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TESTS OF MULTIVARIATE NORMALITY USING SHAPE MEASURES OF THE DISTRIBUTION

ABSTRACT. Karl Pearson, in 1990, proposed two numerical characteristics of the distribution of random variables i.e. asymmetry (skewness) and kurtosis (flatness). Their sample approximations allow to describe partially the empirical distribution, to find out if it differs from a symmetric distribution and if it is exceedingly flat or high.

The measures of shape for distributions with known first four central moments are uniquely defined, in particular, for the univariate normal distribution they are equal to 0 and 3. It allows to compare distributions with known measures of shape with the normal distribution. Such comparisons in univariate case is done by means of standardized tests based on the third and fourth sample central moments. An overview of such tests may be found in the work by D'Agostino and Pearson (1973).

The translation of shape measures to multivariate case was done by Mardia (1970). These measures significantly enriched the statistical description of empirical distributions and allowed to introduce many tests of multivariate normality. The distributions of these tests' statistics using sample multivariate asymmetry and kurtosis are usually derived through central limit theorems.

In the work an overview of multivariate normality tests based on the sample measures of asymmetry and kurtosis is given. The statistical properties of these measures are discussed as well as the usefulness of these tests with respect to power and sample size.

Key words: multivariate normality, statistical tests, shape measures.

I. DEFINITION OF MULTIVARIATE RANDOM VECTOR'S ASYMMETRY AND KURTOSIS

Let X be p-dimensional random vector with a distribution given by the cumulative function $F_p(x, \mu, \Sigma) = F_p$, where $x \in R^p$, μ is a p-dimensional vector of expected values and Σ, by assumption, is a $p \times p$-dimensional covariance

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matrix positive definite. When the distribution considered is multivariate normal its cumulative distribution function is denoted by $N_p(x, \mu, \Sigma) = N_p$.

Multivariate asymmetry and kurtosis denoted with β_{1p}, β_{2p} as random vector's X measures of shape, are Pearson's numerical characteristics $\sqrt{\beta_1}$ and β_2 of univariate case, generalized to p-dimensional case. The numerical characteristics β_{1p}, β_{2p} are defined with a form

(a) bilinear

$$\beta_{1p} = E\left\{ \left[(X - \mu)'\Sigma^{-1}(X_* - \mu) \right]^3 \right\}$$

for independent p-dimensional random vectors X, X_* identically distributed,

(b) quadratic

$$\beta_{2p} = E\left\{ \left[(X - \mu)'\Sigma^{-1}(X_* - \mu) \right]^2 \right\}$$

for p-dimensional random vector X, where Σ^{-1} is the inverse matrix of Σ. In particular, for $p = 1$, we have $\beta_{11} = \beta_1$ and $\beta_{21} = \beta_2$.

The properties of β_{1p}, β_{2p} when $X \sim N_p$ are given by two lemmas.

Lemma 1. If $X \sim N_p$ then $\beta_{1p} = 0$.

Proof. Let $Y = \Sigma^{-1/2}(X - \mu)$. Then $Y \sim N_p(0, I)$, where I is a unit matrix and $\beta_{1p} = E\left\{ (Y'Y_*)^3 \right\} = 0$, because ordinary moments of odd order of the standard normal distribution are equal to zero and each component of two-linear form $Y'Y_*$ contains at least one variable in odd power.

Lemma 2. If $X \sim N_p$ then $\beta_{2p} = p(p + 2) = g$.

Proof. Let us make use of the formulae for the expectation and variance of the quadratic form

$$E(X'AX) = E(X')AE(X) + tr(\Sigma A),$$

$$D^2(X'AX) = 4E(X')A\Sigma AE(X) + 2tr\left[(\Sigma A)^2 \right],$$

where $tr(\cdot)$ stands for matrix trace. After substitutions $X \to (X - \mu)$ and $A \to \Sigma^{-1}$, we get

$$\beta_{2p} = E\left\{ \left[(X-\mu)'\Sigma^{-1}(X_\bullet - \mu)\right]^2 \right\} = D^2\left(\left[(X-\mu)'\Sigma^{-1}(X_\bullet - \mu)\right]^2 \right) +$$
$$+\left\{ E\left[(X-\mu)'\Sigma^{-1}(X_\bullet - \mu)\right] \right\}^2 = 2p + p^2 = p(p+2).$$

When $p = 2$, i.e. for twovariate distributions, we have

$$\beta_{12} = (1-\rho^2)^{-3}\{\gamma_{30}^2 + \gamma_{03}^2 + 3(1+2\rho^2)(\gamma_{12}^2 + \gamma_{21}^2) - 2\rho_{\gamma30\gamma03}^3 +$$
$$+6\rho\{\gamma_{30}(\rho\gamma_{12} - \gamma_{21}) + \gamma_{03}(\rho\gamma_{21} - \gamma_{12}) - (2+\rho^2)\gamma_{12}\gamma_{12}]\},$$
$$\beta_{22} = \left[\gamma_{40} + \gamma_{04} + 2_{\gamma22} + 4\rho^2(\gamma_{22} - \gamma_{13} - \gamma_{31})\right]/(1-\rho^2)^2,$$

where

$$X = (X_1, X_2)', \quad \mu = (\mu_1, \mu_2), \quad \sigma_1^2 = D^2(X_1), \quad \sigma_2^2 = D(X_2),$$
$$\rho = \text{Corr}(X_1, X_2),$$
$$\gamma_{rs} = \mu_{rs}/(\sigma_1^{r/2}\sigma_2^{p/2}), \quad \mu_{rs} = E\{(X_1-\mu_1)^r(X^2-\mu_2)^s\}$$

In particular, when $\rho = 0$, i.e. when the coordinates of the two dimensional random vector are independent, then

$$\beta_{12} = \gamma_{30}^2 + \gamma_{03}^2 + 3(\gamma_{12}^2 + \gamma_{21}^2) \quad \text{and} \quad \beta_{22} = \gamma_{40} + \gamma_{04} + 2\gamma_{22}.$$

Now we will prove lemmas 1 and 2 in the case of $p = 2$, i.e. for the two dimensional normal distribution.

Lemma 3. $\beta_{12} = 0$ and $\beta_{22} = 8$ if $(X_1, X_2)' \sim N_2$.

Proof: We apply the well known formulae for distribution's moments (see e.g. Kendall and Stuart 1963 p. 91) : $\gamma_{12} = \gamma_{21} = \gamma_{30} = 0$ from which it immediately follows that $\beta_{12} = 0$. To prove β_{22} we apply other known formulae $\gamma_{40} = \gamma_{04} = 3$, $\gamma_{31} = \gamma_{13} = 3$ and $\gamma_{22} = 1 + 2\rho^2$, from where we have

$$\beta_{22} = \{3 + 3 + 2(1 + 2\rho^2) + 4\rho^2[1 + 2\rho^2 - 3 - 3]\}/(1-\rho^2)^2 =$$
$$= \{8 - 16\rho^2 + 8\rho^4\}/(1-\rho^2)^2 = 8(1 - 2\rho^2 + \rho^4)/(1-\rho^2)^2 = 8.$$

The formulae for β_{12} and β_{22} for some two dimensional distributions are given in the form of (Mardia 1974, Mardia et al. 1979, Davis 1980):

a) the mixture of two dimensional normal distributions $h(x) = 0.8 f(x, 0, I) + 0.2 f(x, 0, \sigma^2 I)$, where $f(\cdot)$ is the density function and $\beta_{12} = 0$, $\beta_{22} = 8\left(4\dfrac{(\sigma^2 - 1)^2}{(\sigma^2 + 4)^2} + 1\right)$;

b) two dimensional gamma distribution

$$h(x) = \frac{1}{36}(x_1 x_2)^3 e^{-x1-x2}, \text{ for } (x_1, x_2)' \in R_+^2 = (0, \infty) x (0, \infty) - \beta_{12} = 0,$$
$$\beta_{22} = 11,$$

c) two dimensional exponential distribution

$$P(X_1 > x_1, X_2 > x_2) = \exp[-x_1 - x_2 - \max(x_1, x_2)],$$
$$(x_1 x_2) \in R_+^2; \quad \beta_{12} = \frac{3\rho^4 + 9\rho^3 + 15\rho^2 + 12\rho + 4}{2(1 - \rho^2)^3},$$
$$\beta_{22} = \frac{5 + \rho - \rho^2 - 3\rho^3}{4(1 - \rho^2)},$$

d) twodimensional Morgenstern distribution

$$h(x) = 1 + 3\rho(1 - 2x_1)(1 - 2x_2), \quad x \in (0,1) x (0,1), \beta_{12} = 0,$$
$$\beta_{22} = 4(7 - 13\rho^2)/(5(1 - \rho^2)^2).$$

II. MULTIVARIATE SAMPLE ASYMMETRY AND KURTOSIS

The estimators b_{1p} and b_{2p} of β_{1p} and β_{2p} based on p-dimensional sample $U = \{X_1, X_2, \ldots, X_n\}$ are expressed through the powers of bilinear and quadratic form (Mardia 1970, 1974, 1977) in the following way:

a) sample multivariate asymmetry (skewness)

$$b_{1p} = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{j'=1}^{n} g^3_{jj'},$$

b) sample multivariate kurtosis (flatness)

$$b_{2p} = \frac{1}{n}\sum_{j=1}^{n} g^2{}_{jj},$$

where $g_{ij'} = (X_j - \bar{X})' S^{-1}(X_{j'} - \bar{X})$ and $\bar{X}$ and S are the mean vector and covariance matrix based on sample U. The forms $g_{ij'}$ and g_{jj} may also be expressed through scaled residual vectors $Y_j = S^{-1/2}(X_j - \bar{X})$, then $g_{ij'} = Y'_j Y_{j'}$ and $g_{ij} = Y'_{j'} Y_j$. In this notation $S^{-1/2}$ stands for the inverse matrix $S^{1/2}$, so that $S^{1/2}(S^{1/2})' = S$.

The random variables b_{1p} and b_{2p} have distributions implied the distribution of the random vector X whose independent realizations are expressed by matrix U. For these variables the distribution characteristics in the case of the multivariate normal distribution are the following (Mardia 1970, 1974, 1977, Mardia and Kazanawa 1983, Mardia and Foster 1983):

a) $E(b_{1p}) = \dfrac{g((n+1)(p+1)-6)}{(n+1)(n+3)}$, $g = p(p+2)$,

b) $D^2(b_{1p}) = \dfrac{12(p+1)(p+2)}{n^2}$,

c) $E(b_{2p}) = \dfrac{g(n-1)}{n+1}$,

d) $D^2(b_{2p}) = \dfrac{8g(n-3)(n-p-1)(n-p+1)}{(n+1)^2(n+3)(n+5)}$,

e) $E(b_{2p}^3) = g^3 - \dfrac{12g^2(p+2)(p-1)}{n} +$
$+\dfrac{2g(9p^4 + 12p^3 - 192p^2 - 328p + 256)}{n^2}$,

f) $\quad \mu_3(b_{2p}) = \dfrac{64g(p+8)}{n^2}$,

g) $\quad \beta_1(b_{2p}) = \dfrac{8(p+8)^2}{gn^4}\left[\dfrac{(n+1)^2(n+3)(n+5)}{(n-3)(n-p-1)(n-p+1)}\right]^3$,

h) $\quad Cov(b_{1p}, b_{2p}) = 12ph/n^2$, $\quad h = 8p^2 - 13p + 23$,

i) $\quad Corr(b_{1p}, b_{2p}) = \dfrac{3h}{[6n(p+1)(p+2)^2]^{1/2}}$,

j) $\quad E(\sqrt{b_{1p}}) = \dfrac{\sqrt{12v(f)}}{n}$, $\quad f = \begin{pmatrix} p+2 \\ 3 \end{pmatrix}$, $\quad v(f) = \dfrac{\Gamma\left(\dfrac{f+1}{2}\right)}{\Gamma\left(\dfrac{f}{2}\right)}$,

k) $\quad D^2(\sqrt{b_{1p}}) = \dfrac{6f - 12v^2(f)}{n}$,

l) $\quad Cov(\sqrt{b_{1p}}, b_{2p}) = \dfrac{6ph}{n\sqrt{nf}}$,

m) $\quad \beta_1(\sqrt{b_{1p}}) = \dfrac{\sqrt{2v(f)}(1 - 2f + 4v^2(f))}{(f - 2v^2(f))^{3/2}}$,

n) $\quad Corr(\sqrt{b_{1p}}b_{2p}) = \dfrac{\dfrac{\sqrt{6ph}}{n\sqrt{nf}}}{\left[\dfrac{8g(6f - 12v^2 9f)}{n^2}\right]^{1/2}}$.

The above given formulea for the moments of random variables b_{1p} and b_{2p} are correct for big n, and their approximations were given up to the order of $O(n^{-2})$.

III. TESTS OF MULTIVARIATE NORMALITY BASED ON b_{1p} AND b_{2p}

For the distribution N_p we have $\beta_{1p} = 0$, and $\beta_{2p} = p(p+2) = g$. Investigating the p-dimensional empirical distribution P_p by means of independent observable random vectors $X_1, X_2, ..., X_n$ we ask if they come from a multivariate population with $\beta_{1p} = 0$ or $\beta_{2p} = g$ or, simoultaneously, $\beta_{1p} = 0$ and $\beta_{2p} = g$. This leads to define the null hypothesis as $H_0: P_p \in N_p$ against the alternative hypothesis determined by one of the distribution classes:

$$A_1 - \beta_{1p} \neq 0, \; \beta_{2p} = g; \; A_2 - \beta_{1p} = 0, \; \beta_{2p} \neq g; \; A_3 - \beta_{1p} \neq 0, \; \beta_{2p} \neq g.$$

There are many tests of multivariate normality to verify the hypothesis formulated which are based on sample statistics b_{1p} and b_{2p}. We will differentiate between the omnibus and directed tests.

Definition 1. Statistical tests for a determined class of alternative distributions will be called directed tests.

Definition 2. Statistical tests most powerful in the class of possible alternative distributions will be called omnibus tests.

To verify the null hypothesis against $H_1: P_p \in A_1$ or $H_1: P_p \in A_2$ we use the tests of multivariate normality based on the tests' statistics being the equivalents of b_{1p} and b_{2p}. The directed tests used will be most powerful for distribution classes A_1 and A_2. For the omnibus tests and the alternative defined by family A_3 we apply the tests' statistics being functions b_{1p} or b_{2p}. These tests have the property of simoultaneous assessment of the departures of multivariate assymetry and kurtosis of the empirical distribution P_p from $\beta_{1p} = 0$ and $\beta_{2p} = g$. These tests are recommended whenever we do not have any prior information about the distribution specified in the alternative distribution.

The descriptive statistics based on b_{1p} and b_{2p} have distributions known for big n, which are given by lemmas 4 and 5.

Lemma 4. $nb_{1p}/6 \sim \chi_f^2$, $f = \begin{pmatrix} p+2 \\ 3 \end{pmatrix}$, when $U \sim MN_p$ that is when U has the matrix normal distribution (see e.g. Wagner 1990).

Lemma 5. $\left(b_{2p} - E(b_{2p}) \right)/ D(b_{2p}) \sim N(0,1)$.

The proofs of the lemmas may be found in Mardia (1970, 1974), as well as in Domański Wagner (1982).

From the applicational point of view, one differentiates between the tests of multivariate normality based on b_{1p} and b_{2p} with respect to the determined class of alternative distributions, in the following way:

- $A_1 - M_1, C_1, L(b_{1p}), U(b_{1p}), W(b_{1p}), Q_1$ tests;

- $A_2 - M_2, C_2, U(b_{2p}), W(b_{2p}), Q_2$ tests;

- $A_3 - M_3, C_3, C_4, S_L^2, S_N^2, S_W^2, C_N^2, C_R^2, Q$ tests.

In what folows, we review the above mentioned tests, limiting ourselves to mentioning: (1) author (or authors), (2) test's statistic and (3) the distribution of the test's statistic, always assuming that $U \sim MN_p$.

A. Tests for hypothesis $H_0 : P_p \in N_p$ or $H_1 : P_p \in A_1$..

(a) (1) Mardia (1970); (2) $M_1 = nb_{1p}/6$; (3) χ_f^2 (lemma 4);

(b) (2) Bera i John (1983); (2) $C_1 = \dfrac{n}{6} \sum_{i=1}^{p} T_i^2$, (3) χ_p^2,

$T_i = \sum_{j=1}^{n} Y_{ij}^2 / n, i = 1,2..., p$;

$Y_j = S^{-1/2}(X_j - \overline{X}) = (Y_{1j}, Y_{2j}, ..., Y_{pj})'$;

(c) (1) Mardia and Foster (1983); (2) $L(b_{1p}) = \gamma + \delta \ln(b_{1p} - \xi)$,

(3) $N(0,1)$ γ, δ, ξ - S_L Johnson distribution parameters;

(d) (1) Mardia and Foster (1983); (2) $U(b_{1p}) = \dfrac{b_{1p} - 6f/n}{6\sqrt{2f/n^2}}$; (3) N(0,1);

(e) (1) Mardia and Foster (1983); (2) $W(b_{1p}) = \left[\left(\dfrac{4}{3} nf^2 b_{1p} \right)^{1/3} - 3f + \dfrac{2}{3} \right] / \sqrt{2f}$

   (the Wilson-Hilferty approximation of distribution b_{1p}); (3) $N(0,1)$.

B.   Tests for hypothesis $H_0 : P_p \in N_p$ or $H_1 : P_p \in A_2$.

(f) (1) Mardia (1970); (2) $M_2 = \dfrac{(b_{2p} - g)^2}{8g/n}$; (3) χ_1^2;

(g) (1) Bera and John (1983); (2) $C_2 = \dfrac{1}{24} \left[n \sum\limits_{i=1}^{p} (T_{ii} - 3)^2 + \sum\limits_{1 \le i < i' \le p} (T_{ii'} - 1)^2 \right]$;

   (3) $\chi_{p(p+1)/2}^2$; $T_{ii} = \sum\limits_{j=1}^{n} Y_{ij}^4 / n$, $T_{ii'} = \sum\limits_{j=1}^{n} (Y_{ij} Y_{i'j})^2 / n$, Y_{ij} as in (b);

(h) (1) Mardia and Foster (1983); (2) $U(b_{2p}) = \dfrac{b_{2p} - g(n-1)/(n+1)}{\sqrt{8g/n^2}}$;

   (3) N(0,1);

(i) (1) Mardia i Foster (1983);

   (2) $W(b_{2p}) = 3\sqrt{\dfrac{f_1}{2}} \left\{ 1 - \dfrac{2}{9f_1} - \dfrac{1 - 2/9f_1}{1 + a\sqrt{2/(f_1 - 4)}} \right\}^{1/3}$; (3) N(0,1);

   $f_1 = 6 + 4(d + \sqrt{d + d^2})$, $d = \dfrac{np(p+2)}{2(p+8)^2}$, $a = \dfrac{b_{2p} - E(b_{2p})}{D(b_{2p})}$.

C. Tests for hypothesis $H_0 : P_p \in N_p$ or $H_1 : P_p \in A_3$.

(j) (1) Jargune and Mckenzie (1982); (2) $M_3 = M_1 + M_2$; (3) χ_{f+1}^2;

(k) (1) Bera and John (1983); (2) $C_3 = \dfrac{n}{24} \left[4 \sum\limits_{i=1}^{p} T_{ii}^2 + \sum\limits_{i=1}^{p} (T_{ii} - 3)^2 \right]$; (3) χ_{2p}^2;

(l) (1) Bera and John (1983); (2) $C_4 = C_1 + C_2$; (3) $\chi_{p(p+3)/2}^2$;

(m) (1) Mardia and Foster (1983); (2) $S_L^2 = L^2(b_{1p}) + U^2(b_{2p})$,

$S_N^2 = U^2(b_{1p}) + U^2(b_{2p})$, $S_W^2 = W^2(b_{1p}) + W^2(b_{2p})$,

$C_N^2 = b'V^{-1}b$, $C_R^2 = d'V^{-1}d$,

$$b = \begin{bmatrix} b_{1p} - 6f/n \\ b_{2p} - g(n-1)/(n+1) \end{bmatrix}, \quad V = \begin{bmatrix} 72f/n^2 & 12ph/n^2 \\ 12ph/n^2 & 8g/n \end{bmatrix},$$

$$d = \begin{bmatrix} \sqrt{b_{1p}} - E(\sqrt{b_{1p}}) \\ b_{2p} - \dfrac{n-1}{n+1} \end{bmatrix},$$

(3) all statistics mentioned here follow the chi-square distribution with two degrees of freedom;

(n) Small (1980); (2) $Q_1 = Y'_{(1)} U_{(1)}^{-1} Y_{(1)}$, $P_p \in A_1$; $Q_2 = Y'_{(2)} U_{(2)}^{-1} Y_{(2)}$, $P_p \in A_2$

$Q = Q_1 + Q_2$, $P_p \in A_3$,

$$Y_{(1)} = \begin{bmatrix} \delta_1^* \sinh^{-1}(\sqrt{b_1(X_1)}/\lambda_1^*) \\ \cdots \\ \delta_1^* \sinh^{-1}(\sqrt{b_1(X_p)}/\lambda_1^*) \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} \gamma_2 + \delta_2 \sinh^{-1}[(b_2(X_1) - \xi_2)/\lambda_2] \\ \cdots \\ \gamma_2 + \delta_2 \sinh^{-1}[(b_2(X_p) - \xi_2)/\lambda_2] \end{bmatrix},$$

$$U_{(1)} = (r_{ii'}^3), \quad U_{(2)} = (r_{ii'}^4), \quad i, i' = 1,2,\ldots,p,$$

where $r_{ii'}$ are sample coefficients of linear correlation determined from matrix U. The mentioned constants δ_1^* and $1/\lambda_1^*$ can be found in the tables given by D'Agostino and Pearson (1973). The remaining constants $\delta_2, \gamma_2, \lambda_2, \xi_2$ are determined according to the principle of parameter estimation in the S_L Johnson distribution family.

IV. THE PROPERTIES OF MULTIVARIATE NORMALITY TESTS USING b_{1p} AND b_{2p} STATISTICS

In chapter 3 we mentioned many tests of multivariate normality of the directed and omnibus type. Some of them have simple form of tests' statistics, other require additional numerical calculations.

More important properties of multivariate tests of normality using b_{1p} and b_{2p} statistics are as follows:

a) they use scaled vectors of residuals allowing to find big residuals when big were the b_{1p} and b_{2p} statistics;

b) they use the numerical characteristics of the distribution of b_{1p} and b_{2p} statistics mentioned in chapter 3;

c) they make use of constant parameters of the Johnson's family of distributions which are determined by means of special numerical methods;

d) they have the limiting distribution either normal or chi-square;

e) the chi-square degrees of freedom depend only on p;

f) tests are appropriate for big n, because the numerical characteristics of b_{1p} and b_{2p} were given for order $O(n^{-2})$;

g) the omnibus tests using both b_{1p} and b_{2p} will be good for applications in which the correlation between random variables b_{1p} and b_{2p} is sufficiently small i.e. whenever there the condition is $n > 300 \left(8p^2 - 13p + 23 \right)^2 / (p+1)$ met;

h) the Wilson-Hilferty transformation for constructing tests based on b_{1p} may replaced with a transformation corrected by Goldstein (1973);

i) because b_{1p} and b_{2p} are invariant with respect to affine transformations, the tests based on b_{1p} and b_{2p} possess same property ;

j) there are no major numerical difficulties in determining tests' statistics when one assumes that the sample covariance matrix is positive defined;

k) tests based on b_{1p} and b_{2p} have moderate power for undetermined alternative distributions and the power is higher for directed tests.

To illustrate the strength of correlation between b_{1p} and b_{2p} influencing the usefulness of some tests of multivariate normality, in table 1 we present the results of our own computations:

Table 1

Correlation between b_{1p} and b_{2p}

p	n	Corr	p	n	Corr
1	30	0.9487	4	30	1.6499
	50	0.7348		50	1.2781
	100	0.5164		100	0.9087
	200	0.3674		200	0.6390
2	30	0.9360	5	30	2.0605
	50	0.7250		50	1,5960
	100	0.5127		100	1.1285
	200	0.3625		200	0.7980
3	30	1.2522	6	30	0.9533
	50	0.9699		50	0.7784
	100	0.6859		100	0.6029
	200	0.4649		200	0.4767

The above presented calculations prove how important it is to determine the proper sample size with respect to the number of variables investigated p in order to get a reasonable correlation measure. The general coclusion is that this number grows with the number of variables.

V. CONCLUSIONS

We gave an overview of more important tests of multivariate normality based on statistics b_{1p} and b_{2p}, corresponding to multivariate sample asymmetry and curtosis. These tests were also extensively discussed by K. V. Mardia in the seventies and eighties.

The tests mentioned are characterised with good power, especially in the case of directed tests. Many tetsts use the sample vector of means and sample covariance matrix which is assumed to be positive defined. These assumptions may be weakened, then, instead of normal inversion the so called g-inversion is applied. This makes the scope of practical applications of the tests discussed, much wider.

Investigating tests using statistics b_{1p} and b_{2p} has been slightly curtailed, mainly, due to the fact that more powerful tests of multivariate normality have been proposed (e.g. those based on stochastic processes or empirical charteristic functions).

A general overview of other tests of multivariate normality based on ran-
domization principle, union and Roy's intersection, power transformation as
well as radii and angles with the use of multivariate geometry was given in
a monograph by Domański et al. (1998). On the other hand, Wagner (1990),
gives a generalized Shapiro-Wilk test.

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TESTY WIELOWYMIAROWEJ NORMALNOŚCI KORZYSTAJĄCE Z MIAR KSZTAŁTU ROZKŁADU

Miary kształtu rozkładu jedno- i wielowymiarowych zmiennych losowych znajdują powszechne zastosowanie w konstrukcji testów jedno- i wielowymiarowej normalności. Przy ich konstrukcji korzysta się z pierwszych czterech momentów centralnych wyprowadzanych z odpowiednich statystyk próbkowych przy odpowiednich założeniach stochastycznych.

W pracy dokonano przeglądu testów wielowymiarowej normalności opartych na próbkowych miarach asymetrii i kurtozy. Podano różne ich własności statystyczne, uwzględniające wielkości prób oraz postacie przekształcone do jednowymiarowych statystyk próbkowych. Załączone zostały również wyniki badań dotyczące mocy testów.