}$, will be presented. Let $V \subset \mathbb{R}^{\ell}$ be a subspace of dimension $\ell - k,\ k \in \{1, \ldots, \ell - 1\}$. Then there exist linearly independent vectors $h^1, \ldots, h^k \in \mathbb{R}^{\ell}$ ($h^s = \left(h_1^s, \ldots, h_\ell^s\right)$, $s \in \{1, 2, \ldots, k\}$) such that

$$V = \bigcap\nolimits_{s=1}^{k} ker\tilde{h}^s \tag{11}$$

where

$$\tilde{h}^s : \mathbb{R}^{\ell} \ni x \to x_1 h_1^s + \ldots + x_\ell h_\ell^s \in \mathbb{R} \tag{12}$$

are, for every $s \in \{1, 2, \ldots, k\}$, the linear and continuous functions.

Let $E_p = (P, C, \theta, \omega)$ be a Debreu economy. Assume that there exists a proper subspace V of the commodity – price space $\mathbb{R}^{\ell}$ such that

$$\forall i \in I\ X^i \subset V \tag{13}$$

(see Lipieta, 2010). The economy $E_p = (P, C, \theta, \omega)$ in which condition (13) is satisfied for some subspace V of commodity – price space $\mathbb{R}^{\ell}$ will be called the Debreu economy with reduced the consumption sphere. The consumption plans satisfying condition (13) are often seen in the real economy. We can observe the commodities, so called complementary commodities (see Varian, 1999; Lipieta, 2010), which quantities are proportional (or at least approximately proportional). This dependency may be caused by nature of commodities or their application. It may also appear that the consumers are not interested in the consumption at least one of the commodities offered by producers. We show that in the above cases the condition (13) is satisfied.

Let x^i be a consumption plan of consumer i ($x^i \in X^i$). If the quantities of commodities $k_1, k_2 \in \{1, ..., \ell\}$ ($k_1 \neq k_2$) of plan x^i are proportional, then there exists $a > 0$ such that

$$x_{k_1}^i = a \cdot x_{k_2}^i. \tag{14}$$

Hence $\tilde{h}\left(x^i\right) = 0$, where

$$\tilde{h} : \quad \mathbb{R}^{\ell} \ni x \to x_{k_1} - a \cdot x_{k_2}^i \in \mathbb{R} \tag{15}$$

is the linear and continuous function. By the above, $x^i \in ker\tilde{h} \overset{\text{def}}{=} \{x \in \mathbb{R}^{\ell} : \tilde{h}(x) = 0\}$. Let us recall, that set $ker\tilde{h}$ is the linear subspace of $\mathbb{R}^{\ell}$ of dimension $\ell - 1$. If condition (14) is seen for all plans $x^i \in X^i$ then

$$X^i \subset ker\tilde{h}. \tag{16}$$

If condition (16) is hold for every consumer $i \in I$, then condition (13) is satisfied for $V = ker\tilde{h}$. Let us notice that, if commodities $k_1, k_2 \in \{1, 2, \ldots, \ell\}$ satisfy condition (14), then they are (perfect) complementary. Hence the economy with reduced consumption sphere, in which all functionals $\tilde{h}^s$, $s \in \{1, \ldots, k\}$, are of the form (15), is also called the economy with complementary commodities (see Lipieta, 2010).

Now, suppose that there exists vector h, of the form

$$h = \begin{cases} 1 & \text{for} \quad k = k_0 \\ 0 & \text{for} \quad k \neq k_0 \end{cases} \tag{17}$$

for some $k_0 \in \{1, \ldots, \ell - 1\}$ for which $X^i \subset ker\tilde{h}$ and $\tilde{h}$ is of the form (12) with vector h by (17). Then coordinate k_0 is equal 0 in every plan of consumer i. Hence consumer i is not interested in consumption of the commodity k_0.

Notice that productions sets Y^j $(j \in J)$ sometimes also satisfy condition

$$\forall j \in J \ Y^j \subset V \tag{18}$$

Condition (18) can be satisfied, if the quantities of commodities (inputs or outputs) are proportional in production plans or if there exists an commodity which, for every producer, is neither output nor input. Generally production sets do not have to be contained in a subspace of commodity space $\mathbb{R}^\ell$. The economy $E_p = (P, C, \theta, \omega)$, in which condition (18) is satisfied for some subspace V of commodity – price space $\mathbb{R}^\ell$, will be called the Debreu economy with reduced production sphere.

4. THE ECONOMY WITH DYNAMICAL PRODUCTION SYSTEM

Assume that there exists at least two commodities which quantities are proportional in plans of all consumers in a Debreu economy E_p. The producers, observing consumers' activity are aware of this dependency. The change of the production sphere, to get the same dependency in production plans, is the simplest way to avoid the surplus of outputs. Similarly, if there exists a commodity – the output of some producers – which is not the input for any producer and which is not desired by consumers, then producers may, thinking about the increase of the profit, want to eliminate this commodity from their production plans even if there exists an equilibrium in the economy. The existence of undesired or complementary commodities can lead to the changing production plans (notice that the non consumable goods, which are outputs of some producers and inputs for the others, play important role in production sphere and in the trade, so the producers, in their own interest, will not want to give up these commodities from production plans). To sum up, in the spirit of the competitive mechanism, the producers, in some cases, change their production activities (represented by production sets) to meet the consumers' requirements.

The construction of the Debreu economy with reduced both consumption and production spheres on the basis of the economy with reduced consumption sphere presented in Lipieta (2010) leads to the construction of the economy, where production and consumption plans are functions dependent on time (the economy with production and consumption systems changing in time).

Firstly, we present the details about the economy with reduced consumption sphere, which will be of use later. Let $E_p = (P, C, \theta, \omega)$ be a private ownership Debreu economy, where condition (13) is satisfied. Suppose, that subspace V of dimension $\ell - k\,(k \in \{1, \dots, \ell - 1\})$ satisfying assumption (13), is of the form (11) with linearly independent functionals $\tilde{h}^1, \dots, \tilde{h}^k$ satisfying (12). Let $q^1, \dots, q^k \in \mathbb{R}^\ell$ be the solution of the system of equations

$$\tilde{h}^s(q^r) = \delta^{sr} \quad \text{for} \quad s, r \in \{1, \dots, k\}, \tag{19}$$

where

$$\delta^{sr} = \begin{cases} 1 & \text{if} \quad s = r \\ 0 & \text{if} \quad s \neq r \end{cases}$$

is Kronecker delta. Let mapping $Q : \mathbb{R}^\ell \to \mathbb{R}^\ell$ be of the form

$$Q(x) = x - \sum\nolimits_{s=1}^{k} \tilde{h}^s(x) \cdot q^s \tag{20}$$

and mapping $\hat{Q} : \mathbb{R}^\ell \times [0, 1] \to \mathbb{R}^\ell$ be defined by

$$\hat{Q}(x, t) = x - t \cdot \sum\nolimits_{s=1}^{k} \tilde{h}^s(x) \cdot q^s. \tag{21}$$

We say that vectors $q^1, \dots, q^k$ assign the direction of mappings Q and $\hat{Q}$. Let us notice, that mapping Q is the linear and continuous operator and

$$\forall x \in \mathbb{R}^\ell \; Q(x) \in V \quad \text{and} \quad \forall v \in V \; Q(v) = v. \tag{22}$$

By (22), mapping Q is the (linear and continuous by (20) and (12)) projection from $\mathbb{R}^\ell$ into V. Moreover, if mappings Q and $\hat{Q}$ are assigned by the same vectors $q^1, \dots, q^k$ then

$$\forall x \in \mathbb{R}^\ell \; \hat{Q}(x, 1) = Q(x),$$

so

$$\forall v \in V \, \forall t \in [0, 1] \quad \hat{Q}(v, 1) = v.$$

Let subspace $V \subset \mathbb{R}^\ell$ be of the form (11). Notice that if $p \notin V^T$, then the system of equalities

$$\begin{cases} \tilde{h}^s(x) = \delta^{sr} \\ p \circ x = 0 \end{cases} \quad s, r \in \{1, \dots, k\} \tag{23}$$

also, for every $r \in \{1, \dots, k\}$, has a solution.

Keeping the above assumptions and notations, reasoning analogously as in the proof of theorem 4.2 in Lipieta (2010), we get that the following is true:

Theorem 1. Let E_p be a Debreu economy satisfying (13). Let vectors $q^1, \dots, q^k \in \mathbb{R}^\ell$ satisfy (19) and additionally (23) if $p \notin V^T$. Assume that $Q : \mathbb{R}^\ell \to V$ is the projection of the form (20) assigned by vectors $q^1, \dots, q^k$. Then

1. $\forall j \in J$ if $y^{j*} \in \eta^j(p)$, then $Q(y^{j*})$ maximizes at price p the profit of the j–th producer on the modified production set $Q(Y^j)$,
2. $\forall i \in I$ if $x^{i*} \in \varphi^i(p, w^i)$, then x^{i*} maximizes at price p the preference of the i–th consumer on the set $\{x \in X^i : p \circ x \leqslant p \circ e^i + \sum_{j \in J} \theta(i, j) \cdot (p \circ Q\left(y^{j*}\right))\}$.

Analogously to the proof of theorem 4.2 in Lipieta (2010), we can prove the following:

Theorem 2. Let E_p be a Debreu economy satisfying (13). Let vectors $q^1, \ldots, q^k \in \mathbb{R}^\ell$ satisfy (19) and additionally (23) if $p \notin V^T$. Assume that mapping $\hat{Q} : \mathbb{R}^\ell \times [0, 1] \to \mathbb{R}^\ell$ of the form (21) is assigned by vectors $q^1, \ldots, q^k \in \mathbb{R}^\ell$. Then
1. $\forall j \in J$ if $y^{j*} \in \eta^j(p)$ then, for every $t \in [0, 1]$, vector $\hat{Q}(y^{j*} t)$ maximizes (at price p) the profit of producer j on the modified production set $\hat{Q}\left(Y^j, t\right) = \{\hat{Q}(y^j, t) \in \mathbb{R}^\ell : y^j \in Y^j\}$.
2. $\forall i \in I$ if $x^{i*} \in \varphi^i(p, w^i)$ then x^{i*} maximizes (at price p) the preference of consumer i on the set $\{x \in X^i : p \circ x \leqslant p \circ e^i + \sum_{j \in J} \theta(i, j) \cdot p \circ \hat{Q}(y^{j*}, t)\}$.

Now we define the private ownership economy with the commodity – price space of continuous functions defined on the time interval $[0, 1]$, on the basis of Debreu economy E_p satisfying condition (13). The elements of such commodity – price space will be streams of production and constant mappings of consumption and prices.

Let E_p be a Debreu private ownership economy satisfying (13). Let vectors $q^1, \ldots, q^k \in \mathbb{R}^\ell$ be obtained by condition (19). Moreover, if $p^* \notin V^T$, we assume that $q^1, \ldots, q^k$ satisfy additionally system of equalities (23). Let $\hat{Q} : \mathbb{R}^\ell \times [0, 1] \to \mathbb{R}^\ell$ be the mapping of the form (21) assigned by vectors $q^1, \ldots, q^k$. For every $z \in \mathbb{R}^\ell$ we put the constant mapping:

$$\bar{z} : [0, 1] \ni t \to z \in \mathbb{R}^\ell.$$

It is obvious that for every $z \in \mathbb{R}^\ell$, mapping $\bar{z}$ is continuous. The sets of all continuous mappings from $[0, 1]$ into $\mathbb{R}^\ell$ will be noted by $C([0, 1], \mathbb{R}^\ell)$.

Remark 1. The mapping:

$$C\left([0, 1], \mathbb{R}^\ell\right) \times C\left([0, 1], \mathbb{R}^\ell\right) \ni (f, g) \to \int_0^1 f(t) \circ g(t) dt \in \mathbb{R}$$

is the scalar product in space $C\left([0, 1], \mathbb{R}^\ell\right)$, which for every $z, s \in \mathbb{R}^\ell$ satisfies:

$$\bar{z} \circ \bar{s} = z \circ s.$$

Let P be a production system (see def. 2). The following is justified:

Definition 6. The two – range relational system $\check{P} = \left(J, C\left([0,1], \mathbb{R}^\ell\right); \check{y}, \check{p}, \check{\eta}, \check{\pi}\right)$ where

- $J = \{1, 2, \ldots, n\}$ is the finite set of producers from system P,
- $\check{y} : J \ni j \to \check{Y}^j \subset C\left([0,1], \mathbb{R}^\ell\right)$ is the correspondence of production sets changing in (depended on) time, which to every j–th producer ($j \in J$) assigns the production set
$$\check{Y}^j = \{\hat{Q}(y^j, \cdot) \in C\left([0,1], \mathbb{R}^\ell\right) : \ y^j \in Y^j\}$$
where set $\{\hat{Q}(y^j, t) : \ y^j \in Y^j\}$ means the set of producer's plans at time t,
- $\check{p} = \bar{p} \in C\left([0,1], \mathbb{R}^\ell\right)$ is the price (constant) mapping,
- $\check{\eta} : J \ni j \to \check{\eta}^j(\check{p}) \subset C\left([0,1], \mathbb{R}^\ell\right)$ is the correspondence of supply, which to every producer $j \in J$ assigns set $\check{\eta}^j(\check{p})$ of production plans maximizing his profit at the price system $\check{p}$ on set $\check{Y}^j$, $\check{\eta}^j(\check{p}) \overset{\text{def}}{=} \{\hat{Q}(y^{j*}, \cdot) \in C\left([0,1], \mathbb{R}^\ell\right) : \ y^{j*} \in \eta^j(p)\}$,
- $\check{\pi} : J \ni j \to \check{\pi}^j(\check{p}) \in C([0,1], \mathbb{R})$ is the maximal profit function, where for every $j \in J$
$$\left(\check{\pi}^j(\check{p})\right)(t) = \int_0^1 \check{p}(t) \circ \hat{Q}\left(y^{j*}, t\right) dt = \begin{cases} 0 \text{ if } \ p \in V^T \\ \pi^j(p) \ \text{ if } \ \ p \notin V^T \end{cases},$$

is the dynamical production system assigned by mapping $\hat{Q}$ on the basis of production system P (see def. 2).

Now, similarly to the above, we consider the consumption system defined in time, on the basis of consumption system C by definition 4. Let

- $I = \{1, \ldots, m\}$ be the finite set of consumers from system C,
- $\check{\Xi} \subset [0,1] \times \Xi$ be the family of preference relations defined on the commodity space $\mathbb{R}^\ell$ constant in time, where $\Xi \subset \mathbb{R}^\ell \times \mathbb{R}^\ell$ is the family of preference relations on space $\mathbb{R}^\ell$,
- $\check{\chi} : I \ni i \to \check{X}^i \subset C\left([0,1], \mathbb{R}^\ell\right)$ be the correspondence of consumptions sets which to every consumer i assigns the consumption set $\check{\chi}(i) = \check{X}^i$,
$$\check{X}^i = \left\{\hat{Q}\left(x^i, \cdot\right) \in C\left([0,1], \mathbb{R}^\ell\right) : \ x^i \in X^i\right\} = \{\bar{x}^i \in C\left([0,1], \mathbb{R}^\ell\right) : \ x^i \in X^i\}$$
being a subset of the commodity space and representing the consumer's feasible consumption plans $(\check{\chi}(i))(t)$ at time $t \in [0,1]$, with respect to his psycho – physical structure,
- $\check{e} : I \ni i \to e^i \in C\left([0,1], \mathbb{R}^\ell\right)$ be the function of initial endowment which to every consumer i assigns an initial endowment vector e^i at time $t \in [0,1]$; $\check{e}(i) = \bar{e}^i \in \check{X}^i$; precisely $\forall t \in [0,1] \ \ (\check{e}(i))(t) = e^i$,
- $\check{\varepsilon} \subset [0,1] \times I \times \Xi$ be the correspondence of preference relations, which at every time $t \in [0,1]$, to every consumer $i \in I$ assigns a preference relation $\preccurlyeq^i$ from the set Ξ,
- $\check{p} = \bar{p} \in C\left([0,1], \mathbb{R}^\ell\right)$ is the price (constant) mapping.

We assume, as in the economy E_p, that the expenditures of every consumer i at the time $t \in [0,1]$, cannot be greater than the constant value

$$\breve{w}^i(t) = \breve{p}(t) \circ (\breve{e}(i))(t) = p \circ e^i = w^i. \qquad (24)$$

Similarly, (see remark 1) for every $i \in I$, $x^i \in X^i$ and $t \in [0,1]$

$$\left(\breve{p} \circ \breve{x}^i\right)(t) = \breve{p}(t) \circ \breve{x}^i(t) = p \circ x^i.$$

Now, let

$$\breve{\beta}^i\left(\breve{p}, \breve{w}^i\right) \stackrel{\text{def}}{=} \{\breve{x}^i = \bar{x}^i \in \breve{\chi}(i) : \breve{p} \circ \breve{x}^i \leqslant \breve{w}^i\}$$

and

$$\breve{\varphi}^i(\breve{p}, \breve{w}^i) \stackrel{\text{def}}{=} \{\breve{x}^{i*} \in \breve{\beta}^i\left(\breve{p}, \breve{w}^i\right) : \forall \breve{x}^i \in \breve{\beta}^i\left(\breve{p}, \breve{w}^i\right) \; \breve{x}^i \preccurlyeq^i \breve{x}^{i*}, \preccurlyeq^i \in \breve{\Xi}\} \neq \emptyset.$$

Hence for every $t \in [0,1]$ and $i \in I$

$$(\breve{\beta}^i\left(\breve{p}, \breve{w}^i\right))(t) = \beta^i\left(p, w^i\right) \text{ and } (\breve{\varphi}^i\left(\breve{p}, \breve{w}^i\right))(t) = \varphi^i(p, w^i)$$

and consequently, for every $t \in [0,1]$ and $i \in I$,

$$\breve{\beta}^i\left(\breve{p}, \breve{w}^i\right)(t) \neq \emptyset \text{ and } \breve{\varphi}^i\left(\breve{p}, \breve{w}^i\right)(t) \neq \emptyset.$$

In this situation we put the following:

- $\breve{\beta} : I \ni i \to \breve{\beta}^i\left(\breve{p}, \breve{w}^i\right) \subset C\left([0,1], \mathbb{R}^\ell\right)$ – the correspondence of budget sets at time, which to every consumer $i \in I$ assigns his function of budget constrains, at price system $\breve{p}$ and initial endowment $\breve{e}(i)$; moreover

$$\forall i \in I \quad \forall t \in [0,1] \quad \left(\breve{\beta}(i)\right)(t) \subset (\breve{\chi}(i))(t) \wedge \left(\breve{\beta}(i)\right)(t) = \beta(i)$$

- $\breve{\varphi} : I \ni i \to \breve{\varphi}^i\left(\breve{p}, \breve{w}^i\right) \subset C\left([0,1], \mathbb{R}^\ell\right)$ – the demand correspondence, which to every consumer $i \in I$ assigns his consumption plans $(\breve{\varphi}(i))(t)$ maximizing at time $t \in [0,1]$ his preference on budget set $\left(\breve{\beta}(i)\right)(t)$

$$\forall i \in I \, \forall t \in [0,1] \quad (\breve{\varphi}(i))(t) = \varphi(i).$$

The above leads as to the next definition:

Definition 7. The three – range relational system

$$\breve{C} = (I, C\left([0,1], \mathbb{R}^\ell\right), \breve{\Xi}; \breve{\chi}, \breve{e}, \breve{\varepsilon}, \breve{p}, \breve{\beta}, \breve{\varphi})$$

is called the (constant at time) consumption system.

Let $\breve{p} \in C\left([0,1], \mathbb{R}^\ell\right)$ be the price vector not changing in time ($\breve{p} = \bar{p}$), $\breve{P}$ – a production system and $\breve{C}$– a consumption system in space $C\left([0,1], \mathbb{R}^\ell\right)$. Let $\breve{\theta}$:

$[0,1]\times I\times J \to [0,1]$ be the mapping defined on the basis of function θ (by (5)). Then, for every $t \in [0,1]$:

$$\check{\theta}(t,\, i,\, j) = \theta\,(i,\, j) \tag{25}$$

Moreover, by conditions (5) and (25), we get that

$$\forall t \in [0,1]\ \forall j \in J\ \sum\nolimits_{i\in I} \check{\theta}\,(t, i, j) = 1.$$

Every consumer shares in the producers' profits. Number $\check{\theta}\,(t,\, i,\, j)$ indicates that part of the profit of producer j which is owned by consumer i at time t. In this situation function value $\check{w}^i$ in (24) is changed by the rule

$$\check{w}^i\,(t) = \check{p}\,(t) \circ (\check{e}\,(i))\,(t) + \sum\nolimits_{j\in J} \check{\theta}(t, i, j) \cdot \check{\pi}^j(\check{p}) \tag{26}$$

By (24), (25), (6) and (26), for every $i \in I,\ t \in [0,1]$,

$$\check{w}^i\,(t) = w^i.$$

Let

$$\check{\omega} = \check{e}^1 + \ldots + \check{e}^m \in C\left([0,1]\,,\, \mathbb{R}^\ell\right). \tag{27}$$

Mapping (27) is called the mapping of total endowment. In this situation the finally definition may be formulated:

Definition 8. The relational system $\check{E}_p = (\check{P},\, \check{C}, \check{\theta},\, \check{\omega})$ is called the dynamical private ownership economy with changing production. The state of economy $\check{E}_p$ at the time $t \in [0,1]$ will be denoted by $\check{E}_p(t)$.

By definitions 7 and 8, we additionally get that the state of economy $\check{E}_p$ at the time $t \in [0,1]$ (denoted by $\check{E}_p(t)$) is the Debreu private ownership economy.

At the end of that paper we present the structure of action of economy $\check{E}_p$. Let $t = 0$ be the beginning of process of changing of production system. At time $t = 0$, the action of economy $\check{E}_p$ is the same as in the Debreu private ownership economy E_p. Assume that every producer manages to maximize his profit and every consumer manages to maximize his preference on the budget at the same price vector $p \in \mathbb{R}^\ell$. At every (later) point of time t_1, $0 < t_1 \leqslant 1$, the producers change their production which is reflected in modified production sets $\hat{Q}(Y^j,\, t_1)$. The plan $\hat{Q}(y^{j*},\, t_1)$, for every $j \in J$, maximizes the profit of producer j at price vector $\check{p}\,(t_1) = p$. The optimal consumers' plans remain unchanged. At time $t = 1$, economy $\check{E}_p(t)$ is the Debreu economy with reduced production and consumption sphere with the same subspace of space $\mathbb{R}^\ell$. Moreover, if sequence

$$\left(x^{1*}, \ldots, x^{m*}, y^{1*}, \ldots, y^{n*}, p^*\right) \in \mathbb{R}^\ell$$

is the state of Walras equilibrium in the private ownership economy E_p for $p = p^*$, then for every $t \in [0,1]$, the sequence

$$(x^{1*}, \ldots, x^{m*}, \hat{Q}(y^{1*}, t), \ldots, \hat{Q}(y^{n*}, t), p^*) \in \mathbb{R}^\ell$$

is also the state of Walras equilibrium in the private ownership economy $\check{E}_p(t)$ (see theorems 1 and 2, conditions (24)–(27)).

5. CONCLUSIONS

In the contemporary economy, consumers deals with the great number of pairs of complementary commodities. There are also some commodities which are not wanted by all consumers. The Debreu economy with reduced consumption sphere reflects these situations. That two economic properties of consumption sphere leads to the geometrical properties of consumption plans. The producers will want to change their production to offer production plans satisfying the same properties. The dynamical private ownership economy with changing production models such situation. As the result, the Debreu private ownership economy $\check{E}_p(1)$ (see def. 8) is obtained, in which, in case of the existence of the equilibrium, the producers do not have motivation to change their production plans.

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EKONOMIA ZE ZMIENIAJĄCYMI SIĘ W CZASIE SYSTEMAMI PRODUKCJI I KONSUMPCJI

S t r e s z c z e n i e

Załóżmy, że w ekonomii E_p Debreu z własnością prywatną, istnieją przynajmniej dwa towary, których ilości we wszystkich planach konsumpcji są proporcjonalne. Ta zależność powoduje, że producenci zmieniają swoje plany produkcji, tak aby dopasować je do wymagań konsumentów. W artykule analizowany jest proces dostosowawczy producentów, spowodowany przez wspomnianą własność zbiorów konsumpcji. W rezultacie zdefiniowano ekonomię $\check{E}_p$, ze zmieniającymi się w czasie systemami produkcji i konsumpcji, której przestrzeń towarów i cen jest przestrzenią funkcji ciągłych określonych na przedziale czasu $[0, 1]$, o wartościach w przestrzeni towarów i cen początkowej ekonomii E_p.

Słowa kluczowe: ekonomia Debreu, dobra komplementarne, projekcje, równowaga

THE ECONOMY WITH PRODUCTION AND CONSUMPTION SYSTEMS CHANGING IN TIME

A b s t r a c t

Assume that, in a private ownership Debreu economy E_p, among all the commodities there are at least two whose quantities, employed by consumers in all their plans of action, are proportional. This property leads producers change their production plans to satisfy the consumers' requirements. The producers' adjustment process, caused by the special form of consumers' plans, is modeled. As the result, the economy $\check{E}_p$ with dynamical production and consumption systems is defined. The commodity – price space of economy $\check{E}_p$ is the space of continuous functions defined on time interval $[0, 1]$ with values in the commodity – price space of economy E_p.

Key words: Debreu economy, complementary commodities, projections, equilibrium