ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 156, 2002

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TESTING MULTIVARIATE NORMALITY BY DATA TRANSFORMATIONS

Abstract. The problem of testing the hypothesis of multivariate normality is discussed. Several methods of transformations to univariate normal samples are compared. An extensive simulation study for the comparison of various tests is performed under broad range of alternatives. Numerical experiment shows that the testing procedures combining simple approximate transformations to univariate normality and powerful tests for univariate normality give quite interesting results.

Key words: multivariate normality, multivariate beta distribution, *W*-test, *p*-values, combining tests.

1. INTRODUCTION

Although the assumption of multivariate normality is very common in many multivariate data analysis methods (like multivariate regression, principal components, ect.) it is, in practice, seldom verified because of the lack of simple testing procedures. Various tests for the multivariate normality were, of course, proposed: for classical overviews see e.g. Malkovich, Afifi (1973) and Mardia (1980); among tests proposed in the recent years there are: test based on empirical characteristic function (Baringhaus and Henze (1988)), tests based on distance and directions (see Dunn (1995)), methods based on density estimates (Bowman, Foster (1993)), methods for combining independent tests of the univariate normality (Mudholkar, Srivastava, Lin (1995)), test based on interpoint

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distances proposed in Bartoszyński, Pearl and Lawrence (1997) and the test based on the combination of the Shapiro–Wilk test for marginals and the principal components method (Peterson and Stromberg (1998)). However, these tests are too complicated in general. From the practical standpoint it would be more interesting to decompose the problem of testing the multivariate normality into the problem of testing univariate normality.

In this paper we compare various existing schemes of transformations multivariate samples into univariate samples and then we test normality using well known univariate tests. An extensive Monte Carlo study was performed under broad range of alternatives. Numerical experiment shows that the testing procedures combining simple approximate transformations to univariate normality and powerful tests for univariate normality give quite interesting results.

II. TRANSFORMATIONS

Let us denote by $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_n)$ a $p \times n$ matrix of n observations in p-dimensions. A covariance matrix $\mathbf{S}$ is defined as $\mathbf{S} = \frac{1}{n}(\mathbf{X} - \overline{\mathbf{X}})(\mathbf{X} - \overline{\mathbf{X}})'$ where $\overline{\mathbf{X}} = \frac{1}{n}(\mathbf{X}_1 + ... + \mathbf{X}_n)$ is sample mean vector. Below we describe five methods of transformation $\mathbf{X}$ into p independent univariate samples.

Transformation I. Initial multivariate data are transformed into the scaled residuals $\mathbf{Z} = \mathbf{S}^{-1/2}(\mathbf{X} - \overline{\mathbf{X}})$, where $\mathbf{S}^{-1/2}$ is obtained from the equation

$$\mathbf{S}^{1/2}(\mathbf{S}^{1/2})' = \mathbf{S}$$

the decomposition of matrix $\mathbf{S}$ by Choleski. The output data $\mathbf{Z}$ are approximately independent standard normals.

Transformation II. Let $\mathbf{V} = diag(S_{11}^{-1/2}, ..., S_{pp}^{-1/2})$, $\mathbf{C} = \mathbf{VSV}$ (this gives the correlation matrix) and $\Lambda = diag(\lambda_1, ..., \lambda_p)$ is matrix with the eigenvalues of $\mathbf{C}$ on the diagonal. Then we define transformed data matrix as

$$\mathbf{Z} = \mathbf{H}\Lambda^{-1/2}\mathbf{H}'\mathbf{V}(\mathbf{X} - \overline{\mathbf{X}})$$

where the columns of $\mathbf{H}$ are the corresponding eigenvectors, such that $\mathbf{H}'\mathbf{H} = \mathbf{I}_p$ and $\Lambda = \mathbf{H}'\mathbf{C}\mathbf{H}$. This method also gives approximately standard normals. For more details see Doornik (1994).

Transformation III. This method is based on the regression model obtained as the conditional distribution of $\mathbf{X}_i$ given $\mathbf{X}_1, ..., \mathbf{X}_{i-1}$. Mudholkar, Srivastava and Lin (1995) proposed an algorithm which under assumption that random sample **X** of size n is taken from p-variate normal population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ leads to p indepenedent normal vectors $\mathbf{Z}'_1, \mathbf{Z}'_2, ..., \mathbf{Z}'_p$ of sizes n, $(n-2)$, $(n-3)$, ... $(n-p)$, respectively.

We suggest to use their algorithm followed by the Durbin randomization method (see Durbin (1961)), which transforms vectors $\mathbf{Z}'_1, ..., \mathbf{Z}'_p$ into independent samples from standard normal distribution.

Transoformation IV. This method is a modification and simplification of the regression approach by Mudholkar, Srivastava and Lin mentioned above. However, now we obtain p approximately independent normal vectors $\mathbf{Z}'_i, ..., \mathbf{Z}'_p$ each of size n. Then using the Durbin randomization method we may also obtain independent samples from standard normal distribution.

Transformation V. Wagle (1968) proposed to use the multivariate beta distribution and the principle of randomization to transform sample **X** from the multivariate normal population $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with unknown parameters into a sample **Z** from distribution $N_p(\mathbf{0}, \mathbf{I}_p)$. It means that we have pn independent observations from the univariate standard normal distribution.

III. TESTS

Now in order to verify a hypothesis of the multivariate normality we have to combine one of the data transformation ξ described above with a test φ for the univariate normality. Such superposition $T = \varphi \circ \xi$ of the transformation method with a test for the univariate normality also forms a testing procedure. Thus combining different transformations with different tests for the univariate normality we get various tests for the multivariate normality (tests T10–T18). We will investigate their statistical properties in order to find optimal combination. Moreover, we will compare these tests with other known tests for the multivariate normality (tests T1–T9). Here are a detailed list of tests used in our simulation study:

T1 – test based on transformed skewness and kurtosis measures, see Doornik (1994);

T2 – „classical" Mardia's test based on multivariate skewness, see Mardia (1970);

T3 – Mardia's test based on the multivariate kurtosis, see Mardia (1970);

T4 – Mardia's omnibus test based on T1 and T2;

T5 – the Hellwig test, see Hellwig (1977);

T6 – test based on empirical characteristic function (this statistic depends on parameter $\beta > 0$; we used $\beta = 1$), see Baringhaus and Henze (1988);

T7 – test based on multivariate density estimation, see Bowman and Foster (1993);

T8 – test that uses a transformation of the multivariate normal distribution into the uniform distribution of directions on the p-dimensional unit sphere (see Koziol (1982, 1983)), see Dunn (1995);

T9 – test based on testing uniformity of directions, see also Dunn (1995);

T10 – test based on a transformation to the unvariate normal sample followed by the W-test of Shapiro and Wilk (1965); an implementation of W-test by Royston (1982a) was used;

T11 – test based on transformation to univariate normal sample followed by the univariate test of normality using correlation coefficient of the normal probability plot (see Looney, Gulledge (1985));

T12 – test based on transformation to univariate normal sample followed by the Epps-Pulley (1983) test of normality;

T13 – test based on transformation to univariate normal sample followed by the Vasicek (1976) entropy test;

T14 – test based on transformation to univariate normal sample followed by the classical Kolmogorov-Smirnov goodness of-fit test;

T15 – test based on transformation to univariate normal sample followed by the univariate version of the test based on interpoint distances proposed in Bartoszyński, Pearl and Lawrence (1997);

T16 – test based on decomposition of the problem of testing p-variate normality into p independent problems of testing univariate normality; after decomposition test based on interpoint distances was used;

T17 – test based on decomposition of the problem of testing p-variate normality into p independent problems of testing univariate normality using the Shapiro-Wilk test;

T18 – test based on decomposition of the problem of testing p-variate normality into p independent problems of testing univariate normality using the Kolmogorov-Smirnov test.

As it is seen test T16, T17 and T18 are based on the decomposition of the multivariate problem into p independent problems of testing univariate normality. Thus the initial null hypothesis of multivariate normality H_0 is the intersection or the logical conjunction of the univariate hypotheses $H_0^1, ..., H_0^p$. Hence, the problem of testing hypothesis H_0 is equivalent to testing p independent (or approximately independent) hypotheses H_0^k, $k = 1, ..., p$, obtaining the p-values and then combining them in order to get an overall test of H_0.

We may combine independent p-values $P_1, P_2, ..., P_p$ in many ways, e.g. using following well-known statistics (for details see for example Mudholkar and George (1979), Mudholkar, Srivastava, Lin (1995)):

1. Fisher's $\Psi_F = -2\Sigma \log(P_i)$;

2. the logit statistic $\Psi_L = A^{-1/2}\Sigma \log(P_i/(1-P_i))$, where $A = \frac{\pi^2 p(5p+2)}{15p+12}$;

3. Liptak's statistic $\Psi_N = \Sigma\Phi^{-1}(1-P_i)$ where Φ^{-1} denotes inverse cumulative distribution function of the standard normal distribution.

These statistics have known distributions under null hypothesis of normality: Ψ_F has a *chi*-square distribution with $2p$ degress of freedom, Ψ_L is approximated by Student's t distribution with $(5p+4)$ degrees of freedom, and Ψ_N has a normal distribution with mean 0 and variance p. Fisher's method was recommended due to its optimality property of Bahadur efficiency (see Littel and Folks (1971)) and gives best power results according to simulation experiments from Mudholkar, Srivastava, Lin (1995).

T19 – the Peterson and Stromberg test based on the combination of the Shapiro-Wilk test for maginals and the principal components method. Their test requires its own transformation of the data (we denote it as transformation VI), see Peterson and Stromberg (1998).

IV. SIMULATION STUDY

To compare different test of multivariate normality we performed extensive Monte Carlo simulations. The power of these tests at the 5% level against alternatives was estimated by the frequency of samples falling into the corresponding critical regions. We have considered 50 alternatives to multivariate normal distribution: 30 alternatives for bivariate case (series A) and 20 altenatives for general case (series B). The detailed description of all alternatives can be found in Appendix A. For each alternative 1000 samples of sizes n: 10, 25, 50 and dimensions p: 2, 4, 8 were generated to obtain empirical power.

Critical values for all the test were obtained by corresponding percentage points from 10000 samples generated from $N_p(0,\mathrm{I})$ distribution.

The simulations were done in Ox version 2.00 matrix programing language (see Doornik (1998)) on Cray Superserver 6400 computer in Warsaw University of Technology. Additionally some test were implemented in SAS language. Ox and SAS source programs can be obtained from the authors upon request.

We considered two multivariate normal distributions $N_p(0, \Sigma)$, where $\Sigma_{ii} = 1$, $i = 1, ..., p$ and $\Sigma_{ij} = 0{,}5$ for $i \neq j$ and $\Sigma_{ij} = 0{,}9$ for $i \neq j$. Results show that considered tests have acceptable size.

When a new test is suggested an author generally compares it with one or two other test using only few alternatives. This way it is no difficult to show that his test dominates the others. Moreover, the simulation study presentation may be very clear. However, we decided to compare 19 tests using 50 alternatives and 6 data transformations for different sample sizes and dimensions. Thus the first problem is now how to compare all the results? The summarized version of our results is shown in tables 1–8 in Appendix B.

Tables 1–6 show which test and transformation gives maximum power for corresponding alternatives and considered parameters n and p. Next we tried to utilize the nonparametric approach for multiple sample comparison. More precisely, we used the Kruskal–Wallis test which computes mean ranks to distinguish statistically significant differences between samples under study. In our case we rank tests for multivariate normality according to their simulated power. Tables 7–8 show the ranking of tests based on Kruskal–Wallis scores.

V. CONCLUSION

The power comparisons lead to general conclusion that none of the tests is always the best. The behavior of the tests depends on sample size, dimension and alternative. However, one may observe some tendencies. It seems that the Peterson and Stromberg test (T19) is the overall winner in our ranking procedure. Thus this newest test, based on the combination of the Shapiro–Wilk test for marginals and the principal components method, requires further theoretical studies. Next important conclusion is that the idea of transformating data to one-dimensional sample (or samples) following by the univariate Shapiro–Wilk test gives competitive tools for multivariate normality testing. Here we reccommend tests T17 and T12. Other tests which seem to have quite good properties are T9, T10 and the classical Mardia's test based on the multivariate kurtosis (T3).

Additionally, regarding tests T16, T17 and T18 based on combinations of independent univariate tests, it is worth to note that the best method of combinations is Fisher's method, so our experiments confirm results from Littel and Folks (1971) and recommendations of Mudholkar, Srivastava, Lin (1995). According to our simulating study none of the data transformation to distribution $N_p(\mathbf{0}, \mathbf{I})$ is the best too. Numerical experiments indicate only that the regression transformation III by Mudholkar, Srivastava, Lin is the worst.

A short comment on limitations of simulation would be desirable. One should be aware of two general difficulties with simulation: variability between simulations and that the results are often quite specific to the settings we have chosen. The first difficulty is intrinsic to the inference problem and is dealt with by choosing samples large enough. But there is little we can do about the second difficulty. This means that simulations are less satisfactory than theoretical results but nevertheless they provide a very useful suplement to theoretical results and often can be used when theoretical results are unavailable.

Hence the final conclusion is that more both theoretical and simulation studies in this field are still needed.

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TESTOWANIE WIELOWYMIAROWEJ NORMALNOŚCI ZA POMOCĄ TRANSFORMACJI DANYCH

(Streszczenie)

W pracy porównano różne metody testowania hipotezy o wielowymiarowej normalności za pomocą odpowiednich transformacji i znanych testów jednowymiarowej normalności. Wykonano szereg symulacji z wieloma alternatywami w celu zbadania mocy rozważanych testów. Eksperymenty numeryczne pokazały dobre własności i możliwości praktycznego zastosowania idei transformacji połaczonej z kombinacją sprawdzonych jednowymiarowych testów normalności (w tym klasycznego testu Shapiro–Wilka). W implementacji algorytmów i obliczeniach symulacyjnych bardzo efektywnym narzędziem okazał się język programowania macierzowego Ox.

Appendix A

To describe different alternatives to multivariate normal distribution the following notation is used:

$N(0, 1)$ – standard normal distribution, $Exp(1)$ – standard exponential distribution,

$LN(0, 1)$ – logonormal distribution given by $exp(N(0, 1))$, χ^2_f – chi-square distribution with f degrees of freedom, t_f – Student's distribution with f degrees of freedom, $G(a, b)$ – gamma distribution with density $b^{-a}\Gamma(a)x^{a-1}exp(-x/b)$, $Beta(a, b)$ – beta distribution with density $B(a, b)^{-1}x^{a-1}(1-x)^{b-1}(0<x<1)$.

$D_1 \otimes D_2$ is the distribution having independent marginals D_1 and D_2, D^p denotes the product of p independent copies of distribution D.

$PSII_p(m, \Sigma)$ is p-dimensional elliptically symmetric Pearson Type II distribution with the density function

$$f(x) = \frac{\Gamma(p/2+m+1)}{\Gamma(m+1)\pi^{p/2}}|\Sigma|^{-1/2}(1-x'\Sigma^{-1}x)^m \ (0<x'\Sigma^{-1}x<1, m>-1)$$

and $t_f(\Sigma)$ is p-variate t distribution with f degrees of freedom and density

$$f(x) = \frac{\Gamma(f+p)/2}{\Gamma(f/2)(\pi f)^{p/2}}|\Sigma|^{-1/2}(1+f^{-1}x'\Sigma^{-1}x)^{-(f+p)/2}$$

$SPH(Q)$ stands for a spherically symmetric distributed random vector X such that $\|X\|$ has distribution Q. $NMIX\_\ A_p(p_0, \rho_1, \mu_2, \sigma^2, \rho_2)$, denotes the normal mixture model $p_0N_p(0, \Sigma_1)+(1-p_0)N_p(\mu_2, \Sigma_2)$, where

Σ_1 and Σ_2 are positively definite matrices with Σ_1 having all of its diagonal elements equal to 1 and all of its off-diagonal elements equal to ρ_1 and Σ_2 having all of its diagonal elements equal to σ_2^2 and all of its off-diagonal equal to ρ_2. $NMIX\_B_p(\mu_1, \mu_2)$ will be used for a specific bimodal normal mixture of the form $0.5N_p(\mu_1, I) + 0.5N_p(\mu_2, I)$ and $NMIX\_C_p(p_0, d, \Sigma)$ for mixture $p_0 N_p(0, \Sigma) + (1 - p_0)N_p(0, d\Sigma)$ $(d > 0)$. Multivariate chi-square distribution will be denoted by $\chi_p^2(f_1, ..., f_p; f)$; this distribution can be defined as the joint distributions of $W_1, W_2, ..., W_p$, $W_i = V_i + V$ for $i = 1, 2, ..., p$, where $V_1, V_2, .., V_p$ are independent χ^2 variates with degrees of freedom $f_1, f_2, ..., f_p$ respectively, and V is independent of V_i's with χ^{2f} distribution.

$MLN_p(\Sigma)$ stands for multivariate logonormal distribution obtained as $exp(N_p(0, \Sigma))$ (coordinatewise). $MBPL_p(\theta)$ denotes multivariate Burr-Pareto-Logistic distribution with uniform marginals and parameter θ.

$KH1$ and $KH2$ will denote examples of the Khintchine distributions with normal marginals and $GEP1$ and $GEP2$ will denote two variants for Generalized Exponential Power random variables; we used definitions from Horswell and Looney (1992).

For more details concerned with methods of generation of samples from multivariate distributions we refer the reader to Johnson (1987).

We have chosen the following alternative distributions:

A1: $Exp(1)^2$	A16: $N(0,1) \otimes Beta(1, 2)$
A2: $LN(0, 1)^2$	A17: $NMIX\_A_2(0.5, 0.0, 2.0, 1.0, 0.0)$
A3: $G(5,1)^2$	A18: $NMIX\_A_2(0.5, 0.0, 4.0, 1.0, 0.0)$
A4: $(\chi_5^2)^2$	A19: $NMIX\_A_2(0.5, 0.9, 2.0, 1.0, 0.0)$
A5: $(\chi_{15}^2)^2$	A20: $NMIX\_A_2(0.5, 0.9, 0.5, 1.0, 0.0)$
A6: $(t_2)^2$	A21: $NMIX\_A_2(0.5, 0.9, 0.5, 1.0, -0.9)$
A7: $(t_5)^2$	A22: $PSII_2(0, I_2)$
A8: $L(0, 1)^2$	A23: $PSII_2(1, I_2)$
A9: $Beta\ (1, 1)^2$	A24: $t_2(I_2)$
A10: $Beta\ (1, 2)^2$	A25: $t_8(I_2)$
A11: $Beta\ (2, 2)^2$	A26: $SPH(Exp(1))$
A12: $N(0,1) \otimes Exp(1)$	A27: $SPH(G(5, 1))$
A13: $N(0,1) \otimes \chi_5^2$	A28: $SPH\ (Beta\ (1, 1))$
A14: $N(0,1) \otimes t_5$	A29: $SPH\ (Beta\ (1, 2))$
A15: $N(0,1) \otimes Beta\ (1, 1)$	A30: $SPH\ (Beta\ (2, 2))$

B1: $(G(2,1))^p$	B11: $PSII_p(10, \Sigma)$, where $\Sigma_{ii} = 1$, $i = 1, ..., p$ $\Sigma_{ij} = 0,5$ for $i \neq j$
B2: $NMIX\_A_p(0.5,\ 0.0,\ 3.0,\ 1.0,\ 0.0)$	B12: $PSII_p(10, \Sigma)$, where $\Sigma_{ij} = 1$, $i = 1, ..., p$ $\Sigma_{ij} = 0.9$ for $i \neq j$
B3: $NMIX\_B_p(1.5,\ -1.5)$	B13: $t_{10}(I_p)$
B4: $NMIX\_A_p(0.5,\ 0.0,\ 0.0,\ 3.0,\ 0.0)$	B14: $t_{10}(\Sigma)$, where $\Sigma_{ij} = 1$, $i = 1, ..., p$ $\Sigma_{ij} = 0.5$ for $i \neq j$
B5: $NMIX\_A_p(0.5,\ 0.9,\ 1.0,\ 1.0,\ 0.0)$	B15: $t_{10}(\Sigma)$, where $\Sigma_{ii} = 1$, $i = 1, ..., p$, $\Sigma_{ij} = 0.9$ for $i \neq j$
B6: $KH1$	B16: $MBPL_p(2)$
B7: $KH2$	B17: $MLN_p(\Sigma)$, where $\Sigma_{ii} = 0.5$, $i = 1, ..., p$ $\Sigma_{ij} = 0.25$ for $i \neq j$
B8: $GEP(0.1663,\ 0.125)$	B18: $NMIX\_C_p(0.8, 9)$
B9: $GEP(27.905,\ 2.0)$	B19: $NMIX\_C_p(0.9, 16)$
B10: $PSII_p(10, I_p)$	B20: $\chi^2_p(2,\ 2, ..., 2; 3)$

Appendix B

Table 1

Tests and transformations giving maximal power for various alternatives, $n = 10$

p	Alterna-tives	Power	Test	Transfor-mations	p	Alterna-tives	Power	Test	Transfor-mations
2	A1	0.602	12	2	2	B16	0.240	15	2
2	A2	0.801	12	2	2	B17	0.501	12	1
2	A3	0.153	12	1	2	B18	0.344	6	1
2	A4	0.306	12	2	2	B19	0.370	2	1
2	A5	0.142	12	2	2	B20	0.263	7	2
2	A6	0.453	1	4	4	B1	0.507	12	2
2	A7	0.172	6	1	4	B2	0.112	15	1
2	A8	0.125	2	4	4	B3	0.098	15	1
2	A9	0.227	15	1	4	B4	0.483	6	2
2	A10	0.235	13	2	4	B5	0.305	4	5
2	A11	0.123	15	2	4	B6	0.191	4	1
2	A12	0.308	19	6	4	B7	0.519	7	1
2	A13	0.143	17	2	4	B8	0.082	6	2
2	A14	0.124	4	4	4	B9	0.071	13	2
2	A15	0.113	3	5	4	B10	0.072	3	1
2	A16	0.094	13	2	4	B11	0.074	3	5
2	A17	0.102	3	5	4	B12	0.074	5	1
2	A18	0.198	19	6	4	B13	0.124	19	6
2	A19	0.170	9	5	4	B14	0.122	2	1
2	A20	0.141	6	2	4	B15	0.115	2	2
2	A21	0.296	19	6	4	B16	0.269	13	2
2	A22	0.224	3	1	4	B17	0.610	12	2

Table 2

Tests and transformations giving maximal power for various alternatives, $n = 10$ (contd.)

p	Alternatives	Power	Test	Transformations	p	Alternatives	Power	Test	Transformations
2	A23	0.139	3	4	4	B18	0.368	4	2
2	A24	0.467	6	2	4	B19	0.392	19	6
2	A25	0.128	2	2	4	B20	0.395	12	2
2	A26	0.547	7	1	8	B1	0.430	12	2
2	A27	0.097	3	4	8	B2	0.202	19	6
2	A28	0.091	9	4	8	B3	0.152	19	6
2	A29	0.229	6	1	8	B4	0.286	19	6
2	A30	0.110	3	4	8	B5	0.203	2	2
2	B1	0.348	10	2	8	B6	0.235	15	2
2	B2	0.118	15	1	8	B7	0.452	2	1
2	B3	0.126	3	4	8	B8	0.092	1	1
2	B4	0.316	6	2	8	B9	0.070	11	1
2	B5	0.151	2	2	8	B10	0.065	16	1
2	B6	0.101	6	1	8	B11	0.068	14	1
22	B7	0.204	6	2	8	B12	0.070	6	3
2	B8	0.076	2	1	8	B13	0.108	19	6
2	B9	0.070	15	1	8	B14	0.096	19	6
2	B10	0.079	3	1	8	B15	0.097	2	2
2	B11	0.067	9	4	8	B16	0.215	15	2
2	B12	0.067	8	1	8	B17	0.706	19	6
2	B13	0.115	2	2	8	B18	0.394	19	6
2	B14	0.107	7	2	8	B19	0.512	19	6
2	B15	0.113	6	2	8	B20	0.535	12	2

Table 3

Tests and transformations giving maximal power for various alternatives, $n = 25$

p	Alternatives	Power	Test	Transformations	p	Alternatives	Power	Test	Transformations
2	A1	0.995	11	2	2	B16	0.748	13	2
2	A2	1.000	10	2	2	B17	0.988	10	2
2	A3	0.500	12	2	2	B18	0.774	6	1
2	A4	0.836	10	2	2	B19	0.776	6	1
2	A5	0.393	1	4	2	B20	0.696	10	2
2	A6	0.872	6	1	4	B1	0.992	10	2
2	A7	0.742	1	4	4	B2	0.776	19	6
2	A8	0.362	1	4	4	B3	0.760	19	6
2	A9	0.775	13	2	4	B4	0.979	7	2
2	A10	0.765	13	2	4	B5	0.832	7	1
2	A11	0.315	13	1	4	B6	0.552	7	1

Table 3 (contd.)

p	Alterna-tives	Power	Test	Transfor-mations	p	Alterna-tives	Power	Test	Transfor-mations
2	A12	0.818	19	6	4	B7	0.961	7	1
2	A13	0.431	17	2	4	B8	0.093	6	1
2	A14	0.233	1	1	4	B9	0.141	15	1
22	A15	0.316	16	2	4	B10	0.111	3	2
2	A16	0.277	17	2	4	B11	0.121	3	4
2	A17	0.113	3	4	4	B12	0.114	3	2
2	A18	0.828	19	6	4	B13	0.313	6	1
2	A19	0.794	1	4	4	B14	0.323	7	2
2	A20	0.299	7	2	4	B15	0.304	6	2
2	A21	0.738	1	4	4	B16	0.915	15	2
2	A22	0.682	3	4	4	B17	0.998	11	2

Table 4

Tests and transformations giving maximal power for various alternatives, $n = 25$ (contd.)

p	Alterna-tives	Power	Test	Transfor-mations	p	Alterna-tives	Power	Test	Transfor-mations
2	A23	0.298	3	4	4	B18	0.934	6	1
2	A24	0.897	6	1	4	B19	0.871	2	1
2	A25	0.250	7	1	4	B20	0.945	10	2
2	A26	0.953	7	1	8	B1	1.000	10	2
2	A27	0.170	15	1	8	B2	0.990	19	6
2	A28	0.121	14	2	8	B3	0.988	19	6
2	A29	0.492	7	1	8	B4	1.000	6	1
2	A30	0.174	3	4	8	B5	0.994	7	1
2	B1	0.894	10	2	8	B6	0.905	6	1
2	B2	0.304	16	1	8	B7	0.999	6	1
2	B3	0.294	19	6	8	B8	0.089	13	2
2	B4	0.772	7	2	8	B9	0.162	15	2
2	B5	0.297	7	2	8	B10	0.150	3	2
2	B6	0.184	7	2	8	B11	0.158	3	5
22	B7	0.450	9	5	8	B12	0.209	12	3
2	B8	0.090	7	2	8	B13	0.444	2	1
2	B9	0.095	16	2	8	B14	0.449	4	4
2	B10	0.079	13	1	8	B15	0.471	4	5
2	B11	0.083	15	1	8	B16	0.968	15	2
2	B12	0.070	13	2	8	B17	1.000	10	2
2	B13	0.221	6	1	8	B18	0.973	2	1
2	B14	0.199	4	1	8	B19	0.899	4	5
2	B15	0.218	4	1	8	B20	1.000	10	2

Table 5

Tests and transformations giving maximal power for various alternatives, $n = 50$

p	Alternatives	Power	Test	Transformations	p	Alternatives	Power	Test	Transformations
2	A1	1.000	1	1	2	B6	0.320	6	1
2	A2	1.000	1	1	2	B7	0.822	9	5
2	A3	0.898	10	1	2	B8	0.136	14	1
2	A4	0.994	10	2	2	B9	0.134	15	2
2	A5	0.710	10	1	2	B10	0.094	15	1
2	A6	0.996	7	2	2	B11	0.084	16	2
2	A7	0.654	6	1	2	B12	0.082	16	2
2	A8	0.388	1	3	2	B13	0.390	6	2
2	A9	0.996	13	2	2	B14	0.362	7	1
2	A10	0.992	10	1	2	B15	0.348	6	1
2	A11	0.668	13	2	2	B16	0.978	13	2
2	A12	1.000	19	6	2	B17	1.000	1	1
2	A13	0.810	17	2	2	B18	0.968	4	2
2	A14	0.372	1	5	2	B19	0.958	4	1
22	A15	0.690	16	2	2	B20	0.962	10	2
2	A16	0.658	17	2	4	B1	1.000	17	1
2	A17	0.156	3	4	4	B2	1.000	19	6
2	A18	1.000	19	6	4	B3	1.000	19	6
2	A19	0.856	9	4	4	B4	0.872	17	3
2	A20	0.482	6	1	4	B5	0.709	19	6

Table 6

Tests and transformations giving maximal power for various alternatives, $n = 50$ (contd.)

p	Alternatives	Power	Test	Transformations	p	Alternatives	Power	Test	Transformations
2	A21	0.940	19	6	4	B6	0.296	19	6
2	A22	0.982	3	4	4	B7	0.554	19	6
2	A23	0.664	3	1	4	B8	0.120	18	2
2	A24	0.996	7	1	4	B9	0.180	16	2
2	A25	0.478	6	2	4	B10	0.088	16	2
2	A26	1.000	6	1	4	B11	0.092	16	2
2	A27	0.250	16	2	4	B12	0.082	16	1
2	A28	0.386	13	2	4	B13	0.330	19	6
2	A29	0.786	7	1	4	B14	0.301	19	6
2	A30	0.358	3	4	4	B15	0.282	17	3
2	B1	1.000	10	1	4	B16	0.992	16	2
2	B2	0.836	19	6	4	B17	1.000	17	1
2	B3	0.856	19	6	4	B18	0.980	19	6
2	B4	0.980	6	1	4	B19	0.981	19	6
2	B5	0.532	4	2	4	B20	0.996	17	2

Table 7

Best ranked tests based on Kruskal–Wallis scores

$n=20$	$n=25$	$n=50$	$p=2$	$p=4$	$p=8$
19	19	19	19	19	19
17	3	3	10	17	9
9	10	10	17	9	12
12	12	11	12	12	2
2	17	17	3	16	11
10	11	12	11	11	17
4	9	15	15	10	10
6	2	1	16	18	8
11	15	16	9	8	4
7	4	9	1	2	6
8	16	4	4	6	18
18	1	14	2	4	16
16	8	2	14	7	7
1	6	7	18	1	15
14	7	18	7	14	14
15	14	13	6	15	1
13	18	6	13	3	13
3	13	8	8	13	3
5	5	5	5	5	5

Table 8

Best ranked tests based on Kruskal–Wallis scores

$n=10$		$n=25$		$n=50$		$p=2$		$p=4$		$p=8$	
19	6	19	6	19	6	19	6	19	6	19	6
9	5	3	4	3	1	10	1	9	5	12	3
17	5	10	1	3	4	17	5	9	1	11	3
17	4	3	2	10	1	10	2	18	1	9	5
9	4	3	1	10	2	17	4	9	4	2	3
9	1	3	5	3	2	10	5	17	1	9	4
17	1	10	5	3	5	11	1	17	5	6	1
6	1	10	2	11	1	10	4	17	3	9	1
6	2	17	5	11	4	16	1	17	4	17	4
18	1	12	4	10	4	12	1	9	2	17	5
9	2	11	1	10	5	12	2	16	1	9	2
7	1	10	4	17	4	17	1	6	1	10	3
12	2	16	1	15	1	3	1	8	1	16	1
7	2	10	3	17	5	17	3	11	1	17	3
4	1	12	1	11	5	3	4	11	2	8	1