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A SEQUENTIAL RATIO TEST FOR THE LOGISTIC DISTRIBUTION

Abstract. There is given a population with a random variable X, subject to the logistic distribution

$$f(x;\ \rho,\ \sigma,\ \Theta) = \frac{\Theta\rho}{\sigma} e^{-x/\sigma} (1 + \rho e^{-x/\sigma})^{-(\Theta+1)},\ x \in R$$

with positive parameters. We verify the hypothesis $H : \Theta = \Theta_0$ against the alternative $H_1 : \Theta = \Theta_1$ using the sequential ratio test to this problem. There are also presented the OC and ASN functions of the test considered.

Key words: sequential tests, logistic distribution.

INTRODUCTION

The logistic distribution, proposed in 1838 and 1845 by P. F. Verhulst [V e r h u l s t (1838, 1845)] is well known and widely applied nowadays in biology (e.g. to the determination of the increase of a population), economy, medicine and survival analysis.

J. C. Ahujà and S. W. Nash [A h u j a, N a s h (1967)] generalized the logistic and Gompertz distributions (also applied in ecology) by introducing an additional parameter. The problem of estimating a parameter of the logistic distribution was considered by T. Gerstenkorn [G e r s t e n k o r n (1992)].

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Assume that a characteristic X of elements of a general population $\mathcal{X}$ has a logistic distribution with positive parameters ρ, σ, θ. The density of X is given by

$$f(x;\ \rho, \sigma, \theta) = f(x) = \frac{\theta\rho}{\sigma} e^{-x/\sigma}(1 + \rho e^{-x/\sigma})^{-(\theta+1)};\ x \in \mathbb{R}, \qquad (1)$$

where the parameters ρ and σ are known, while θ is an unknown parameter.

1. EXAMINATION OF THE TEST ASSUMPTIONS

Let $\theta_0 < \theta_1$. We shall verify the hypothesis H_0: $\theta = \theta_0$ against the alternative hypothesis H_1: $\theta = \theta_1$. We construct a sequential ratio test for this problem.

At first, we check the assumptions of the test. Let

$$Z = \ln \frac{f(X;\ \rho, \sigma, \theta_1)}{f(X;\ \rho, \sigma, \theta_0)}. \qquad (2)$$

In view of (1), we have

$$Z = (\theta_0 - \theta_1)\ln(1 + \rho e^{-X/\sigma}) + \ln(\theta_1/\theta_0). \qquad (3)$$

LEMMA. If a random variable X has logistic distribution (1), then a random variable

$$W = \ln(1 + \rho e^{-X/\sigma}) \qquad (4)$$

has an exponential distribution.

P r o o f. With (1), the density of $Y = \rho e^{-X/\sigma}$ is given by

$$\frac{\sigma}{y} f(-\sigma \ln(y/\rho;\ \rho, \sigma, \theta), \quad y > 0.$$

After simple calculations we obtain

$$f_Y(y) = \theta(1 + y)^{-(\theta+1)}, \quad y > 0.$$

If $V = 1 + Y$, then

$$f_V(v) = \theta v^{-(\theta+1)}, \quad v > 0,$$

and finally, if $W = \ln V$, then

$$f_W(w) = \theta e^{-\theta w}, \quad w > 0.$$

Therefore

$E(W) = 1/\theta$ and $Var(W) = 1/\theta^2$.

Q.E.D.

In consequence,

$$E(Z) = (\theta_0 - \theta_1)/\theta + \ln(\theta_1/\theta_0), \tag{5}$$

$$Var(Z) = ((\theta_1 - \theta_0)/\theta)^2. \tag{6}$$

The variance of the variable Z is finite, thus, on the ground of a theorem [G i r s h i c k (1946), p. 127] the sequential procedure in the ratio test leads to a decision making for a finite sample with probability 1.

It is evident that $E(Z) = 0$ if

$$\theta = (\theta_1 - \theta_0)/\ln(\theta_1/\theta_0), \quad \theta_1 \neq \theta_0. \tag{7}$$

We calculate the expectation $E(e^{hZ})$, $h \in \mathbb{R}$. We have

$$e^{hZ} = (\theta_1/\theta_0)^h e^{-h(\theta_1-\theta_0)W},$$

where W is given by (4) and has the exponential distribution. Therefore

$$E(e^{hZ}) = (\theta_1/\theta_0)^h \cdot \theta \int_0^\infty e^{-[h(\theta_1-\theta_0)+\theta]w} dw.$$

We consider two cases: (a) $h(\theta_1 - \theta_0) + \theta > 0$ and (b) $h(\theta_1 - \theta_0) + \theta \leq 0$. Let us denote the examined integral by I. Condition (a) is fulfilled if $h > \theta/(\theta_0 - \theta_1)$. Let us put: $t = [h(\theta_1 - \theta_0) + \theta]w$. Let a function $\phi: (0, +\infty) \to \mathbb{R}$ be defined as

$$\phi(t) = t/[h(\theta_1 - \theta_0) + \theta]w.$$

The function is differentiable and monotonic in $(0, +\infty)$ and $\phi'(t)$ is in this interval integrable and

$$\lim_{t\to 0} \phi(t) = 0, \qquad \lim_{t\to\infty} \phi(t) = \infty,$$

then we have

$$I = 1/[h(\theta_1 - \theta_0) + \theta] \int_0^\infty e^{-t} dt = 1/[h(\theta_1 - \theta_0) + \theta].$$

Therefore, in case (a), we have

$$E(e^{hZ}) = (\theta_1/\theta_0)^h \cdot \theta/[h(\theta_1 - \theta_0) + \theta].$$

In case (b), i.e. if $h \leqslant \theta/(\theta_0 - \theta_1)$, the integral I is divergent.

As the expected value $E(e^{hZ}) = g(h)$ exists only in the interval $(\theta/(\theta_0 - \theta_1), +\infty)$, we observe it in this interval. We note that

$$\lim_{h \to a+} g(h) = \lim_{h \to \infty} g(h) = \infty \quad \text{where} \quad a = \theta/(\theta_0 - \theta_1) < 0.$$

Furthermore, we state that

$$g''(h) = g(h)\left\{\left[\ln (\theta_1/\theta_0) + \frac{\theta_0 - \theta_1}{h(\theta_1 - \theta_0) + \theta}\right]^2 + \left[\frac{\theta_1 - \theta_0}{h(\theta_1 - \theta_0) + \theta}\right]^2\right\} > 0.$$

There exists $0 \neq h_0 \in (\theta/(\theta_0 - \theta_1), +\infty)$ such that

$$E(e^{h_0 Z}) = 1. \tag{8}$$

One can see that all assumptions of the sequential test are fulfilled [W a l d (1963), p. 158- 159].

2. CONSTRUCTION OF THE TEST

Let $(x_1, x_2, \ldots, x_m)$ be an m-element sample from X and let z_i be a value of the variable Z given by (2) if $X = x_i$ $(i = 1, 2, \ldots, m)$.

Hence, by virtue of (3) and (4),

$$\sum_{i=1}^{m} z_i = (\theta_0 - \theta_1) \sum_{i=1}^{m} \ln w_i + m \ln (\theta_1/\theta_0).$$

Let α and β denote the probabilities of errors of the first and the second kind, respectively. Let A and B be constant determined by the conditions

$$A \approx \frac{1 - \beta}{\alpha}, \qquad B \approx \frac{\beta}{1 - \alpha}. \tag{9}$$

Therefore, if a characteristic X of elements of a general population $\mathcal{X}$ has logistic distribution (1), then the sequential ratio test has the form:

if

$$\sum_{i=1}^{m} w_i \leqslant (\theta_0 - \theta_1)^{-1}(\ln A - m \ln (\theta_1/\theta_0)) = C, \tag{10}$$

we reject the hypothesis H_0 in verification in favour of the alternative hypothesis H_1;

if

$$\sum_{i=1}^{m} w_i \geqslant (\theta_0 - \theta_1)^{-1}(\ln B - m \ln(\theta_1/\theta_0)) = D, \tag{11}$$

we accept the hypothesis H_0;

however, if

$$C < \sum_{i=1}^{m} w_i < D$$

we draw to the sample one element x_{m+1} more and again take into account inequalities (10) or (11).

3. THE OC AND ASN FUNCTIONS

As is known, the OC function (operating characteristic function) has the form

$$L(\theta) \approx \frac{A^{h_0(\theta)} - 1}{A^{h_0(\theta)} - B^{h_0(\theta)}} \text{ for } \theta \text{ different from that given by (7),}$$

where $h_0(\theta)$ is determined by equality (8) and A and B - by (9); if - on the other hand - (7) takes place, then

$$L(\theta) \approx \ln A/\ln(A/B).$$

The ASN function (average sample number function) is of the form

$$E_\theta(n) \approx \theta \frac{(A^{h_0(\theta)} - 1) \ln B + (1 - B^{h_0(\theta)}) \ln A}{(A^{h_0(\theta)} - B^{h_0(\theta)}) \cdot [\theta \ln(\theta_1/\theta_0) + \theta_0 - \theta_1]}$$

after taking into consideration the known formula

$$E_\theta(n) \approx [L(\theta) \ln B + (1 - L(\theta)) \ln A]/E(Z)$$

and the conditions $\theta \neq (7)$ and (5), (9).

Assuming (7), we have

$$E_\theta(n) \approx - \ln A \ln B/E(Z^2),$$

which, because of the fact that in this case

$$E(Z^2) = \text{Var}\ (Z) = (6),$$

gives

$$E_\theta(n) \approx -\ln A \ln B/(\ln(\theta_1/\theta_0))^2.$$

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SEKWENCYJNY TEST ILORAZOWY DLA ROZKŁADU LOGISTYCZNEGO

Podana jest populacja ze zmienną losową X, podlegająca rozkładowi logistycznemu

$$f(x;\ \rho, \sigma, \theta = \frac{\theta\rho}{\sigma}\ e^{-x/\sigma}\ (1 + \rho e^{-x/\sigma}),\ x \in R$$

z parametrami dodatnimi. Sprawdzamy hipotezę $H = \theta = \theta_0$ względem alternatywy $H_1 : \theta = \theta_1$, stosując sekwencyjny test ilorazowy. Przedstawiono także funkcje OC i ASN rozważanego testu.