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THE MANN-WHITNEY TEST. SOME PROBLEMS CONCERNING APPROXIMATION OF CRITICAL VALUES AND TIED RANKS

Abstract. The nonparametric Mann-Whitney test [Mann-Whitney (1947)] is one of commonly used two sample tests. The problem of tied ranks exists in the case of discrete distributions. It makes the analysis of distribution of Mann-Whitney test quite complicated.

There is continuing interest in the properties of this test's statistics. It can be seen from many of papers concerning the distribution of this statistic according to the assumption of equality of distributions about the critical values tables and test's power.

In this paper we shall present an approximation of Mann-Whitney statistics, critical values, compared with the most often used approximation by normal distribution. We will determine the unconditional distribution of U-statistics for $p = \frac{1}{2}$ in the case of tied ranks. We will compare the interpolated quantiles of these distributions with the case of continuous distribution without the tied ranks.

Key words: Mann-Whitney-test, approximation of critical values of Mann-Whitney, unconditional distribution of U-statistics.

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1. THE BASIC PROPERTIES OF U STATISTIC AND INTERPOLATED QUANTILES

Let $X_1, \ldots, X_n$ be the sample composed of n independent observations from population with the continuous distribution function F and let $Y_1, \ldots, Y_m$ - be the sample composed of n independent observations from population with the continuous distribution function G.

We have to verify the hypothesis:

$$H_0: F = G \tag{1}$$

The Mann-Whitney statistic is defined as the U number of these pairs (X_i, Y_j) among all nm pairs when $X_i < Y_j$. Formally we can define U in the following form:

$$U = \sum_{i=1}^{n} \sum_{j=1}^{m} D_{ij}, \tag{2}$$

where

$$D_{ij} = \begin{cases} 1 & \text{if} \quad X_i < Y_j \\ 0 & \text{if} \quad X_i > Y_j. \end{cases}$$

According to the assumptions of continuous distribution functions the possibility $X_i = Y_j$ need not be considered (because $P(X_i = Y_j) = 0$). A large number of pairs $X_i < Y_j$ accept the alternative hypothesis:

$$H_1: F(x) > G(x), \tag{3}$$

so for this hypothesis the critical area is right-sided. On the other hand, in the case of hypothesis:

$$H_1: F(x) < G(x), \tag{4}$$

the critical area is left-sided.

Two sided critical area is applied for a more general hypothesis:

$$H_1: F(x) \neq G(x). \tag{5}$$

The distribution of U statistics is discret. Due to this, we cannot assort critical value $u_c(n, m, \alpha)$ with α level of significance in such a way that $P(U \leq u_c(n, m, \alpha)) = \alpha$.

In applications we have

$$u_c(n, m, \alpha) = \max \{u: P(U \leq u) \leq \alpha\},$$

hence, in general

$P(U \leq u_c(n, m, \alpha)) < \alpha.$

This fact makes the analysis of the power of the test quite complicated (reservations concerning of tests with different sizes).

The randomizing tests are more useful in the power analysis. The randomizing left-sided test of Mann-Whitney is based upon

- rejection of H_0 when $U \leq u_c(n, m, \alpha)$
- acceptance of H_0 when $U > u_c(n, m, \alpha)$
- acceptance of H_0 with the probability

$$p_{rand}(n, m, \alpha) = \frac{\alpha - P(U \leq u_c(n, m, \alpha))}{P(U = u_c(n, m, \alpha) + 1)} \tag{6}$$

when

$U = u_c(n, m, \alpha) + 1.$

Our considerations will concerning the left-sided test corresponding to the alternative hypothesis

H_1: $F(x) < G(x)$.

(Analogously for the right-sided and two-sided analysis).

The left-sided interpolated quantile [introduced by Domański and Tomaszewicz (1986)] is defined by the sum of

$$u_i(n, m, \alpha) = u_c(n, m, \alpha) + p_{rand}(n, m, \alpha). \tag{7}$$

Mann and Whitney shown that the distribution of U-statistic converges to the normal distribution. This fact suggests the possibility approximation of critical values by means of this distribution.

The approximation formula for the interpolated quantile is of the form:

$$u_i(n, m, \alpha) \approx u_i^n(n, m, \alpha) = \Phi^{-1}(\alpha)\ \sigma_U^2(n, m) + \mu_U(n, m) + \frac{1}{2}, \tag{8}$$

where $\sigma_U^2(n, m) = \frac{mn}{12}(m + n + 1)$ and $\mu_U(n, m) = \frac{mn}{2}$, while Φ^{-1} is a normal distribution function $N(0, 1)$, $\frac{1}{2}$ is associated with the applied adjustment of continuity [see, for example, Jacobson (1963)]. Of course, it is only one of the possible approximations.

The most often applied include those developed by means Edgeworth's extension [see, among others Fellingham and Stoker (1964)].

For the randomized tests, there is a very simple approximation of the interpolated quantiles. Let us reduce our analysis to $m = n$. Hence, we shall apply simplified symbols

$u_c(n, \alpha)$, $u_i(n, \alpha)$, $p_{rand}(n, \alpha)$.

The approximation formula is of the form

$$u_i(n, m, \alpha) \approx u_i^a(n, m, \alpha) = \tag{9}$$

$$\gamma_{-2}(\alpha)n^{-2} + \gamma_{-1}(\alpha)n^{-1} + \gamma_0(\alpha) + \gamma_1(\alpha)n^1 + \gamma_2(\alpha)n^2.$$

For each of the selected function significance levels α the function (9) we have adjusted $u_i(n, \alpha)$ using the least squares method.

The determined coefficients $\gamma_j(\alpha)$ are shown in Table 1.

The size of the test determined by formulas is equal to

$$\alpha^a = P(U \le u_i^a(n, \alpha)) + p^a_{rand}(n, \alpha)\ P(U) = u_c^a(n, \alpha) + 1). \tag{10}$$

As the measure of the approximation error we have $\alpha^a - \alpha$. Analogously, for the approximation (8) we have, as the measure of approximation $\alpha^n - \alpha$.

The difference

$$d_u^a(n, \alpha) = u_i^a(n, \alpha) - u_i(n, \alpha) \tag{11}$$

is worse measure.

The interpolated quantiles, their approximation, errors of approximation for the levels of significance:

$\alpha = 0.01$, $\alpha = 0.05$, $\alpha = 0.10$

we present in Tables 1-7.

Table 1

The parameters $\gamma_j(\alpha)$ for approximation of the interpolated quantiles

α	$\gamma_{-2}(\alpha)$	$\gamma_{-1}(\alpha)$	$\gamma_0(\alpha)$	$\gamma_1(\alpha)$	$\gamma_2(\alpha)$
$\alpha_1 = 0.00010$	959.42	-400.567	67.4469	-7.28112	0.409107
$\alpha_2 = 0.00020$	933.38	-389.904	64.6332	-6.97648	0.413871
$\alpha_3 = 0.00025$	916.77	-384.064	63.5316	-6.86969	0.415398
$\alpha_4 = 0.00050$	847.84	-359.897	59.5044	-6.50602	0.420066
$\alpha_5 = 0.00100$	765.07	-330.940	55.0016	-6.10933	0.424852
$\alpha_6 = 0.00200$	665.13	-294.868	49.7592	-5.66769	0.429703
$\alpha_7 = 0.00250$	643.44	-286.494	48.3503	-5.52964	0.431438
$\alpha_8 = 0.00500$	574.71	-258.470	43.7067	-5.07482	0.437026
$\alpha_9 = 0.01000$	533.46	-236.862	39.4750	-4.60228	0.443259
$\alpha_{10} = 0.02000$	471.88	-208.833	34.5768	-4.07153	0.449946
$\alpha_{11} = 0.02500$	455.34	-200.711	33.0452	-3.89249	0.452287
$\alpha_{12} = 0.05000$	379.85	-167.565	27.4343	-3.27133	0.459967
$\alpha_{13} = 0.10000$	291.18	-129.249	21.0641	-2.55111	0.468806
$\alpha_{14} = 0.20000$	193.59	-85.430	13.6530	-1.68000	0.479546

Table 2

The errors of approximation of interpolated quantiles by function (9) (multiplied by 10^5)

n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}
5	201	159	144	113	95	115	118	116	199	273	232	478	490	472
6	-10	-20	-25	-33	-20	-32	-52	-94	-156	-211	-243	-362	-468	-405
7	-10	-8	-6	-8	-19	-32	-47	-68	-114	-153	-167	-272	-320	-357
8	0	-2	-2	-5	-10	-14	-20	-22	-40	-62	-79	-94	-153	-142
9	1	0	-0	-1	-3	-0	2	4	2	-1	-1	10	13	7
10	1	1	1	2	3	9	9	19	27	43	44	65	89	82
11	1	2	2	3	6	11	12	21	35	53	62	80	115	105
12	1	2	2	3	6	11	13	22	32	55	59	83	112	114
13	1	1	2	3	5	9	11	18	30	47	52	71	95	102
14	1	1	1	2	4	7	9	14	23	37	41	59	79	76
15	0	1	1	2	3	5	6	10	16	25	29	43	52	59
16	0	1	1	1	2	3	4	6	10	16	18	26	35	34
17	0	0	0	1	1	2	2	3	5	7	9	12	16	17

Table 2 (contd)

n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}
18	-0	0	0	0	0	0	0	0	0	-0	1	1	0	2
19	-0	-0	-0	-0	-0	-1	-1	-2	-4	-6	-6	-8	-12	-12
20	-0	-0	-0	-0	-1	-2	-2	-4	-6	-10	-10	-15	-21	-21
21	-0	-0	-0	-1	-1	-2	-3	-5	-8	-12	-13	-20	-26	-27
22	-0	-0	-0	-1	-1	-3	-3	-6	-9	-14	-15	-23	-30	-30
23	-0	-0	-0	-1	-1	-3	-3	-6	-9	-14	-16	-24	-31	-32
24	-0	-0	-0	-1	-1	-3	-3	-6	-9	-14	-16	-23	-32	-32
25	-0	-0	-0	-1	-1	-3	-3	-5	-9	-13	-15	-23	-30	-31
26	-0	-0	-0	-1	-1	-2	-3	-5	-8	-12	-14	-20	-28	-28
27	-0	-0	-0	-1	-1	-2	-2	-4	-7	-11	-12	-18	-24	-25
28	-0	-0	-0	-1	-1	-2	-2	-4	-6	-9	-11	-16	-21	-21
29	-0	-0	-0	-0	-1	-1	-2	-3	-5	-7	-8	-12	-16	-16
30	-0	-0	-0	-0	-1	-1	-1	-2	-3	-5	-6	-9	-12	-12
31	-0	-0	-0	-0	-0	-1	-1	-1	-2	-3	-4	-6	-7	-8
32	-0	-0	-0	-0	-0	-0	-0	-1	-1	-2	-2	-3	-3	-3
33	0	0	0	-0	0	0	0	0	0	0	0	1	1	0
34	0	0	0	0	0	0	0	1	1	2	2	3	4	4
35	0	0	0	0	0	1	1	1	2	3	4	5	7	7
36	0	0	0	0	0	1	1	2	3	5	5	8	10	10
37	0	0	0	0	1	1	1	2	3	6	6	9	12	12
38	0	0	0	0	1	1	1	2	4	6	7	10	14	14
39	0	0	0	0	1	1	1	3	4	7	7	11	15	15
40	0	0	0	0	1	1	2	3	4	7	8	11	15	15
41	0	0	0	0	1	1	1	3	4	6	7	11	14	15
42	0	0	0	0	1	1	1	2	4	6	7	10	13	14
43	0	0	0	0	1	1	1	2	3	5	6	9	12	12
44	0	0	0	0	0	1	1	2	3	4	5	7	9	10
45	0	0	0	0	0	1	1	1	2	3	3	5	6	6
46	0	0	0	0	0	0	0	0	1	1	1	2	2	2
47	-0	-0	-0	-0	-0	-0	-0	-0	-1	-1	-1	-2	-2	-2
48	-0	-0	-0	-0	-0	-1	-1	-1	-2	-3	-4	-6	-7	-8
49	-0	-0	-0	-0	-1	-1	-1	-2	-4	-6	-7	-10	-13	-14
50	-0	-0	-0	-1	-1	-2	-2	-3	-6	-9	-10	-15	-20	-20

Table 3

The errors of approximation of the interpolated quantiles by means of normal distribution (multiplied by 10^5)

n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}
5	-10	-20	-25	-50	-100	-112	-27	162	479	1112	1503	2989	4887	7197
6	-10	-20	-25	-50	-78	-35	-37	67	284	859	1113	2144	3733	5510
7	-10	-20	-25	-43	-47	-42	-41	21	221	596	806	1643	2891	4423
8	-10	-20	-22	-32	-43	-44	-36	16	163	461	625	1312	2381	3635
9	-10	-16	-18	-28	-38	-37	-33	6	124	382	506	1092	1989	3060
10	-9	-14	-17	-26	-33	-35	-32	1	97	315	424	918	1695	2625
11	-8	-13	-16	-23	-31	-33	-31	-4	78	264	356	784	1469	2284
12	-7	-12	-14	-21	-29	-31	-30	-8	63	223	308	682	1288	2013
13	-7	-11	-13	-20	-26	-30	-28	-9	52	193	267	604	1138	1800
14	-6	-11	-12	-18	-25	-28	-27	-11	42	170	235	541	1027	1617
15	-6	-10	-12	-17	-23	-26	-26	-12	35	151	209	486	925	1467
16	-6	-9	-11	-16	-22	-25	-25	-13	30	133	188	439	845	1337
17	-6	-9	-10	-15	-21	-24	-24	-14	25	120	168	401	772	1229
18	-5	-8	-10	-15	-20	-23	-23	-14	21	109	154	368	709	1132
19	-5	-8	-9	-14	-19	-22	-22	-14	18	98	139	338	657	1049
20	-5	-8	-9	-13	-18	-22	-22	-14	15	90	128	313	610	977
21	-5	-7	-8	-13	-17	-21	-21	-14	13	82	118	291	569	912
22	-4	-7	-8	-12	-17	-20	-20	-15	11	75	108	271	532	855
23	-4	-7	-8	-12	-16	-19	-20	-15	9	69	101	253	500	803
24	-4	-6	-7	-11	-15	-19	-19	-14	7	64	94	237	471	757
25	-4	-6	-7	-11	-15	-18	-19	-14	6	59	87	223	444	715
26	-4	-6	-7	-10	-14	-18	-18	-14	5	55	82	211	421	677
27	-4	-6	-7	-10	-14	-17	-18	-14	4	51	77	199	398	642
28	-4	-6	-7	-10	-14	-17	-17	-14	3	48	72	188	379	610
29	-3	-5	-6	-9	-13	-16	-17	-14	2	45	68	179	360	581
30	-3	-5	-6	-9	-13	-16	-16	-14	1	42	64	170	343	555
31	-3	-5	-6	-9	-12	-15	-16	-14	1	40	60	162	328	530
32	-3	-5	-6	-9	-12	-15	-16	-13	-0	37	57	154	313	507
33	-3	-5	-6	-8	-12	-15	-15	-13	-0	35	54	148	300	487
34	-3	-5	-5	-8	-11	-14	-15	-13	-1	33	52	141	288	467
35	-3	-5	-5	-8	-11	-14	-15	-13	-1	31	49	135	277	449
36	-3	-4	-5	-8	-11	-14	-14	-13	-2	30	47	130	266	432
37	-3	-4	-5	-8	-11	-13	-14	-13	-2	28	45	124	256	416
38	-3	-4	-5	-7	-10	-13	-14	-13	-2	27	42	120	247	401
39	-3	-4	-5	-7	-10	-13	-13	-12	-3	25	41	115	238	387
40	-3	-4	-5	-7	-10	-13	-13	-12	-3	24	39	111	230	373
41	-3	-4	-5	-7	-10	-12	-13	-12	-3	23	37	107	222	361
42	-2	-4	-4	-7	-9	-12	-13	-12	-3	22	36	103	215	350
43	-2	-4	-4	-7	-9	-12	-12	-12	-4	21	34	100	208	339
44	-2	-4	-4	-6	-9	-12	-12	-12	-4	20	33	96	201	328
45	-2	-4	-4	-6	-9	-11	-12	-12	-4	19	32	93	195	318
46	-2	-4	-4	-6	-9	-11	-12	-11	-4	18	30	90	190	309
47	-2	-4	-4	-6	-9	-11	-12	-11	-4	17	29	88	184	300
48	-2	-3	-4	-6	-8	-11	-11	-11	-4	17	28	85	179	291
49	-2	-3	-4	-6	-8	-11	-11	-11	-4	16	27	82	174	283
50	-2	-3	-4	-6	-8	-10	-11	-11	-4	15	26	80	169	276

Table 4

The approximation of the interpolated quantiles of Mann-Whitney statistic for $5 \leq n \leq 50$ and $\alpha = 0.0001$.

n	$u_1(n,\alpha)$	$u_1^a(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^a-\alpha$	$u_1^n(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^n-\alpha$
5	-0.9748	-0.4675	0.5073	0.00201	-4.8037	-3.8289	-0.00010
6	-0.9076	-1.6225	-0.7149	-0.00010	-4.7256	-3.8180	-0.00010
7	-0.6568	-1.1185	-0.4617	-0.00010	-4.1064	-3.4496	-0.00010
8	0.2870	0.3009	0.0139	0.00000	-2.9127	-3.1997	-0.00010
9	2.2873	2.3918	0.1044	0.00001	-1.1176	-3.4049	-0.00010
10	4.9251	5.0839	0.1589	0.00001	1.3013	-3.6238	-0.00009
11	8.1181	8.3705	0.2524	0.00001	4.3629	-3.7552	-0.00008
12	11.9794	12.2670	0.2875	0.00001	8.0838	-3.8957	-0.00007
13	16.4914	16.7957	0.3043	0.00001	12.4781	-4.0133	-0.00007
14	21.7189	21.9793	0.2604	0.00001	17.5585	-4.1604	-0.00006
15	27.6319	27.8388	0.2069	0.00000	23.3364	-4.2956	-0.00006
16	34.2560	34.3927	0.1367	0.00000	29.8218	-4.4342	-0.00006
17	41.5881	41.6569	0.0687	0.00000	37.0240	-4.5641	-0.00006
18	49.6475	49.6449	-0.0026	-0.00000	44.9514	-4.6962	-0.00005
19	58.4349	58.3685	-0.0664	-0.00000	53.6115	-4.8234	-0.00005
20	67.9741	67.8376	-0.1366	-0.00000	63.0116	-4.9625	-0.00005
21	78.2352	78.0606	-0.1746	-0.00000	73.1580	-5.0771	-0.00005
22	89.2552	89.0448	-0.2104	-0.00000	84.0570	-5.1982	-0.00004
23	101.0389	100.7965	-0.2424	-0.00000	95.7140	-5.3249	-0.00004
24	113.5676	113.3211	-0.2465	-0.00000	108.1345	-5.4331	-0.00004
25	126.8777	126.6232	-0.2545	-0.00000	121.3233	-5.5544	-0.00004
26	140.9555	140.7070	-0.2485	-0.00000	135.2851	-5.6704	-0.00004
27	155.8023	155.5760	-0.2263	-0.00000	150.0243	-5.7779	-0.00004
28	171.4302	171.2333	-0.1969	-0.00008	165.5451	-5.8852	-0.00004
29	187.8489	187.6817	-0.1673	-0.00000	181.8512	-5.9977	-0.00003
30	205.0534	204.9235	-0.1299	-0.00000	198.9465	-6.1069	-0.00003
31	223.0469	222.9609	-0.0860	-0.00000	216.8345	-6.2125	-0.00003
32	241.8323	241.7960	-0.0364	-0.00000	235.5185	-6.3139	-0.00003
33	261.4155	261.4302	0.0147	0.00000	255.0017	-6.4138	-0.00003
34	281.8045	281.8652	0.0607	0.00000	275.2871	-6.5174	-0.00003
35	302.9989	303.1023	0.1034	0.00000	296.3778	-6.6211	-0.00003
36	324.9961	325.1428	0.1467	0.00000	318.2764	-6.7197	-0.00003
37	347.7995	347.9878	0.1883	0.00000	340.9857	-6.8138	-0.00003
38	371.4166	371.6382	0.2215	0.00000	364.5081	-6.9085	-0.00003
39	395.8526	396.0949	0.2424	0.00000	388.8463	-7.0063	-0.00003
40	421.1036	421.3589	0.2554	0.00000	414.0025	-7.1011	-0.00003
41	447.1723	447.4308	0.2585	0.00000	439.9790	-7.1933	-0.00003
42	474.0650	474.3114	0.2464	0.00000	466.7781	-7.2869	-0.00002
43	501.7779	502.0011	0.2233	0.00000	494.4017	-7.3761	-0.00002
44	530.3177	530.5007	0.1830	0.00000	522.8521	-7.4656	-0.00002
45	559.6861	559.8107	0.1246	0.00000	552.1311	-7.5550	-0.00002
46	589.8858	589.9314	0.0456	0.00000	582.2407	-7.6452	-0.00002
47	620.9157	620.8635	-0.0522	-0.00000	613.1827	-7.7330	-0.00002
48	652.7771	652.6072	-0.1700	-0.00000	644.9588	-7.8183	-0.00002
49	685.4740	685.1629	-0.3111	-0.00000	677.5709	-7.9031	-0.00002
50	719.0120	718.5311	-0.4809	-0.00000	711.0206	-7.9914	-0.00002

Table 5

The approximation of the interpolated quantiles of Mann-Whitney statistic for $5 \leq n \leq 50$ and $\alpha = 0.001$

n	$u_i(n,\alpha)$	$u_i^a(n, \alpha)$	$d_u(n, \alpha)$	$\alpha^a - \alpha$	$u_i^n(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^n - \alpha$
5	1.2600	1.5110	0.2510	0.00199	1.8635	0.6035	0.00479
6	3.4480	3.1599	-0.2881	-0.00156	3.9720	0.5240	0.00284
7	6.2880	6.0282	-0.2598	-0.00114	6.7934	0.5054	0.00221
8	9.8868	9.7529	-0.1339	-0.00040	10.3487	0.4619	0.00163
9	14.2188	14.2264	0.0076	0.00002	14.6547	0.4358	0.00124
10	19.3108	19.4266	0.1158	0.00027	19.7253	0.4145	0.00097
11	25.1887	25.3602	0.1715	0.00035	25.5725	0.3838	0.00078
12	31.8480	32.0431	0.1951	0.00032	32.2065	0.3585	0.00063
13	39.2954	39.4927	0.1973	0.00030	39.6362	0.3408	0.00052
14	47.5492	47.7250	0.1758	0.00023	47.8696	0.3204	0.00042
15	56.6158	56.7544	0.1386	0.00016	56.9138	0.2980	0.00035
16	66.4974	66.5929	0.0955	0.00010	66.7750	0.2776	0.00030
17	77.2055	77.2511	0.0456	0.00005	77.4591	0.2536	0.00025
18	88.7362	88.7376	0.0013	0.00000	88.9712	0.2350	0.00021
19	101.1038	101.0597	-0.0441	-0.00004	101.3162	0.2124	0.00018
20	114.3008	114.2238	-0.0770	-0.00006	114.4985	0.1977	0.00015
21	128.3424	128.2351	-0.1072	-0.00008	128.5222	0.1798	0.00013
22	143.2305	143.0982	-0.1323	-0.00009	143.3910	0.1605	0.00011
23	158.9694	158.8170	-0.1524	-0.00009	159.1086	0.1392	0.00009
24	175.5505	175.3947	-0.1558	-0.00009	175.6781	0.1277	0.00007
25	192.9976	192.8343	-0.1634	-0.00009	193.1028	0.1051	0.00006
26	211.2920	211.1382	-0.1538	-0.00008	211.3854	0.0934	0.00005
27	230.4508	230.3087	-0.1421	-0.00007	230.5288	0.0780	0.00004
28	250.4735	250.3477	-0.1258	-0.00006	250.5356	0.0621	0.00003
29	271.3620	271.2568	-0.1052	-0.00005	271.4081	0.0460	0.00002
30	293.1203	293.0375	-0.0828	-0.00003	293.1487	0.0284	0.00001
31	315.7449	315.6911	-0.0538	-0.00002	315.7597	0.0148	0.00001
32	339.2432	339.2188	-0.0244	-0.00001	339.2430	-0.0002	-0.00000
33	363.6149	363.6216	0.0067	0.00000	363.6008	-0.0141	-0.00000
34	388.8652	388.9004	0.0352	0.00001	388.8348	-0.0303	-0.00001
35	414.9931	415.0561	0.0630	0.00002	414.9470	-0.0461	-0.00001
36	441.9993	442.0894	0.0901	0.00003	441.9391	-0.0601	-0.00002
37	469.8854	470.0009	0.1155	0.00003	469.8128	-0.0726	-0.00002
38	498.6544	498.7913	0.1369	0.00004	498.5696	-0.0848	-0.00002
39	528.3096	528.4611	0.1516	0.00004	528.2111	-0.0984	-0.00003
40	558.8516	559.0108	0.1592	0.00004	558.7388	-0.1128	-0.00003
41	590.2793	590.4409	0.1617	0.00004	590.1541	-0.1252	-0.00003
42	622.5962	622.7518	0.1556	0.00004	622.4584	-0.1378	-0.00003
43	655.8046	655.9438	0.1392	0.00003	655.6530	-0.1516	-0.00004
44	689.9042	690.0173	0.1131	0.00003	689.7391	-0.1651	-0.00004
45	724.8957	724.9727	0.0770	0.00002	724.7181	-0.1776	-0.00004
46	760.7805	760.8101	0.0296	0.00001	760.5911	-0.1893	-0.00004
47	797.5603	797.5299	-0.0304	-0.00001	797.3593	-0.2010	-0.00004
48	835.2376	835.1323	-0.1053	-0.00002	835.0238	-0.2138	-0.00004
49	873.8118	873.6175	-0.1942	-0.00004	873.5856	-0.2262	-0.00004
50	913.2836	912.9858	-0.2978	-0.00006	913.0458	-0.2377	-0.00004

Table 6

The approximation of the interpolated quantiles of Mann-Whitney statistic for $5 \leq n \leq 50$ and $\alpha = 0.05$

n	$u_1(n,\alpha)$	$u_1^a(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^a - \alpha$	$u_1^n(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^n - \alpha$
5	4.0857	4.2579	0.1722	0.00478	5.1259	1.0401	0.02989
6	7.1778	6.9891	-0.1887	-0.00362	8.2279	1.0501	0.02144
7	11.0868	10.8876	-0.1992	-0.00272	12.1270	1.0402	0.01643
8	15.7766	15.6911	-0.0855	-0.00094	16.8379	1.0613	0.01312
9	21.3109	21.3208	0.0099	0.00010	22.3724	1.0615	0.01092
10	27.6781	27.7597	0.0816	0.00065	28.7406	1.0625	0.00918
11	34.8949	35.0117	0.1168	0.00080	35.9508	1.0560	0.00784
12	42.9550	43.0876	0.1326	0.00083	44.0103	1.0553	0.00682
13	51.8648	51.9994	0.1346	0.00071	52.9254	1.0606	0.00604
14	61.6351	61.7582	0.1231	0.00059	62.7016	1.0665	0.00541
15	72.2774	72.3741	0.0967	0.00043	73.3440	1.0666	0.00486
16	83.7903	83.8554	0.0651	0.00026	84.8571	1.0668	0.00439
17	96.1772	96.2096	0.0324	0.00012	97.2449	1.0677	0.00401
18	109.4394	109.4428	0.0033	0.00001	110.5112	1.0717	0.00368
19	123.5864	123.5600	-0.0264	-0.00008	124.6593	1.0729	0.00338
20	138.6183	138.5657	-0.0526	-0.00015	139.6923	1.0740	0.00313
21	154.5376	154.4637	-0.0739	-0.00020	155.6132	1.0755	0.00291
22	171.3482	171.2571	-0.0911	-0.00023	172.4246	1.0764	0.00271
23	189.0545	188.9487	-0.1059	-0.00024	190.1290	1.0745	0.00253
24	207.6499	207.5407	-0.1092	-0.00023	208.7288	1.0789	0.00237
25	227.1476	227.0354	-0.1122	-0.00023	228.2262	1.0786	0.00223
26	247.5414	247.4343	-0.1071	-0.00020	248.6231	1.0818	0.00211
27	268.8403	268.7390	-0.1012	-0.00018	269.9216	1.0814	0.00199
28	291.0421	290.9509	-0.0911	-0.00016	292.1235	1.0814	0.00188
29	314.1466	314.0712	-0.0754	-0.00012	315.2305	1.0838	0.00179
30	338.1589	338.1009	-0.0580	-0.00009	339.2442	1.0852	0.00170
31	363.0804	363.0410	-0.0394	-0.00006	364.1662	1.0858	0.00162
32	388.9108	388.8921	-0.0187	-0.00003	389.9980	1.0871	0.00154
33	415.6511	415.6552	0.0041	0.00001	416.7410	1.0899	0.00148
34	443.3056	443.3307	0.0252	0.00003	444.3965	1.0909	0.00141
35	471.8748	471.9194	0.0446	0.00005	472.9659	1.0911	0.00135
36	501.3570	501.4217	0.0648	0.00008	502.4503	1.0934	0.00130
37	531.7569	531.8381	0.0812	0.00009	532.8510	1.0941	0.00124
38	563.0749	563.1691	0.0942	0.00010	564.1691	1.0943	0.00120
39	595.3091	595.4149	0.1058	0.00011	596.4057	1.0966	0.00115
40	628.4638	628.5760	0.1122	0.00011	629.5618	1.0980	0.00111
41	662.5394	662.6527	0.1134	0.00011	663.6384	1.0990	0.00107
42	697.5364	697.6453	0.1089	0.00010	698.6366	1.1001	0.00103
43	733.4560	733.5540	0.0981	0.00009	734.5571	1.1012	0.00100
44	770.2991	770.3792	0.0801	0.00007	771.4010	1.1020	0.00096
45	808.0671	808.1209	0.0537	0.00005	809.1692	1.1020	0.00093
46	846.7585	846.7794	0.0209	0.00002	847.8624	1.1039	0.00090
47	886.3762	886.3549	-0.0213	-0.00002	887.4815	1.1052	0.00088
48	926.9219	926.8476	-0.0743	-0.00006	928.0272	1.1053	0.00085
49	968.3931	968.2576	-0.1355	-0.00010	969.5004	1.1073	0.00082
50	1010.7939	1010.5851	-0.2087	-0.00015	1011.9017	1.1079	0.00080

Table 7

The approximation of the interpolated quantiles of Mann-Whitney statistic for $5 \leq n \leq 50$ and $\alpha = 0.01$

n	$u_1(n,\alpha)$	$u_1^a(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^a-\alpha$	$u_1^n(n,\alpha)$	$d_u(n,\alpha)$	$\alpha^n-\alpha$
5	5.6889	5.8260	0.1371	0.00490	6.8650	1.1762	0.04887
6	9.3357	9.1812	-0.1545	-0.00468	10.4967	1.1610	0.03733
7	13.8027	13.6561	-0.1466	-0.00320	14.9703	1.1676	0.02891
8	19.1294	19.0523	-0.0771	-0.00153	20.2972	1.1678	0.02381
9	25.3031	25.3112	0.0081	0.00013	26.4867	1.1837	0.01989
10	32.3554	32.4205	0.0650	0.00089	33.5467	1.1912	0.01695
11	40.2868	40.3839	0.0971	0.00115	41.4835	1.1967	0.01469
12	49.1038	49.2101	0.1063	0.00112	50.3029	1.1991	0.01288
13	58.8016	58.9086	0.1069	0.00095	60.0097	1.2081	0.01138
14	69.3916	69.4880	0.0964	0.00079	70.6085	1.2169	0.01027
15	80.8835	80.9563	0.0728	0.00052	82.1029	1.2194	0.00925
16	93.2686	93.3200	0.0513	0.00035	94.4966	1.2280	0.00845
17	106.5577	106.5847	0.0270	0.00016	107.7927	1.2350	0.00772
18	120.7546	120.7554	0.0008	0.00000	121.9940	1.2395	0.00709
19	135.8593	135.8359	-0.0235	-0.00012	137.1034	1.2440	0.00657
20	151.8735	151.8297	-0.0438	-0.00021	153.1230	1.2495	0.00610
21	168.7994	168.7397	-0.0598	-0.00026	170.0553	1.2559	0.00569
22	186.6400	186.5683	-0.0717	-0.00030	187.9023	1.2623	0.00532
23	205.3982	205.3177	-0.0805	-0.00031	206.6659	1.2677	0.00500
24	225.0774	224.9897	-0.0877	-0.00032	226.3481	1.2706	0.00471
25	245.6727	245.5858	-0.0869	-0.00030	246.9503	1.2776	0.00444
26	267.1925	267.1075	-0.0850	-0.00028	268.4744	1.2819	0.00421
27	289.6341	289.5559	-0.0782	-0.00024	290.9218	1.2877	0.00398
28	313.0037	312.9321	-0.0717	-0.00021	314.2938	1.2901	0.00379
29	337.2950	337.2369	-0.0581	-0.00016	338.5920	1.2970	0.00360
30	362.5154	362.4712	-0.0442	-0.00012	363.8175	1.3021	0.00343
31	388.6651	388.6357	-0.0294	-0.00007	389.9715	1.3064	0.00328
32	415.7445	415.7310	-0.0136	-0.00003	417.0552	1.3107	0.00313
33	443.7546	443.7577	0.0030	0.00001	445.0698	1.3152	0.00300
34	472.6964	472.7162	0.0198	0.00004	474.0162	1.3198	0.00288
35	502.5711	502.6071	0.0361	0.00007	503.8955	1.3244	0.00277
36	533.3800	533.4308	0.0509	0.00010	534.7086	1.3287	0.00266
37	565.1244	565.1876	0.0631	0.00012	566.4565	1.3320	0.00256
38	597.8034	597.8778	0.0744	0.00014	599.1400	1.3366	0.00247
39	631.4187	631.5017	0.0831	0.00015	632.7600	1.3413	0.00238
40	665.9732	666.0597	0.0865	0.00015	667.3173	1.3441	0.00230
41	701.4631	701.5519	0.0887	0.00014	702.8126	1.3495	0.00222
42	737.8942	737.9785	0.0843	0.00013	739.2468	1.3526	0.00215
43	775.2634	775.3399	0.0764	0.00012	776.6206	1.3572	0.00208
44	813.5733	813.6361	0.0628	0.00009	814.9347	1.3614	0.00201
45	852.8250	852.8674	0.0424	0.00006	854.1897	1.3647	0.00195
46	893.0183	893.0339	0.0155	0.00002	894.3863	1.3680	0.00190
47	934.1528	934.1357	-0.0172	-0.00002	935.5252	1.3724	0.00184
48	976.2306	976.1730	-0.0576	-0.00007	977.6069	1.3764	0.00179
49	1019.2520	1019.1459	-0.1061	-0.00013	1020.6321	1.3802	0.00174
50	1063.2176	1063.0545	-0.1631	-0.00020	1064.6014	1.3838	0.00169

2. THE PROBLEMS CONCERNING THE TIED RANKS

In its classical form the Mann-Whitney test verifies the hypothesis that the samples come from population with continuous distributions. The problem of tied ranks exist in the case of discret distributions.

Now we decline the assumption of continuity of F and G and the test statistic we define as follows:

$$U = \sum_{i=1}^{n} \sum_{j=1}^{m} D_{ij}, \tag{12}$$

where

$$D_{ij} = \begin{cases} 1 & \text{if} \quad X_i < Y_j \\ 0 & \text{if} \quad X_i > Y_j \\ \frac{1}{2} & \text{if} \quad X_i = Y_j \end{cases}.$$

Assume that both of variables X_i and Y_j have the binary distribution with parameter p.

Let n_o and n_1 be number of zeros and ones among the variables X_i and let m_o, m_1 be a number zeros and ones among the variables Y_j.

The value of statistic U is

$$U = \frac{1}{2}\,(n_o m + m_1 n) \tag{13}$$

Thus, the U-statistic is linear combination of two independent variables n_o, m_1. These variables have binominal distribution.
Hence:

$$P(U = u) = \Sigma \binom{n}{n_o} \binom{m}{m_o} p^{n-n_o+m_1} \; q^{m-m_1+n_o}, \tag{14}$$

where the summation is performed over all pairs (n_o, m_1) such that

$$\frac{1}{2}\, mn_o + \frac{1}{2}\, nm_1 = u.$$

The recursive formula for probabilities is of the form:

$$P(n, m, m_o, u) = p\, P(n - 1, m, m_o, u - (m - \tfrac{1}{2}\, m_o)) + \\ + q\, P(n - 1, m, m_o, u - \tfrac{1}{2}\,(m - m_o)) \tag{15}$$

under the initial condition

Table 8

The interpolated quantiles of U-statistic for p = 0.5

n	m	α=0.01	α=0.05	α=0.10	n	m	α=0.01	α=0.05	α=0.10
5	5	2.310	4.733	5.967	13	13	42.529	53.709	59.788
6	5	4.131	7.603	8.357	14	5	14.338	19.394	23.514
6	6	4.271	7.715	9.670	14	6	18.161	24.629	28.210
7	5	5.199	8.365	10.716	14	7	21.265	28.516	32.509
7	6	6.664	11.854	12.811	14	8	26.606	34.741	40.025
7	7	7.556	11.721	14.293	14	9	31.733	40.738	45.442
8	5	6.497	10.322	12.563	14	10	35.934	45.547	50.664
8	6	9.011	12.831	15.291	14	11	40.991	51.148	56.656
8	7	10.884	15.022	18.336	14	12	44.843	56.843	63.050
8	8	11.703	16.696	19.697	14	13	52.519	60.735	67.357
9	5	7.404	11.778	14.108	14	14	51.255	63.919	70.659
9	6	9.852	14.474	17.017	15	5	14.193	20.266	23.598
9	7	12.465	18.359	20.527	15	6	19.342	26.288	30.125
9	8	16.261	20.969	24.870	15	7	24.723	32.176	35.996
9	9	15.902	22.619	25.796	15	8	30.565	38.517	42.580
10	5	8.043	12.515	14.897	15	9	33.854	43.558	48.466
10	6	11.611	16.574	19.347	15	10	38.411	48.483	53.981
10	7	15.159	20.321	23.699	15	11	44.276	55.268	60.917
10	8	17.368	24.163	26.817	15	12	49.384	60.142	66.526
10	9	22.052	27.968	32.365	15	13	54.606	67.813	74.432
10	10	21.383	28.964	32.864	15	14	58.096	72.195	79.393
11	5	10.061	15.155	17.770	15	15	61.091	75.093	82.464
11	6	14.084	19.591	22.532	16	5	16.339	22.730	25.841
11	8	16.821	22.730	26.324	16	6	21.141	28.461	33.033
11	7	20.350	26.590	30.406	16	7	26.048	35.538	40.033
11	9	23.644	30.053	34.340	16	8	29.957	39.093	44.146
11	10	26.170	35.927	40.688	16	9	37.570	46.650	51.654
11	11	27.979	36.214	40.888	16	10	42.529	53.221	58.626
12	5	12.071	15.977	18.989	16	11	48.449	59.338	64.964
12	6	13.752	19.528	22.754	16	12	52.685	64.888	71.486
12	7	18.906	25.263	28.709	16	13	57.827	72.017	78.709
12	8	21.831	28.898	32.820	16	14	65.291	78.554	84.961
12	9	25.560	33.759	37.969	16	15	69.416	84.679	92.443
12	10	30.560	37.909	42.935	16	16	71.962	86.644	94.834
12	11	34.034	44.693	46.463	17	5	17.411	24.568	28.244
12	12	35.007	44.469	49.864	17	6	23.284	31.562	34.948
13	5	12.330	18.141	20.899	17	7	29.105	37.584	41.431
13	6	16.449	22.701	27.087	17	8	33.314	44.534	48.950
13	7	20.349	27.156	30.724	17	9	39.497	51.544	56.537
13	8	24.596	32.669	36.712	17	10	45.488	56.461	63.552
13	9	28.783	37.291	41.781	17	11	51.723	63.709	69.598
13	10	32.885	42.585	46.977	17	12	57.369	69.658	76.811
13	11	36.386	46.886	52.532	17	13	64.036	76.984	83.724
13	12	42.899	50.396	56.356	17	14	68.545	84.012	91.244

Table 9

The interpolated quantiles of U-statistic for p = 0.5

n	m	α =0.01	α=0.05	α=0.10	n	m	α=0.01	α =0.05	α=0.10
17	15	74.480	88.453	96.654	20	17	104.982	123.870	134.316
17	16	81.801	98.146	106.486	20	18	112.438	131.665	141.614
17	17	82.602	99.144	108.180	20	19	117.163	136.956	153.531
18	5	19.014	25.843	30.087	20	20	121.558	142.639	154.036
18	6	23.469	31.413	36.031	21	5	22.181	30.321	35.266
18	7	29.882	39.528	44.402	21	6	29.103	38.115	43.414
18	8	36.563	45.864	51.281	21	7	35.138	45.336	50.914
18	9	40.761	51.352	57.344	21	8	43.835	54.766	60.850
18	10	47.969	60.851	66.404	21	9	50.693	62.448	69.686
18	11	54.467	67.378	74.271	21	10	60.030	71.228	77.342
18	12	60.094	73.306	80.587	21	12	64.805	80.383	86.348
18	13	67.657	81.344	90.043	21	12	72.701	88.126	95.602
18	14	73.546	88.493	96.271	21	13	80.396	96.296	104.473
18	15	79.970	95.960	104.302	21	14	86.602	103.438	112.608
18	16	84.974	101.756	110.526	21	15	94.976	112.479	122.439
18	17	95.132	112.555	121.487	21	16	103.106	121.643	132.096
18	18	94.462	112.651	122.499	21	17	111.574	130.654	141.112
19	5	19.820	27.012	31.697	21	18	116.949	137.395	149.367
19	6	26.362	33.939	39.527	21	19	121.639	148.105	158.698
19	7	32.950	42.291	46.727	21	20	132.975	153.832	164.609
19	8	38.537	49.244	54.740	21	21	136.709	159.144	171.251
19	9	45.552	55.459	63.861	22	5	24.552	33.005	36.578
19	10	53.177	63.510	72.226	22	6	31.247	40.409	45.765
19	11	58.556	72.065	77.982	22	7	38.670	49.456	56.127
19	12	64.410	79.195	85.975	22	8	46.076	57.508	64.414
19	13	70.973	86.472	95.565	22	9	53.357	66.672	73.294
19	14	79.508	93.883	103.116	22	10	61.358	75.745	82.604
19	15	84.834	101.677	110.379	22	11	66.700	81.381	89.376
19	16	93.512	109.856	119.254	22	12	75.835	92.110	102.055
19	17	98.169	116.194	125.545	22	13	84.781	102.096	110.840
19	18	103.247	127.514	137.357	22	14	92.496	110.274	120.187
19	19	107.464	127.152	137.785	22	15	100.236	118.713	129.534
20	5	20.413	28.090	32.478	22	16	107.983	127.661	177.829
20	6	27.455	36.527	40.936	22	17	116.575	136.481	147.641
20	7	34.348	44.396	50.704	22	18	124.863	146.255	157.309
20	8	40.643	51.349	57.353	22	19	132.317	154.150	165.409
20	9	47.206	60.282	65.734	22	20	143.037	165.429	176.771
20	10	52.669	65.580	72.403	22	21	149.862	171.789	182.994
20	11	62.411	74.742	83.289	22	22	151.787	146.544	189.427
20	12	68.425	82.828	90.844	23	5	24.667	33.543	38.470
20	13	75.487	91.736	98.975	23	6	32.556	43.508	47.437
20	14	82.725	98.929	108.422	23	7	39.707	50.800	57.739
20	15	89.626	106.616	115.839	23	8	47.462	59.484	66.776
20	16	96.732	114.668	124.253	23	9	53.035	70.070	76.913

Table 10

The interpolated quantiles of U-statistic for p = 0.5

n	m	α=0.01	α=0.05	α=0.10	n	m	α=0.01	α=0.05	α=0.10
23	10	63.805	79.137	85.943	25	15	115.532	135.853	146.918
23	11	72.476	89.086	95.368	25	16	125.645	146.493	158.449
23	12	82.035	99.088	105.752	25	17	134.648	155.969	168.401
23	13	89.652	105.743	115.818	25	18	143.436	168.096	180.575
23	14	97.331	115.746	125.166	25	19	153.805	178.652	191.542
23	15	105.804	124.816	136.012	25	20	162.170	186.979	200.352
23	16	113.737	133.623	144.761	25	21	171.366	198.251	213.151
23	17	122.459	144.579	154.271	25	22	183.212	209.576	222.340
23	18	131.275	152.335	165.053	25	23	190.816	216.480	236.636
23	19	139.667	163.564	175.187	25	24	196.725	231.709	244.592
23	20	148.400	171.604	183.319	25	25	203.325	233.121	249.606
23	21	154.542	183.471	195.897	26	5	28.688	37.929	43.400
23	22	167.716	190.774	202.500	26	6	36.930	48.457	54.764
23	23	167.838	194.620	208.566	26	7	47.025	58.157	65.252
24	5	26.909	34.903	41.052	26	8	56.120	69.205	77.092
24	6	33.185	43.309	49.120	26	9	66.040	79.542	88.131
24	7	43.091	53.869	60.430	26	10	73.750	89.496	98.402
24	8	49.001	61.600	68.643	26	11	82.914	99.496	110.108
24	9	58.999	72.430	80.104	26	12	92.387	110.933	119.615
24	10	68.109	82.432	90.452	26	13	99.289	118.382	128.737
24	11	77.539	91.235	101.694	26	14	113.194	133.148	141.958
24	12	82.086	98.891	108.257	26	15	121.292	141.943	154.566
24	13	93.652	111.849	120.178	26	16	131.326	152.734	165.302
24	14	103.037	120.955	131.221	26	17	141.168	162.959	175.793
24	15	110.822	130.484	141.409	26	18	150.122	175.239	188.093
24	16	118.253	138.954	150.105	26	19	159.821	185.346	198.487
24	17	127.984	150.386	161.822	26	20	170.304	196.305	210.147
24	18	136.547	159.134	171.248	26	21	180.166	206.389	221.630
24	19	147.541	169.401	183.114	26	22	189.481	216.838	233.055
24	20	154.279	179.583	192.792	26	23	198.518	230.037	243.382
24	21	165.205	189.272	201.937	26	24	210.564	237.317	252.430
24	22	172.060	196.976	215.687	26	25	216.843	253.591	267.135
24	23	186.404	210.756	223.043	26	26	222.692	253.879	271.230
24	24	185.031	213.365	228.665	27	5	29.825	40.600	45.753
25	5	26.533	35.974	41.198	27	6	28.811	49.869	56.436
25	6	36.023	45.813	51.869	27	7	48.330	61.718	68.646
25	7	43.767	56.294	63.193	27	8	57.327	70.974	78,994
25	8	52.499	65.161	73.054	27	9	65.425	80.748	89.010
25	9	61.371	76.739	83.151	27	10	76.575	93.773	102.389
25	10	69.885	85.270	93.547	27	11	87.195	103.879	114.209
25	11	79.672	96.436	104.791	27	12	96.793	114.398	125.179
25	12	90.635	104.486	115.800	27	13	105.583	125.469	138.230
25	13	96.339	115.146	127.196	27	14	116.918	137.475	150.578
25	14	107.150	126.639	137.728	27	15	127.029	148.565	159.996

Table 11

The left-sided interpolated quantiles and test's size

for α = 0.01

n	Quantiles				Quantiles			
	cont. distr.	binominal distr.			cont. distr.	binominal distr.		
		p=0.5	p=0.25	p=0.1		p=0.5	p=0.25	p=0.1
5	1.26	2.31	3.05	5.26	0.0100	0.0196	0.0287	0.0848
6	3.45	4.27	6.13	9.06	0.0100	0.0150	0.0343	0.0916
7	6.29	7.56	9.35	13.03	0.0100	0.0164	0.0299	0.0830
8	9.89	11.70	13.50	17.91	0.0100	0.0175	0.0287	0.0789
9	14.22	15.90	18.80	23.92	0.0100	0.0154	0.0101	0.0797
10	19.31	21.38	25.16	30.99	0.0100	0.0157	0.0325	0.0826
11	25.19	27.98	31.59	39.08	0.0100	0.0167	0.0306	0.0866
12	31.85	35.01	38.93	48.18	0.0100	0.0166	0.0296	0.0909
13	39.30	42.53	47.44	57.60	0.0100	0.0159	0.0300	0.0895
14	47.55	51.25	57.03	67.61	0.0100	0.0160	0.0311	0.0863
15	56.62	61.09	67.66	78.77	0.0100	0.0166	0.0327	0.0853
16	66.50	71.96	78.26	91.02	0.0100	0.0175	0.0315	0.0858
17	77.21	82.60	89.82	104.33	0.0100	0.0166	0.0307	0.0871
18	88.74	94.46	102.53	118.67	0.0100	0.0164	0.0308	0.0888
19	101.10	107.46	116.34	134.04	0.0100	0.0165	0.0313	0.0909
20	114.30	121.56	131.21	150.42	0.0100	0.0170	0.0321	0.0932
21	128.34	136.71	147.11	167.39	0.0100	0.0176	0.0332	0.0939
22	143.23	151.79	162.87	184.58	0.0100	0.0171	0.0323	0.0917
23	158.97	167.84	179.66	202.90	0.0100	0.0169	0.0318	0.0906
24	175.55	185.03	197.59	222.31	0.0100	0.0169	0.0317	0.0903
25	193.00	203.33	216.61	242.80	0.0100	0.0171	0.0319	0.0905
26	211.29	222.69	236.70	264.34	0.0100	0.0174	0.0324	0.0911
27	230.45	243.11	257.83	286.92	0.0100	0.0179	0.0330	0.0919
28	250.47	263.34	280.00	310.53	0.0100	0.0175	0.0337	0.0930
29	271.36	284.63	301.90	335.16	0.0100	0.0173	0.0330	0.0943
30	293.12	307.05	324.93	360.81	0.0100	0.0173	0.0326	0.0956

Table 12

The left-sided interpolated quantiles and test's size for $\alpha = 0.05$

n	Quantiles				Quantiles			
	cont. distr.	binominal distr.			cont. distr.	binominal distr.		
		p=0.5	p=0.25	p=0.1		p=0.5	p=0.25	p=0.1
5	4.09	4.73	5.47	7.45	0.0500	0.0680	0.0921	0.1797
6	7.18	7.72	9.14	11.14	0.0500	0.0605	0.0940	0.1608
7	11.09	11.72	13.32	15.88	0.0500	0.0598	0.0894	0.1554
8	15.78	16.70	18.31	21.63	0.0500	0.0613	0.0855	0.1550
9	21.31	22.62	24.31	28.38	0.0500	0.0637	0.0852	0.1567
10	27.68	28.96	31.30	36.10	0.0500	0.0612	0.0864	0.1594
11	34.89	36.21	39.26	44.80	0.0500	0.0600	0.0884	0.1626
12	42.96	44.47	48.17	54.47	0.0500	0.0601	0.0908	0.1660
13	51.86	53.71	57.57	65.11	0.0500	0.0609	0.0893	0.1693
14	61.64	63.92	67.81	76.23	0.0500	0.0621	0.0877	0.1670
15	72.28	75.09	79.06	88.19	0.0500	0.0636	0.0872	0.1643
16	83.79	86.64	91.31	101.18	0.0500	0.0624	0.0876	0.1630
17	96.18	99.14	104.56	115.18	0.0500	0.0617	0.0883	0.1627
18	109.44	112.65	118.78	130.18	0.0500	0.0616	0.0894	0.1630
19	123.59	127.15	133.98	146.18	0.0500	0.0619	0.0906	0.1637
20	138.62	142.64	150.14	163.17	0.0500	0.0625	0.0920	0.1647
21	154.54	159.10	166.73	181.14	0.0500	0.0632	0.0911	0.1660
22	171.35	176.54	184.22	200.11	0.0500	0.0641	0.0903	0.1674
23	189.05	194.62	202.73	220.05	0.0500	0.0642	0.0900	0.1688
24	207.65	213.37	222.23	240.98	0.0500	0.0636	0.0900	0.1703
25	227.15	233.12	242.73	262.89	0.0500	0.0633	0.0903	0.1719
26	247.54	253.88	264.22	285.53	0.0500	0.0633	0.0907	0.1723
27	268.84	275.63	286.70	308.74	0.0500	0.0635	0.0913	0.1710
28	291.04	298.37	310.17	332.97	0.0500	0.0639	0.0920	0.1701
29	314.15	322.10	334.61	358.21	0.0500	0.0643	0.0928	0.1696
30	338.16	346.81	360.03	384.46	0.0500	0.0648	0.0937	0.1694

Table 13

The left-sided interpolated quantiles and test's size
for $\alpha = 0.01$

n	Quantiles				Quantiles			
	cont. distr.	binominal distr.			cont. distr.	binominal distr.		
		p=0.5	p=0.25	p=0.1		p=0.5	p=0.25	p=0.1
5	5.69	5.97	6.70	8.06	0.1000	0.1099	0.1418	0.2143
6	9.34	9.67	10.50	12.45	0.1000	0.1101	0.1375	0.2173
7	13.80	14.29	15.25	17.79	0.1000	0.1117	0.1370	0.2200
8	19.13	19.70	20.94	24.07	0.1000	0.1112	0.1381	0.2233
9	25.30	25.80	27.58	31.01	0.1000	0.1079	0.1404	0.2185
10	32.36	32.86	35.15	38.82	0.1000	0.1069	0.1426	0.2140
11	40.29	40.89	43.40	47.61	0.1000	0.1171	0.1411	0.2120
12	49.10	49.86	52.49	57.40	0.1000	0.1180	0.1391	0.2114
13	58.80	59.79	62.55	68.17	0.1000	0.1193	0.1384	0.2117
14	69.39	70.66	73.59	79.91	0.1000	0.1107	0.1385	0.2126
15	80.88	82.46	85.60	92.64	0.1000	0.1121	0.1390	0.2138
16	93.27	94.83	98.56	106.33	0.1000	0.1108	0.1399	0.2152
17	106.56	108.18	112.50	121.00	0.1000	0.1102	0.1410	0.2166
18	120.75	122.50	127.39	136.65	0.1000	0.1101	0.1422	0.2182
19	135.86	137.78	143.24	153.26	0.1000	0.1103	0.1434	0.2198
20	151.87	154.04	160.05	170.85	0.1000	0.1107	0.1447	0.2213
21	168.80	171.25	177.45	189.41	0.1000	0.1113	0.1439	0.2229
22	186.64	189.43	195.80	208.90	0.1000	0.1120	0.1433	0.2240
23	205.40	208.57	215.13	228.99	0.1000	0.1128	0.1430	0.2227
24	225.08	228.66	235.45	250.07	0.1000	0.1137	0.1430	0.2218
25	245.67	249.61	256.73	272.14	0.1000	0.1141	0.1432	0.2213
26	264.19	271.23	279.00	295.20	0.1000	0.1137	0.1435	0.2211
27	289.63	293.83	302.23	319.26	0.1000	0.1134	0.1439	0.2210
28	313.00	317.41	326.44	344.30	0.1000	0.1133	0.1444	0.2211
29	337.29	341.97	351.62	370.33	0.1000	0.1134	0.1450	0.2214
30	362.52	367.49	377.78	397.34	0.1000	0.1136	0.1456	0.2217

$$P(1, m, m_o, u) = \begin{cases} \binom{m}{m_o} p^{m-m_o} q^{m_o+1} & \text{for } u = \frac{1}{2}(m - m_o) \\ \binom{m}{m_o} p^{m-m_o+1} q^{m_o} & \text{for } u = m - \frac{1}{2} m_o \\ 0 & \text{for add } u. \end{cases} \qquad (16)$$

From this formulae for $p = \frac{1}{2}$ and $m, n = 1, \ldots, 30$ we have obtained unconditional distribution of U-statistic.

Their left-sided interpolated quantiles for two significance levels $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.10$ are presented in Table 8-10. They differ from the respective quantiles of U-statistic for continuous distribution (see Table 9-11). The general analysis of differences is not easy. The numbers from Table 9-11 allow to catch some idea of range of differences between the quantiles and test size errors for m = n and some chosen probabilities P.

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TEST MANNA-WHITNEYA. PROBLEMY ZWIĄZANE Z APROKSYMACJĄ WARTOŚCI KRYTYCZNYCH I RANGAMI WIĄZANYMI

Nieparametryczny test Manna-Whitneya (1947) należy do najczęściej stosowanych testów dla dwóch prób. W tym artykule prezentujemy dokładną aproksymację wartości krytycznych statystyki Manna-Whitneya w porównaniu z aproksymacją za pomocą rozkładu normalnego.

Tablice 2 i 3 zawierają błędy aproksymacji dla 14 poziomów istotności. Obliczone kwantyle interpolowane, ich aproksymację za pomocą rozkładu normalnego oraz błędy aproksymacji dla czterech poziomów istotności prezentujemy w tablicach 4-7.