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ON SOME INTEGRAL INEQUALITIES FOR (*h, m*)-CONVEX FUNCTIONS

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Abstract. In this paper we establish several Hadamard type inequalities for (*h*, *m*) convex functions.

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1. Introduction

A function $f: I \to R$, $I \subseteq R$ is an interval, said to be a convex function on I if

$$f\left(tx+\left(1-t\right)y\right) \le tf\left(x\right)+\left(1-t\right)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1.1) holds, then f is concave.

Many important inequalities have been established for the class of convex functions, but the most famous is Hermite-Hadamard's inequality. This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.2}$$

where $f: [a,b] \to R$, is a convex function. The above inequalities are in reversed order if f is a concave function.

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In 1978, Breckner introduced the *s*-convex function as a generalization of the convex function (Breckner 1978). Such a function is defined in the following way: a function $f:[0,\infty]\to R$ is said to be *s*-convex in the second sense if

$$f\left(tx+\left(1-t\right)y\right)\le t^{s}f(x)+\left(1-t\right)^{s}f(y) \tag{1.3}$$

holds for all $x,y\in[0,\infty]$, $t\in[0,1]$ and for fixed $s\in[0,1]$.

In (Dragomir, Fitzpatrick 1999) Dragomir and Fitzpatrick proved the following variant of Hermite-Hadamard's inequality which holds for *s*-convex functions in the second sense.

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1}\,. \tag{1.4}$$

In the paper (Varošanec 2007) a large class of non-negative functions, the so-called *h*-convex functions, is considered. This class contains several well-known classes of functions such as non-negative convex functions and s-convex in the second sense. This class is defined in the following way: a non-negative function $f:I\to R$, $I\subseteq R$ is an interval, called *h*-convex if

$$f\left(tx+(1-t)y\right)\le h(t)f(x)+h(1-t)f(y) \tag{1.5}$$

holds for all $x,y\in I$, $t\in(0,1)$, where $h:J\to R$ is a non-negative function, $h\not\equiv 0$ and J is an interval, $(0,1)\subseteq J$.

In (Sarikaya, Saglam, Yildirim 2008) the authors proved that for *h*-convex function the following variant of Hadamard inequality is fulfilled

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\left[f(a)+f(b)\right]\cdot\int_0^1 h(t)dt \tag{1.6}$$

In 1988, Weir and Mond (1998) introduced the preinvex function. Such a function is defined in the following way: a function f on the invex set X is said to be preinvex with respect to η, if

$$f\left(u+t\eta(v,u)\right)\le(1-t)f(u)+tf(v) \tag{1.7}$$

for each $u,v\in X$ and $t\in[0,1]$, where $\eta: X\times X\to R$.

Noor in (Noor 2009) proved the Hermite-Hadamard inequality for the preinvex functions:

$$f\left(a+\frac{1}{2}\eta(b,a)\right)\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\le\frac{f(a)+f(b)}{2}. \quad (1.8)$$

Matłoka introduced in (Matłoka 2013) the *h*-preinvex function in the following way: The non-negative function f on the invex set X is said to be *h*-preinvex with respect to η, if

$$f\left(u+t\eta(v,u)\right)\le h(1-t)f(u)+h(t)f(v) \quad (1.9)$$

for each $u,v\in X$ and $t\in[0,1]$.

In the same paper Matłoka proved the Hermite-Hadamard inequality for the *h*-preinvex functions:

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}f\left(a+\tfrac{1}{2}\eta(b,a)\right)&\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\\&\le\left[f(a)+f(b)\right]\int_0^1 h(t)dt.\end{aligned} \quad (1.10)$$

Toader (1985) defined *m*-convexity in the following way: the function $f:\ [0,b]\to R$, $b>0$, is said to be *m*-convex, where $m\in[0,1]$, if

$$f\left(tx+m(1-t)y\right)\le tf(x)+m(1-t)f(y) \quad (1.11)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

In (Dragomir, Toader 1993) the authors proved the following Hadamard type inequality for *m*-convex functions:

$$\frac{1}{b-a}\int_a^b f(x)dx\le\min\left\{\frac{f(a)+m\,f\left(\frac{b}{m}\right)}{2},\frac{f(b)+m\,f\left(\frac{a}{m}\right)}{2}\right\}. \quad (1.12)$$

In this paper we introduce the concept of the (*h*, *m*)-convex function. The main purpose of this paper is to establish new inequalities of the class of (*h*, *m*)-convex functions.

2. Inequalities for (*h, m*)-convex functions

Definition 2.1. Let $h:[0,1]\to R$ be a nonnegative function, $h\not\equiv 0$. The non-negative function $f:[0,b]\to R$, $b>0$, is said to be (*h*, *m*)-convex, where $m\in[0,1]$, if we have

$$f\left(tx+m(1-t)y\right)\le h(t)f\left(x\right)+mh\left(1-t\right)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

If the above inequality is reversed, then *f* is said to be (*h*, *m*)-preconcave.

Note that if *h*(*t*) = *t* then the *f* above definition reduces to the definition of *m*-convex function.

Definition 2.2. The function $f:[0,b]\to R$, $b>0$, is said to be (*h*, *m*)-logarithmic convex, where $m\in[0,1]$, if

$$\log f(tx+m(1-t)y)\le h(t)\log f\left(x\right)+mh(1-t)\log f(y)$$

for all $x,y\in[0,b]$, $t\in[0,1]$, where $f(\cdot)>0$.

If the above inequality is reversed, then *f* is said to be (*h*, *m*)-logarithmic concave.

From now on we suppose that all the integrals of function *h* considered below exist.

Theorem 2.1. Let $f:[0,\infty]\to R$ be a (*h*, *m*)-convex function with $m\in(0,1]$. If $0\le a<mb<\infty$ and $f\in L^1([a,mb])$, $h\in L^1([0,1])$ then

$$\frac{1}{mb-a}\int_a^{mb}f\left(x\right)dx\le\left[f\left(a\right)+mf\left(b\right)\right]\cdot\int_0^1 h\left(t\right)dt. \tag{2.1}$$

Proof. From the (*h*, *m*)-convexity of f we have

$$f\left(ta+m(1-t)b\right)\le h(t)f\left(a\right)+mh\left(1-t\right)f(b).$$

Thus by integrating over $[0,1]$ we obtain

$$\int_0^1 f\left(ta+m(1-t)b\right)dt\le f(a)\int_0^1 h\left(t\right)dt+mf(b)\int_0^1 h\left(1-t\right)dt.$$

Since,

$$\int_0^1 f\left(ta + m\left(1-t\right)b\right)dt = \frac{1}{mb-a}\int_a^{mb} f\left(x\right)dx$$

then

$$\frac{1}{mb-a}\int_a^{mb} f\left(x\right)dx \le \left[f\left(a\right) + mf\left(b\right)\right]\cdot\int_0^1 h\left(t\right)dt$$

which completes the proof.

Remark 2.1.

– if $m = 1$ and $h(t) = t$ then inequality (2.1) reduces to the right hand of the Hermite-Hadamard inequality for convex function.

– if $m = 1$ and $h(t) = t^s$, $s \in [0, 1]$ then we obtain the right hand of a variant of the Hadamard inequality (1.4) for *s*-convex function in the second sense.

– if $m = 1$ then inequality (2.1) reduces to the right hand of the Hadamard inequality (1.6) for *h*-convex function (see Sarikaya et. al. 2008).

In an analogous way we can prove the following inequality for (h, m)-logarithmic convex function

$$\frac{1}{mb-a}\int_a^{mb} \log f\left(x\right)dx \le \left[\log f\left(a\right) + m\log f\left(b\right)\right]\cdot\int_0^1 h\left(t\right)dt. \tag{2.2}$$

Theorem 2.2. Let f be a (h_1, m)-convex and g a (h_2, m)-convex functions such that $f \cdot g \in L^1\left([a,b]\right)$ and $h_1 \cdot h_2 \in L^1\left([0,1]\right)$. Then the following inequality holds:

$$\begin{aligned}\frac{1}{mb-a}\int_a^{mb} f(x)g(x)dx \le \left[f(a)g(a) + m^2 f(b)g(b)\right]\cdot\int_0^1 h_1(t)h_2(t)dt \\ + m\left[f(a)g(b) + f(b)g(a)\right]\cdot\int_0^1 h_1(t)h_2\left(1-t\right)dt.\end{aligned} \tag{2.3}$$

Proof. Using the fact that *f* and *g* are (h_1, m)-convex and (h_2, m)-convex respectively we have

$$\begin{aligned}&(f\cdot g)\big(ta+m(1-t)b\big)\\ &\le\big[h_1(t)f(a)+mh_1(1-t)f(b)\big]\cdot\big[h_2(t)g(a)+mh_2(1-t)g(b)\big]\\ &=h_1(t)h_2(t)f(a)g(a)+m^2h_1(1-t)h_2(1-t)f(b)g(b)\\ &+mh_1(t)h_2(1-t)f(a)g(b)+mh_1(1-t)h_2(t)f(b)g(a).\end{aligned}$$

Thus, by integrating with respect to t over $[0,1]$, we obtain

$$\int_0^1(f\cdot g)\big(ta+m(1-t)b\big)dt\le\big[f(a)g(a)+m^2f(b)g(b)\big]\int_0^1 h_1(t)h_2(t)dt$$

$$+m\big[f(a)g(b)+f(b)g(a)\big]\int_0^1 h_1(t)h_2(1-t)dt.$$

Since

$$\int_0^1(f\cdot g)\big(ta+m(1-t)b\big)dt=\frac{1}{mb-a}\int_a^{mb}f(x)g(x)dx$$

then we obtain the inequality (2.3).

Theorem 2.3. Let f be a (h_1, m_1)-convex and g a (h_2, m_2)-convex functions such that $f\cdot g\in L^1([a,mb])$ and $h_1\cdot h_2\in L^1([0,1])$. Then the following inequality holds:

$$\begin{aligned}&\frac{1}{mb-a}\int_a^{mb}f(x)g(x)dx\\ &\le\min\Big\{M_1\cdot\int_0^1 h_1(t)h_2(t)dt+M_2\int_0^1 h_1(t)h_2(1-t)dtM_3\\ &\cdot\int_0^1 h_1(t)h_2(t)dt+M_4\int_0^1 h_1(t)h_2(1-t)dt\Big\},\end{aligned}\tag{2.4}$$

where

$$M_1=f(a)g(a)+m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right),$$

$$M_2 = m_2\, f(a)g\left(\frac{b}{m_2}\right) + m_1\, f\left(\frac{b}{m_1}\right)g(a),$$

$$M_3 = m_1 m_2\, f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right) + f\left(b\right)g(b),$$

$$M_4 = m_1\, f\left(\frac{a}{m_1}\right)g\left(b\right) + m_2 f\left(b\right)g\left(\frac{a}{m_2}\right).$$

Proof. Using the fact that f i g are (h_1, m_1)-convex and (h_2, m_2)-convex respectively we have

$$f\left(ta+(1-t)b\right)\cdot g\left(ta+(1-t)b\right) = f\left(ta + m_1(1-t)\frac{b}{m_1}\right)$$

$$\cdot g\left(ta + m_2(1-t)\frac{b}{m_2}\right) \le \left[h_1\left(t\right) f\left(a\right) + m_1 h_1\left(1-t\right) f\left(\frac{b}{m_1}\right)\right]$$

$$\cdot\left[h_2\left(t\right) g\left(a\right) + m_2 h_2\left(1-t\right) g\left(\frac{b}{m_2}\right)\right]$$

$$= h_1(t) f(a) h_2(t) g(a) + m_1 m_2 h_1\left(1-t\right) h_2\left(1-t\right) f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)$$

$$+ m_2 h_1(t) f(a) h_2\left(1-t\right) g\left(\frac{b}{m_2}\right) + m_1 h_1\left(1-t\right) f\left(\frac{b}{m_1}\right) h_2(t) g(a).$$

Integrating both sides of the above inequality over $[0,1]$ we obtain

$$\int_0^1 f\left(ta+(1-t)b\right) g\left(ta+(1-t)b\right)dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx$$

$$\le \left[f(a)g(a) + m_1 m_2 f\left(\frac{b}{m_1}\right) g\left(\frac{b}{m_2}\right)\right]\int_0^1 h_1(t)\, h_2(t)dt$$

$$+\left[m_2 f\left(a\right) g\left(\frac{b}{m_2}\right) + m_1 f\left(\frac{b}{m_1}\right) g(a)\right]\int_0^1 h_1\left(t\right) h_2\left(1-t\right)dt.$$

Analogously we obtain

$$\frac{1}{b-a}\int_a^b f(x)g(x) \le \left[m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) + f(b)g(b)\right]\int_0^1 h_1(t)\,h_2(t)dt$$

$$+\left[m_1\, f\left(\frac{a}{m_1}\right) g(b) + m_2\, f(b) g\left(\frac{a}{m_2}\right)\right]\int_0^1 h_1(t)h_2\left(1-t\right)dt$$

which completes the proof.

Let us note that from the inequality (2.2) it follows the following inequality for (h_1, m)-log-convex function f and (h_2, m)-log-convex function g:

$$\frac{1}{mb-a}\int_a^{mb} \log\left(f(x)\cdot g(x)\right)dx$$

$$\le \left[\log f(a) + m\log f(b)\right]\int_0^1 h_1(t)dt$$

$$+\left[\log g(a) + m\log g(b)\right]\int_0^1 h_2(t)dt.$$

Moreover, if f is (h_1, m)-log-convex and g is (h_2, m)-log-concave then from the some inequality it follows that

$$\frac{1}{mb-a}\int_a^{mb} \log\frac{f(x)}{g(x)}dx \le \left[\log f(a) + m\log f(b)\right]\cdot\int_0^1 h_1(t)dt$$

$$-\left[\log g(a) + m\log g(b)\right]\cdot\int_0^1 h_2(t)dt.$$

Using the technique and ideas of Bakula, Özdemir and Pećarić (2008, Theorem 2.1), one can prove the following theorem.

Theorem 2.4. Let I be an open real interval such that $[0,\infty]\subset I$. Let $f: I \to R$ be a differentiable function on I such that $f' \in L^1\left(\left[a,b\right]\right)$, where $0 \le a < b < \infty$. If $\left|f'\right|^q$ is (h, m)-convex on $\left[a,b\right]$ for some fixed $m \in (0, 1]$ and $q \in \left[1,\infty\right]$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\cdot\left[\int_{\frac{1}{2}}^{1} h(1-t)(2t-1)\,dt+\int_{\frac{1}{2}}^{1} h(t)(2t-1)\,dt\right]^{\frac{1}{q}}$$

$$\cdot\min\left\{\left(|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}},\ \left(m\left|f'\left(\frac{a}{m}\right)\right|^q+\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Proof. First let us note that for a differentiable mapping f such that $f'\in L^1([a,b])$ the following equation holds

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)\,f'\big(ta+(1-t)b\big)dt.$$

First let us suppose that $q = 1$. Then from the above equation we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{b-a}{2}\int_0^1\left|(1-2t)\right|\cdot\left|f'\big(ta+(1-t)b\big)\right|dt.$$

Since $|f'|$ is (h, m)-convex on $[a,b]$ we know that

$$\left|f'\big(ta+(1-t)b\big)\right|=\left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right|\leq h(t)|f'(a)|+mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|,$$

hence

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{2}\int_0^1|1-2t|\cdot\left[h(t)|f'(a)|+mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt=$$

$$= \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}}(1-2t)\left[h(t)\left|f'(a)\right| + mh(1-t)\cdot\left|f'\left(\frac{b}{m}\right)\right|\right]dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1}(2t-1)\left[h(t)\left|f'(a)\right| + mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right]dt\right\}$$

$$= \frac{b-a}{2}\left[\left|f'(a)\right| + m\left|f'\left(\frac{b}{m}\right)\right|\right]\cdot\left[\int_{\frac{1}{2}}^{1}h(1-t)(2t-1)\,dt + \int_{\frac{1}{2}}^{1}h(t)(2t-1)\,dt\right],$$

where we have used the fact that

$$\int_0^{\frac{1}{2}}(1-2t)h(1-t)\,dt = \int_{\frac{1}{2}}^{1}h(t)(2t-1)\,dt$$

and

$$\int_0^{\frac{1}{2}}(1-2t)h(t)dt = \int_{\frac{1}{2}}^{1}h(1-t)(2t-1)\,dt.$$

Analogously we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$= \frac{b-a}{2}\cdot\left[m\left|f'\left(\frac{a}{m}\right)\right| + \left|f'(b)\right|\right]\cdot\left[\int_{\frac{1}{2}}^{1}h(1-t)(2t-1)\,dt + \int_{\frac{1}{2}}^{1}h(t)(2t-1)\,dt\right],$$

which completes the proof for $q = 1$.

Suppose now that $q > 1$. Since $\left|f'\right|^q$ is (*h*, *m*)-convex on $[a,b]$

$$\left|f'(ta+(1-t)b)\right|^q \le h(t)\left|f'(b)\right|^q + mh(1-t)\cdot\left|f'\left(\frac{b}{m}\right)\right|^q$$

hence using the well-known Hőlder inequality we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$=\frac{b-a}{2}\left(\int_0^1|1-2t|dt\right)^{\frac{q-1}{q}}\cdot\left(\int_0^1|1-2t|\cdot\left|f'\left(ta+m(1-t)\frac{b}{m}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le\frac{b-a}{2}\cdot\left(\left[\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt+\int_{\frac{1}{2}}^1 h(t)(2t-1)dt\right]\cdot\left[|f'(a)|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right]\right)^{\frac{1}{q}}$$

and analogously

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{b-a}{2}\cdot\left[\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt+\int_{\frac{1}{2}}^1 h(t)(2t-1)dt\right]^{\frac{1}{q}}\cdot\left[m\left|f'\left(\frac{a}{m}\right)\right|^q+|f'(b)|^q\right]^{\frac{1}{q}}$$

which completes the proof.

Using the identity

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx=\frac{1}{b-a}\int_a^b S(x)f'(x)dx,$$

where

$$S(x)=\begin{cases}x-a, & x\in\left[a,\ \frac{a+b}{2}\right]\\ x-b, & x\in\left[\frac{a+b}{2},\ b\right]\end{cases}$$

(see (Pearce, Pećarić 2000, Theorem 2)) we can prove the following theorem.

Theorem 2.5. Let I be an open real interval such that $[0,\infty)\subset I$. Let $f: I\to R$ be a differentiable function on I such that $f'\in L^1([a,b])$, where $0\le a<b<\infty$. If $|f'|$ is (*h*, *m*)-convex on $[a,b]$ for some fixed $m\in[0,1]$, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le(b-a)\left(\int_0^{\frac{1}{2}} th(t)dt+\int_0^{\frac{1}{2}} th(1-t)dt\right)\min\left\{|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|;m\left|f'\left(\frac{a}{m}\right)\right|+|f'(b)|\right\}.$$

Proof.

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\le\frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}}(x-a)|f'(x)|dx+\int_{\frac{a+b}{2}}^b(b-x)|f'(x)|dx\right]$$

$$=(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{1}{2}}^1(1-t)\left|f'(ta+(1-t)b)\right|dt\right]$$

$$\le(b-a)\left[\int_0^{\frac{1}{2}} t\left(h(t)\left|f'(a)\right|+mh(1-t)\left|f\left(\frac{b}{m}\right)\right|\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1(1-t)\left(h(t)|f'(a)|+mh(1-t)\left|f'\left(\frac{b}{m}\right)\right|\right)dt\right]$$

$$=(b-a)\left(|f'(a)|+m\left|f'\left(\frac{b}{m}\right)\right|\right)\cdot\left(\int_0^{\frac{1}{2}} th(t)dt+\int_0^{\frac{1}{2}} th(1-t)dt\right)$$

and analogously

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left(m\left|f'\left(\frac{a}{m}\right)\right|+m\left|f'(b)\right|\right)\cdot\left(\int_0^{\frac{1}{2}} th(t)dt+\int_0^{\frac{1}{2}} th(1-t)dt\right)$$

which completes the proof.

Now, let us note that it can be easy to prove the following two lemmas.

Lemma 2.1. Let $f: I \to R$, $I \subset R$, be a differentiable mapping on I°, and $a,b \in I$, $m \in [0,1]$ and $a < mb$. If $f' \in L^1([a,mb])$, then

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx=\frac{mb-a}{2}\int_0^1 (1-2t)\, f'\big(ta+m(1-t)b\big)dt.$$

Lemma 2.2. Let $f: I \to R$, $I \subset R$, be a differentiable mapping on I°, and $a,b \in I$, $m \in [0,1]$ and $a < mb$. If $f' \in L^1([a,mb])$, then

$$\frac{1}{mb-a}\int_a^{mb} f(x)dx - f\left(\frac{a+mb}{2}\right)$$

$$=(mb-a)\left[\int_0^{\frac{1}{2}} tf'\big(ta+m(1-t)b\big)dt+\int_{\frac{1}{2}}^1 (t-1)\, f'\big(ta+m(1-t)b\big)dt\right].$$

Theorem 2.6. Let $f: I \to R$, be a differentiable function on I° that $f' \in L^1([a,mb])$, where $a,b \in I$, $m \in [0,1]$ and $a < mb$. If $|f'|$ is (*h*, *m*)-
-convex function, then we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq \frac{mb-a}{2}\big[|f'(a)|+m|f'(b)|\big]\cdot\left[\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt+\int_{\frac{1}{2}}^1 h(t)(2t-1)dt\right].$$

Proof. Using Lemma 2.1 and the (h, m)-convexity of $|f'|$ we have

$$\left|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\right|$$

$$\leq \frac{mb-a}{2}\int_0^1 |1-2t|\left(h(t)|f'(a)|+mh(1-t)|f'(b)|\right)dt$$

$$=\frac{mb-a}{2}\left[\int_0^{\frac{1}{2}}\left(|1-2t|h(t)|f'(a)|+mh(1-t)\left(|f'(b)|\right)\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1 (2t-1)\left(h(t)|f'(a)|+mh(1-t)|f'(b)|\right)dt\right]$$

$$=\frac{mb-a}{2}\left[|f'(a)|\int_0^{\frac{1}{2}}(1-2t)h(t)dt+m|f'(b)|\int_0^{\frac{1}{2}}(1-2t)h(1-t)dt\right.$$

$$\left.+|f'(a)|\int_{\frac{1}{2}}^1(2t-1)h(t)dt+m|f'(b)|\int_{\frac{1}{2}}^1(2t-1)h(1-t)dt\right]$$

$$=\frac{mb-a}{2}\left[|f'(a)|\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt+m|f'(b)|\int_{\frac{1}{2}}^1 h(t)(2t-1)dt\right.$$

$$\left.+|f'(a)|\int_{\frac{1}{2}}^1 h(t)(2t-1)dt+m|f'(b)|\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt\right]$$

$$=\frac{mb-a}{2}\left[|f'(a)|+m|f'(b)|\right]\left[\int_{\frac{1}{2}}^1 h(1-t)(2t-1)dt+\int_{\frac{1}{2}}^1 h(t)(2t-1)dt\right],$$

which completes the proof.

Theorem 2.7. Let $f : I \to R$, be a differentiable function on I°, with $a,b \in I$, $m \in [0,1]$ and $a < mb$. If $|f'|$ is (*h*, *m*)-convex, then we have

$$\left|\frac{1}{mb-a}\int_a^{mb} f(x)dx - f\left(\frac{a+mb}{2}\right)\right|$$

$$=(mb-a)\left[|f'(a)|+m|f'(b)|\right]\cdot\left[\int_0^{\frac{1}{2}} th(t)dt+\int_0^{\frac{1}{2}} th(1-t)dt\right].$$

Proof. Using Lemma 2.2 and the (*h*, *m*)-convexity of $|f'|$, it follows that

$$\left|\frac{1}{mb-a}\int_a^{mb} f(x)dx - f\left(\frac{a+mb}{2}\right)\right|$$

$$\le (mb-a)\left[\int_0^{\frac{1}{2}} |t|\left|f'\left(ta+m(1-t)b\right)\right|dt+\int_{\frac{1}{2}}^{1} |t-1|\left|f'\left(ta+m(1-t)b\right)\right|dt\right].$$

$$\le (mb-a)\left[\int_0^{\frac{1}{2}} |t|\left(h(t)|f'(a)|+mh(1-t)|f'(b)|\right)dt\right.$$

$$\left.+\int_{\frac{1}{2}}^{1} |t-1|\left(h(t)|f'(a)|+mh(1-t)|f'(b)|\right)dt\right]$$

$$=(mb-a)\left[|f'(a)|\int_0^{\frac{1}{2}} t\,h(t)dt+m|f'(b)|\int_0^{\frac{1}{2}} t\,h(1-t)dt\right.$$

$$\left.+|f'(a)|\int_0^{\frac{1}{2}} t\,h(1-t)dt+m|f'(b)|\int_0^{\frac{1}{2}} t\,h(t)dt\right]$$

$$=(mb-a)\left[|f'(a)|+m|f'(b)|\right]\cdot\left[\int_0^{\frac{1}{2}} th(t)dt+\int_0^{\frac{1}{2}} th(1-t)dt\right]$$

which completes the proof.

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