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EXACT SIMILAR TESTS FOR THE ROOT OF A FIRST-ORDER AUTOREGRESSIVE REGRESSION MODEL

Abstract. A procedure is developed for testing whether or not the coefficient of the lagged dependent variable in a first-order autoregressive multiple regression model equals a particular value, such as zero or unity or any other arbitrary stable or unstable value. Under the null hypothesis the estimate of this coefficient is found to be distributed as the ratio of two quadratic forms in standard normal variables, when it is obtained from a particular auxiliary regression model where in addition to the exogenous regressors also some redundant transformed regressors ara included. This null distribution is found to be independent of any nuisance parameters. So, this estimate is easily calculated and it can directly be used as a test statistic; its type I errors can be controled exactly, whereas this test is similar and also invariant. Particular unit root tests developed by Dickey and Fuller appear to be simple examples of our test for very specific regressor matrices. We provide extended tables of exact critical values for these and for some other forms. Finally we illustrate the usefulness of our general test procedure in the dynamic specification of econometric time series models.

Key words: Autoregressive models, non-stationarity, unit roots, exact tests, similar tests, dynamic specification.

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1. INTRODUCTION

In dynamic regression equations where lags of the dependent variable occur as explanatory variables most of the available procedures for statistical inference lack precision. The presence of autoregressive explanatory variables in linear regression models complicates the analysis of its parameters dramatically, especially in small samples. Even the asymptotic behaviour of statistics of interest in such models is quite complex and crucially depends on whether all the roots of the lag polynomial of the dependent variable are located outside the unit circle or not. The values of these roots, which are determined by the lagged dependent variable coefficients, also characterize key aspects of the dynamics of the relationship which the model purports to describe. Therefore tests for the actual values of these coefficients, and more particularly for the presence of unit or unstable roots, are important and have recently received considerable attention.

In the first order autoregressive model the only root of the lag polynomial is simply the inverse of the coefficient of the lagged dependent variable (and hence here a unit root conforms to a unit coefficient value). In this paper we shall develop tests on the value of this coefficient which are exact and similar in finite samples under quite general conditions. These conditions entail normality of the disturbances and strict exogeneity of the regressors (apart from the lagged dependent variable), and also that all regressors are mutually linearly independent but otherwise arbitrary (stationarity is not required). So these assumptions correspond to those made in the classical static linear regression model in order to achieve inference (based on t and F statistics) on coefficient restrictions, which is exact and similar in finite samples. Note that a test is called 'similar' if its null distribution is independent of the value of any of the nuisance parameters, and that it is called 'exact' if its critical values are determined precisely so that one can effectively control the actual type I errors at exactly the chosen nominal value.

The tests developed here have wide applicability, both in single econometric time series regression models and in pure time series analysis, since they yield accurate inference on dynamics in general, and on (unit) roots of lag polynomials in particular, in samples of a small size. Particular well-known Dickey-Fuller tests for unit roots in the random walk (with drift) model are shown to be very specific variants of the procedure devised in this paper.

The paper is organized as follows. In section 2 we introduce the model and indicate some of its complexities, and we refer to some recent literature. Then in section 3 we develop an exact test for the presence of these complications, viz. an exact test for the significance of the lagged dependent variable coefficient. The test statistic is a straightforward least-squares coefficient estimate in an auxiliary regression equation. In section 4 an exact procedure is developed to test whether the lagged dependent variable coefficient equals some arbitrary real value. This test statistic is calculated by applying generalized least-squares to an auxiliary regression equation with first order moving average disturbances, where the moving average coefficient is known. In section 5 we show that the test statistic can also be calculated directly by employing ordinary least-squares to an appropriately augmented regression specification. Under the null hypothesis the test statistic is not dependent on any of the parameter values of the model, and it is distributed as the ratio of two quadratic forms in standard normal variables. Hence, exact critical values can be calculated numerically. In section 6 we focus on testing for unit roots, notably on situations where upto now (non-augmented) DF tests - [see Dickey and Fuller (1979, 1981)] - are usually employed. We show that particular DF tests, viz. for cases where the fixed regressors are void or simply the constant (and possibly a linear trend), are variants of our procedure. In section 7 we present some tables of exact critical values for this procedure. These allow to test for a zero or for a unit value of the coefficient of the lagged dependent variable in particular models; they have been calculated by numerical methods instead of Monte Carlo simulation. Finally we illustrate the use of our test procedure in a first-order dynamic econometric model, and construct exact similar confidence regions for the coefficient of the lagged dependent variable.

2. THE MODEL

We consider the linear first-order autoregressive model

$$y_t = \lambda y_{t-1} + x_t'\beta + u_t \quad t = \ldots, 0, 1, \ldots, T \tag{2.1}$$

where x_t and β are Kx1 vectors. We assume that $u_t \sim NID(0, \sigma^2)$ and do not make specific assumptions regarding the value of λ; so the relationship may either be stable ($|\lambda| < 1$) or unstable ($|\lambda| \geq 1$). Nor do we make particular assumptions yet on the series of vectors $\{x_t; t = \ldots, 0, 1, \ldots, T\}$ except that they are strictly exogenous, i.e. completely independent of the disturbances, so $Ex_t u_i = 0$, $\forall t, i \leq T$.

We assume that observations on x_t and y_{t-1} are available for $t \geq 1$ only. The last T equations of (2.1) are collected in

$$y = \lambda y_{-1} + X\beta + u \tag{2.2}$$

where y, y_{-1}, and u are stochastic $T \times 1$ vectors with $u \sim N(0, \sigma^2 I_T)$, whereas X is a T x K matrix of regressors. These regressors are treated as fixed, hence we condition on the realizations of the possibly stochastic x_t. The elements of x_t may be either stationary or non-stationary, i.e. realizations of a pure deterministic process or of some stochastic stationary, trend-stationary or difference-stationary process. So among the columns of X we may find for instance: a constant (column of unit elements), a linear trend, polynomial trends, sets of dummy variables etc., and also (lags of) other (non-artificial) variables, possibly generated by (non) zero mean ARIMA processes.

The standard (asymptotic) test procedure for hypotheses on λ is based on the ordinary least-squares (OLS) estimator $\hat{\lambda}$. Because of the normality assumption this OLS estimator is equivalent to the maximum-likelihood estimator (conditional upon y_0). We have

$$\hat{\lambda} = \frac{y_{-1}'M(X)y}{y_{-1}'M(X)y_{-1}} \tag{2.3}$$

where $M(X) = I_T - X(X'X)^{-1}X'$ is the well-known projection matrix onto the null-space of X. Substituting (2.2) in (2.3) yields

$$\hat{\lambda} - \lambda = \frac{y_{-1}'M(X)u}{y_{-1}'M(X)y_{-1}} \tag{2.4}$$

Because of the stochastic nature of y_{-1} the estimator $\hat{\lambda}$ is biased in general. Its finite sample moments and distribution cannot be established in a straightforward way. This hampers the development of exact inference techniques on the value of λ (and so for β).

Under various sets of more specific regularity conditions than those we have adopted here a nondegenerate limiting distribution of $g(X)(\hat{\lambda} - \lambda)$ can be derived, where $g(X)$ is some appropriate scaling factor. This limiting distribution can then be employed for the construction of an asymptotic test for H_0: $\lambda = \lambda_0$. Often such a test is based on the 'studentized' statistic

$$\frac{\hat{\lambda} - \lambda_0}{[\hat{\sigma}^2(y_{-1}'M(X)y_{-1})^{-1}]^{1/2}} = \frac{(T - K - 1)y_{-1}'M(X)u}{[(y_{-1}'M(X)y_{-1})(u'M(y_{-1}:X)u)]^{1/2}}$$

for $\lambda = \lambda_0$ (2.5)

where $M(y_{-1}:X)$ projects onto the null-space of $[y_{-1}:X]$ and $y'M(y_{-1}:X)y/(T - K - 1)$ is the least-squares estimator of σ^2 with degrees of freedom correction. Whether or not (2.5) is asymptotically standard normal under the null depends on the specific regularity assumptions made, notably on the value of λ_0 and on the asymptotic behaviour of the series $\{x_t\}$.

For the very specific case $K = 0$ (no exogenous regressors, so $M(X) = I$) the distribution of statistics such as (2.3) and (2.5) are examined extensively in [E v a n s and S a v i n (1981)]. For the case $K = 1$ where the one and only regressor is a constant term the distribution of the statistics (2.3) and (2.5) are investigated in [E v a n s and S a v i n (1984)] and in [N a n k e r v i s and S a v i n (1985)] respectively. In these articles it is shown that - in models with an intercept and y_0 fixed - tests based on (2.4) or (2.5) are non-similar in general, and that despite comforting asymptotic results their null-distribution in finite samples can be very unlike the normal or Student's t. In [N a n k e r v i s and S a v i n (1987)] the finite sample distribution of (2.5) is estimated by Monte Carlo methods for $K = 2$, where model (2.1) contains a constant term and one explanatory variable which is either stable AR(1), or a random walk, or a linear time trend. Again the accuracy of the relevant asymptotic approximations is found to be extremely dependent on nuisance characteristics of the data generating process.

In the studies mentioned above various values of λ_0 are investigated, including the case $\lambda_0 = 1$. The research into tests for this particular value (unit root tests) has a somewhat longer history; it goes back to [F u l l e r (1976)]. For particular simple X matrices and for some specific values of particular parameters he derives the asymptotic distributions of (2.3) and (2.5) and obtains percentiles of their empirical distribution in finite samples from Monte Carlo simulation; see also the review in [D i c k e y et al. (1986)] and the concise summarizing table in [H a l d r u p and H y l l e b e r g (1989)]. Exact similar tests are available for the case K = 0 with $y_0 = 0$ or y_0 random (there are no nuisance parameters). The nonsimilarity of the straightforward DF test in the K = 1 model and the poorness of its asymptotic approximation are illustrated patently in [H y l l e b e r g and M i z o n (1989)]. However, when the DF test is applied by calculating (2.3) or (2.5) from the (overparametrized) model with K = 2 (constant plus trend) a similar unit root test is obtained for the K = 1 model. Exact unit root tests are also given in [B h a r g a v a (1986)] for the model with y_0 random and K = 0 or K = 1 (constant drift); these tests are locally most powerful invariant against one-sided alternatives.

In more general regression models attention has been paid in the literature to a possible unit root in the disturbance process, for instance by [B h a r g a v a (1986)], but to the best of our knowledge no exact similar test procedure for a unit root in the dependent variable for models with K > 1 are available yet. Nor are there exact similar tests for arbitrary (non unit) values of the autoregressive coefficient λ in the geometric lag model (2.1). It is evident that practicable tools for exact statistical inference (hypothesis tests and confidence regions) in finite samples for such models would be extremely useful.

In what follows we shall not consider the (important) case with higher order dynamics (augmented DF-type tests) nor shall we discuss the approach of a nonparametric nature, see [P h i l l i p s (1987)] and [P h i l l i p s and P e r r o n (1988)], where the degree is examined to which (modifications of) simple first-

-order techniques are vindicated approximately and asymptotically in models with possibly higher order autoregressive terms or general ARMA or nonnormal disturbances. Also the assessment of the robustness and of the power of our exact test procedure is deferred to a future paper.

3. TESTING H_0: $\lambda = 0$

By applying OLS to a slightly adapted specification of the model we obtain an estimator for λ which has a distribution not depending on any unknown parameters if $\lambda = 0$; hence, an exact test procedure for the specific case H_0: $\lambda = 0$ follows straightforwardly. Consider the extended regression

$$y_t = \lambda y_{t-1} + x_t'\beta + x_{t-1}'\beta_* + u_t \qquad (3.1)$$

where the lags of all the exogenous regressors have been added to the specification. The coefficients of (3.1) can be estimated by OLS in a straightforward way only if the first sample observation is set aside (assuming x_0 is not available) and if the extra regressors do not induce extreme multicollinearity.

A feasible extended regression model conforming to (3.1) is

$$\tilde{y} = \lambda\tilde{y}_{-1} + X^*\beta^* + \tilde{u} \qquad (3.2)$$

where $\tilde{y}$, $\tilde{y}_{-1}$ and $\tilde{u}$ are (T-1) x 1 vectors which lack the first element of y, y_{-1} and u respectively, and where X^* is a (T - 1) x x (K + L) full-column rank matrix of regressors with (K + L) x 1 coefficient vector β^*, whereas $0 \leqslant L \leqslant K$ since the matrics X^* is such that its columns span the K + L dimensional subspace in R^{T-1} which is spanned by the following two sets of K columns: the first set consists of the X matrix upon deletion of its first row x_1', which can be denoted by $\tilde{X} = [x_2, \ldots, x_T]$, and the second set is composed of the columns of the matrix $\tilde{X}_{-1} = [x_1, \ldots, x_{T-1}]'$. So, we assume that rank $(\tilde{X}) = K$, and if $\tilde{X}$ contains a column of unit elements, or a set of seasonal dummies, or a linear trend, or if it includes in addition to the j-th column $X_{.j}$ also the lag of $X_{.j}$, then these appear only once in the matrix X^*.

The OLS estimator of λ in model (3.2), which is not fully efficient because of the inclusion of L redundant regressors and the omission of one sample observation, will be denoted by $\tilde{\lambda}^*$ and is found to be

$$\tilde{\lambda}^* = \frac{\tilde{y}'_{-1}M(X^*)\tilde{y}}{\tilde{y}'_{-1}M(X^*)\tilde{y}_{-1}} = \lambda + \frac{\tilde{y}'_{-1}M(X^*)\tilde{u}}{\tilde{y}'_{-1}M(X^*)\tilde{y}_{-1}}, \tag{3.3}$$

where $M(X^*) = I - X^*(X^{*\prime}X^*)^{-1}X^{*\prime}$ is a $(T - 1) \times (T - 1)$ matrix. Since the first $T - 1$ rows of (2.2) state that $\tilde{y}_{-1} = \lambda\tilde{y}_{-2} + \tilde{X}_{-1}\beta + \tilde{u}_{-1}$, we have

$$M(X^*)\tilde{y}_{-1} = \lambda M(X^*)\tilde{y}_{-2} + M(X^*)\tilde{u}_{-1} \tag{3.4}$$

where we used $M(X^*)\tilde{X}_{-1} = 0$. So, under H_0: $\lambda = 0$ we find that $M(X^*)\tilde{y}_{-1} = M(X^*)\tilde{u}_{-1}$ and therefore we obtain for (3.3)

$$\tilde{\lambda}^* = \frac{\tilde{u}'_{-1}M(X^*)\tilde{u}}{\tilde{u}'_{-1}M(X^*)\tilde{u}_{-1}}, \text{ under } \lambda = 0 \tag{3.5}$$

Now let

$$v = \frac{1}{\sigma} u \sim N(0, I_T),$$

$$B_0 = \begin{bmatrix} 1 & 0 & & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & & & & \\ & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot & \\ 0 & & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} 0 & 1 & 0 & & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & 0 & & & & \\ & & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & & \cdot & \cdot & \cdot \\ \cdot & & & & & & \cdot & \cdot & 0 \\ 0 & & & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

where B_0 and B_1 are $(T - 1) \times T$ matrices. Then $\tilde{u} = B_1 u = \sigma B_1 v$ and $\tilde{u}_{-1} = B_0 u = \sigma B_0 v$, and thus we can rewrite (3.5) as

$$\tilde{\lambda}^* = \frac{v'[B_0' M(X^*)B_1]v}{v'[B_0' M(X^*)B_0]v}, \text{ under } \lambda = 0 \tag{3.6}$$

This is a ratio of two quadratic forms in the $T \times 1$ standard normal vector v, and it does not depend on σ and β. Hence an exact similar test procedure for H_0: $\lambda = 0$ can simply be based directly on the OLS estimator $\tilde{\lambda}^*$ of λ in (3.2). Critical values can be calculated by using a method such as [I m h o f's (1961)].

The distribution of $\tilde{\lambda}^*$ cannot be tabulated once and for all, since it is determined by the space spanned by the regressor matrix X^*. In section 7 we present tables of percentiles of this test statistic for some very special X matrices, viz. those where $X^* = X$ since X only contains a constant and/or a linear trend, or X is void. If X is more general the $\tilde{\lambda}^*$ test procedure is best applied as follows. Instead of calculating the exact critical value for some particular significance level α less computational effort is required if for a given model and accompanying $\tilde{\lambda}^*$ estimate, say $\bar{\lambda}^*$, only the Prob-value is calculated. This value equals

$$P\left\{\frac{v'[B_0'M(X^*)B_1]v}{v'[B_0'M(X^*)B_0]v} > \bar{\lambda}^*\right\} = P\left\{v'B_0'M(X^*)\left(B_1 - \bar{\lambda}^*B_0\right)v > 0\right\} \quad (3.7)$$

and it is fully (and exactly) informative on whether or not $H_0: \lambda = 0$ should be rejected against either $\lambda < 0$, $\lambda > 0$, or $\lambda \neq 0$ at any chosen significance level.

Note that regression (3.2) and the estimator of the coefficient of y_{-1} conform to those which [D u r b i n (1960)] suggested to examine in the static regression model with first-order serial correlation. Our manipulations show that in this way exact and similar inference on the serial correlation coefficient can be obtained easily; this was noticed first by [P h i l l i p s and M c C a b e (1988), p. 42].

4. TESTING H_0: $\lambda = \lambda_0$

For the more general case, where λ is tested against an arbitrary value λ_0, the procedure can be adapted in the following way. Consider the Cochrane-Orcutt type transformation of model (2.1), viz.

$$(y_t - \lambda_0 y_{t-1}) = \lambda(y_{t-1} - \lambda_0 y_{t-2}) + [x_t' - \lambda_0 x_{t-1}']\beta + (u_t - \lambda_0 u_{t-1}) \quad (4.1)$$

where λ_0 is a given real value, which may differ from zero. Note that, although the Cochrane-Orcutt transformation was originally meant to remove correlation of the disturbances, here it produces moving-average errors (like the Koyck transformation usually does). Transformation (4.1) of model (2.2) is performed by the $(T-1) \times T$ matrix

$$D = \begin{bmatrix} -\lambda_0 & 1 & 0 & . & . & . & 0 \\ 0 & -\lambda_0 & 1 & 0 & & & . \\ 0 & 0 & . & . & & & . \\ . & & . & . & . & & . \\ .. & & & & . & . & 0 \\ 0 & . & . & . & 0 & -\lambda_0 & 1 \end{bmatrix} = B_1 - \lambda_0 B_0 \tag{4.2}$$

We adapt our notation a bit, and from now on we have

$$\tilde{y} = Dy, \quad \tilde{y}_{-1} = Dy_{-1} \quad \text{and} \quad \tilde{u} = Du \tag{4.3}$$

Note that this notation is in line with the foregoing section where we had $\lambda_0 = 0$ and hence $D = B_1$.

Like we did in (3.2) we now examine estimating λ in model

$$\tilde{y} = \lambda\tilde{y}_{-1} + X^*\beta^* + \tilde{u} \tag{4.4}$$

where X^* is still the $(T-1) \times (K+L)$ full column rank matrix containing the K columns B_1X and also $L \leqslant K$ columns of B_0X. Note that (4.1) can be obtained by imposing L linear coefficient restrictions on β^* of (4.4).

Despite the MA(1) structure of the disturbances of (4.4), we first consider the OLS estimator of λ and denote it by $\hat{\lambda}^*$. We obtain

$$\hat{\lambda}^* = \frac{\tilde{y}'_{-1}M(X^*)\tilde{y}}{\tilde{y}'_{-1}M(X^*)\tilde{y}_{-1}} = \lambda + \frac{\tilde{y}'_{-1}M(X^*)\tilde{u}}{\tilde{y}'_{-1}M(X^*)\tilde{y}_{-1}} \tag{4.5}$$

Since $B_1y_{-1} = B_0y$ we have

$$\tilde{y}_{-1} = Dy_{-1} = [B_1 - \lambda_0 B_0]y_{-1} = B_0[y - \lambda_0 y_{-1}]$$

and upon using (2.2) we thus find that

$$\tilde{y}_{-1} = B_0(X\beta + u), \quad \text{under} \quad H_0\colon \lambda = \lambda_0 \tag{4.6}$$

and, since B_0X lies in the column space of X^*, we obtain

$$M(X^*)\tilde{y}_{-1} = M(X^*)B_0u, \quad \text{under} \quad \lambda = \lambda_0 \tag{4.7}$$

Substitution of (4.7), and of $\tilde{u} = B_1u - \lambda_0 B_0 u$ and $u = \sigma v$ in (4.5) yields

$$\hat{\lambda}^* = \lambda_0 + \frac{u'B_0'M(X^*)\tilde{u}}{u'B_0'M(X^*)B_0u} = \frac{v'B_0'M(X^*)B_1v}{v'B_0'M(X^*)B_0v}, \quad \text{under} \quad \lambda = \lambda_0 \tag{4.8}$$

Result (4.8) is completely in line with (3.6). So, whatever the chosen value of λ_0 is (zero or not), the OLS estimate of the coefficient of $\tilde{y}_{-1}$ in the auxiliary regression (4.4) has under $\lambda = \lambda_0$ exactly the same distribution. This distribution does not involve any nuisance parameters and thus $\hat{\lambda}^*$ can be used as an exact and similar test statistic for $H_0: \lambda = \lambda_0$.

However, since the MA(1) structure of the disturbances in (4.2) leads to joint-dependence with the regressor $\tilde{y}_{-1}$, the estimator $\hat{\lambda}^*$ will - under standard regularity conditions - be inconsistent. Although the test for $H_0: \lambda = \lambda_0$ based on $\hat{\lambda}^*$ is exact this inconsistency, which arises in cases where $\lambda_0 \neq 0$, will adversely affect the power of the test. One can derive that for $\lambda = \lambda_0$ we have plim $\hat{\lambda}^* = 0$, hence $\hat{\lambda}^*$ is consistent for λ only in case $\lambda = \lambda_0 = 0$; for $\lambda > \lambda_0 > 0$ we find plim $\hat{\lambda}^* > 0$, but for $\lambda_0 > \lambda > 0$ the value of plim $\hat{\lambda}^*$ can be either positive or negative. Hence, if the unit root hypothesis $\lambda_0 = 1$ is tested by $\hat{\lambda}^*$ then the power of this test may be quite poor if in fact $\lambda < 1$; most probably this test is biased (power may be smaller than significance level). Therefore it seems worthwile to examine a test procedure where the non-scalar covariance matrix of the disturbance $\tilde{u}$ is properly taken into account.

Since λ_0 is known the coefficients of (4.4) can be estimated consistently by generalized least-squares (GLS). We have

$$E\tilde{u}\tilde{u}' = \sigma^2 V = \sigma^2 DD' \tag{4.9}$$

with V a known (T - 1) x (T - 1) matrix. If the (T - 1) x (T - 1) matrix P is such that $V^{-1} = P'P$ then the application of GLS to (4.4) is equivalent to OLS estimation of

$$P\tilde{y} = \lambda P\tilde{y}_{-1} + PX^*\beta^* + P\tilde{u} \tag{4.10}$$

As we shall see in due course it will not lead to confusion if we denote the OLS estimator of λ in (4.10) and the equivalent GLS estimator of λ in (4.4) by $\tilde{\lambda}^*$, like we did in section 3 for the special case $\lambda_0 = 0$. The estimator $\tilde{\lambda}^*$ of λ is now

$$\tilde{\lambda}^* = \frac{\tilde{y}_{-1}'P'M(PX^*)P\tilde{y}}{\tilde{y}_{-1}'P'M(PX^*)P\tilde{y}_{-1}} = \lambda + \frac{\tilde{y}_{-1}'P'M(PX^*)P\tilde{u}}{\tilde{y}_{-1}'P'M(PX^*)P\tilde{y}_{-1}} \tag{4.11}$$

From (4.6) it follows that

$$M(PX^*)P\tilde{y}_{-1} = M(PX^*)PB_0u, \quad \text{under} \quad \lambda = \lambda_0. \tag{4.12}$$

Therefore we obtain

$$\tilde{\lambda}^* = \lambda_0 + \frac{u'B_0'P'M(PX^*)P[B_1 - \lambda_0 B_0]u}{u'B_0'P'M(PX^*)PB_0u} \tag{4.13}$$

$$= \frac{v'[B_0'P'M(PX^*)PB_1]v}{v'[B_0'P'M(PX^*)PB_0]v}, \quad \text{under} \quad \lambda = \lambda_0,$$

with

$$P' M(PX^*)P = V^{-1} - V^{-1}X^*(X^{*\prime}V^{-1}X^*)^{-1}X^{*\prime}V^{-1}$$

So the coefficient estimator $\tilde{\lambda}^*$ at the same time constitutes an exact similar test statistic for H_0: $\lambda = \lambda_0$ in (2.2). This test procedure is also invariant with respect to linear transformations of the exogenous regressors; it is obvious that the distribution of $\tilde{\lambda}^*$ is not determined by X as such, but only by the subspace spanned by the columns of X. If we chose $\lambda_0 = 0$ the test based on $\tilde{\lambda}^*$ simplifies to the procedure presented in section 3.

It can be proved that under standard regularity assumptions the estimator $\tilde{\lambda}^*$ is consistent for λ, not only when $\lambda = \lambda_0$ but for any λ_0. Obviously the GLS estimator $\tilde{\lambda}^*$ is not most efficient; afterall we still have a lagged dependent variable in the regression, and in addition we lost the first observation and have L redundant coefficients in (4.4). Hopefully as a test it has power characteristics which are reasonable in comparison with other - usually approximate or nonsimilar - test procedures.

5. AN ALTERNATIVE DERIVATION AND EXPRESSION FOR THE TEST STATISTIC

We shall show now that there is an alternative and simpler way to derive and to express a similar test statistic for the hypothesis H_0: $\lambda = \lambda_0$. It is found from an augmented regression equation where the Cochrane-Orcutt transformation and GLS estimation are not required. We shall also prove that this test statistic can be constructed such that it is equivalent with $\tilde{\lambda}^*$.

Consider the augmented regression

$$y = \lambda y_{-1} + X\beta + Z\gamma + u, \tag{5.1}$$

where Z is a T x H matrix of redundant regressors whereas $[y_{-1}:X:Z]$ has full column rank. We denote the OLS estimator of λ by $\hat{\lambda}_z$ and

$$\hat{\lambda}_z = \frac{y_{-1}'M(X:Z)y}{y_{-1}'M(X:Z)y_{-1}} = \lambda + \frac{y_{-1}'M(X:Z)u}{y_{-1}'M(X:Z)y_{-1}} \tag{5.2}$$

The matrix Z is chosen such that it renders $\hat{\lambda}_z$ invariant with respect to nuisance parameters. This is achieved as follows.

Upon recursive substitution of (2.1) we obtain

$$\begin{aligned} y_t &= \lambda(\lambda y_{t-2} + x_{t-1}'\beta + u_{t-1}) + x_t'\beta + u_t \\ &= \lambda^2(\ldots) + \lambda x_{t-1}'\beta + x_t'\beta + \lambda u_{t-1} + u_t = \ldots \end{aligned}$$

and this yields

$$y_{-1} = y_0\iota(\lambda) + C(\lambda)X\beta + C(\lambda)u \tag{5.3}$$

where

$$\iota(\lambda) = \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda^{T-1} \end{bmatrix} \quad \text{and } C(\lambda) = \begin{bmatrix} 0 & & . & . & . & 0 \\ 1 & 0 & & & & \\ \lambda & 1 & 0 & & & . \\ \lambda^2 & & . & . & & . \\ . & & & . & . & . \\ . & & & & . & . \\ \lambda^{T-2} & & . & . & \lambda & 1 & 0 \end{bmatrix}$$

So, if Z is such that the K+H columns of [X:Z] span the same subspace in R^T as is spanned by $[X:\iota(\lambda):C(\lambda)X]$ then we have

$$M(X:Z)y_{-1} = M(X:Z)C(\lambda)u \tag{5.4}$$

Therefore a similar test for $H_0: \lambda = \lambda_0$ is obtained by collecting the H extra regressors from the K+1 columns of $[\iota(\lambda_0):C(\lambda_0)X]$. This leads to

$$\hat{\lambda}_z = \lambda_0 + \frac{u'C(\lambda_0)'M(X:Z)u}{u'C(\lambda_0)'M(X:Z)C(\lambda_0)u}, \quad \text{under } \lambda = \lambda_0 \tag{5.5}$$

Note that this test statistic is similar with respect to β and σ, but also with respect to y_0, and therefore assumptions on whether y_0 is fixed or random are not required in order to assess the distribution of $\hat{\lambda}_z$. The critical values of (5.5) can again be found by Imhof's method.

We shall prove now that statistic (5.5), which results from applying OLS to a model with $K + H + 1$ regressors of T observations, is equivalent with the $\tilde{\lambda}^*$ estimator (4.13), which results from applying GLS to a model with $K + L + 1$ regressors of $T - 1$ observations. We also prove that $H = L + 1$.

Below we shall (in line with the notation $D = B_1 - \lambda_0 B_0$) simply write ι and C for $\iota(\lambda_0)$ and $C(\lambda_0)$ respectively. We recall that $\tilde{\lambda}^*$ is obtained by applying GLS to (4.4), or to

$$Dy = \lambda Dy_{-1} + X^*\beta^* + Du \tag{5.6}$$

and that X^* spans the same $K + L$ dimensional subspace as is spanned by $[B_1X:B_0X]$. The latter space is also spanned by $[(B_1-\lambda_0B_0)X::B_0X]$, which conforms to $[DX:B_0X] = [DX:DCX] = D[X:CX]$, since $DC = B_0$ as is easily verified. Now let Q^* be a $T \times (K + L)$ full-column rank matrix which is spanned by $[X:CX]$, whereas DQ^* spans the same subspace as X^*, then $\tilde{\lambda}^*$ can also be obtained by applying GLS to

$$Dy = \lambda Dy_{-1} + DQ^*\beta^{**} + Du \tag{5.7}$$

This yields

$$\tilde{\lambda}^* = \frac{y'_{-1}D'[V^{-1} - V^{-1}DQ^*(Q^{*\prime}D'V^{-1}DQ^*)^{-1}Q^{*\prime}D'V^{-1}]Dy}{y'_{-1}D'[V^{-1} - V^{-1}DQ^*(Q^{*\prime}D'V^{-1}DQ^*)^{-1}Q^{*\prime}D'V^{-1}]Dy_{-1}} \tag{5.8}$$

where $V^{-1} = (DD')^{-1}$. Note that $D'V^{-1}D = D'(DD')^{-1}D$ is a peculiar matrix; it projects onto the $(T - 1)$ dimensional subspace of R^T spanned by the columns of D'. Since $M(\iota)$ projects onto the $(T-1)$ dimensional null-space of ι, whereas $D\iota = 0$, we have

$$D'(DD')^{-1}D = M(\iota) \tag{5.9}$$

Hence, we may write the matrix of the quadratic forms in the numerator and in the denominator of (5.8) as

$$M(\iota)[I - Q^*(Q^{*\prime}M(\iota)Q^*)^{-1}Q^{*\prime}]M(\iota) = M(\iota)M\big(M(\iota)Q^*\big) \tag{5.10}$$

Since $D\iota = 0$ and DQ^* has full column rank the $T \times (K + L + 1)$ matrix $[\iota:Q^*]$ has full column rank too, whereas $H = L + 1$. From straightforward application of the inversion rules for partitioned matrices one can derive that

$$\begin{aligned} M(\iota:Q^*) &= I - [\iota:Q^*]\big([\iota:Q^*]'[\iota:Q^*]^{-1}[\iota:Q^*]\big) \\ &= M\big(M(\iota)Q^*\big) + M(\iota) - I \end{aligned} \tag{5.11}$$

and hence

$$M(\iota)M(M(\iota)Q^*) = M(\iota)M(\iota:Q^*) = M(\iota:Q^*) \tag{5.12}$$

So (5.8) can be written as

$$\tilde{\lambda}^* = \frac{y_{-1}'M(\iota:Q^*)y}{y_{-1}'M(\iota:Q^*)y_{-1}} \tag{5.13}$$

This conforms to $\hat{\lambda}_z$ of (5.5) which also results from the regression of y on y_{-1} and regressors which are spanned by $[\iota:Q^*]$ or $[\iota:X:CX]$.

Note that for $\lambda_0 = 0$ the vector ι constitutes the dummy regressor which annihilates the first observation, whereas CX produces X lagged one period. The equivalence with the procedure of Section 3 is obvious. For $\lambda_0 \neq 0$ the correspondence with the procedure of Section 4 is not all that straightforward, but application of the test via the augmented OLS regression is much more attractive indeed.

6. TESTING FOR UNIT ROOTS

We recall that the test procedure based on $\tilde{\lambda}^*$ or $\hat{\lambda}_z$ is exact, invariant with respect to y_0, and similar with respect to β and σ, provided that the regressors are exogenous and the disturbances in (2.1) are uncorrelated and identically normally distributed. No restrictive assumptions concerning the actual value of λ and the behaviour of (x_t) had to be made, and therefore our procedure is also suitable for unit root testing. Unit root tests are important in time-series analysis in general, but have recently become very popular in econometrics too. Following [N e l s o n and P l o s s e r (1982)] many papers have examined whether particular economic series can be described as stationary changes around a deterministic trend, or as random walks with drift, i.e. as dynamic stochastic processes with a unit root. In the context of cointegration analysis, see [E n g l e and G r a n g e r (1987)], it is also very important to test for unit roots in autoregressive processes describing either individual time-series or residuals from cointegrating regressions.

Many recent studies on unit root tests - see section 2 - deal with test procedures for H_0: $\lambda = 1$ in our model (2.2) for some particular X matrices and under particular assumptions on the starting value y_0 (viz. being zero, arbitrary but fixed, or stochastic) and they often highlight the fact that serious small sample problems may arise. This will not be the case for our test procedure. If the assumptions underlying (2.2) are correct then it provides an exact test which is invariant with respect to y_0 and also similar and therefore easy to carry out in practice.

Some popular variants of the unit root tests suggested by Dickey and Fuller are calculated from regressions with very specific forms of the x_t vector (viz. x_t is empty, or contains a constant and possibly a linear trend) and they also may lead to similar tests; so it seems worthwhile to make a comparison with our procedure for these particular situations. To that end we consider the following three different specific forms of model (2.1), viz.

$$y_t = \lambda y_{t-1} + u_t \qquad t = 1, \ldots, T \qquad (6.1a)$$

$$y_t = \lambda y_{t-1} + \beta_1 + u_t \qquad t = 1, \ldots, T \qquad (6.1b)$$

$$y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + u_t \qquad t = 1, \ldots, T \qquad (6.1c)$$

Upon using reparametrization $\delta = \lambda - 1$ these regressions can be rewritten as respectively. The test for a unit root $\lambda = 1$ in (6.1)

$$\Delta y_t = \delta y_{t-1} + u_t \qquad t = 1, \ldots, T \qquad (6.2a)$$

$$\Delta y_t = \delta y_{t-1} + \beta_1 + u_t \qquad t = 1, \ldots, T \qquad (6.2b)$$

$$\Delta y_t = \delta y_{t-1} + \beta_1 + \beta_2 t + u_t \qquad t = 1, \ldots, T \qquad (6.2c)$$

conforms now to testing $\delta = 0$ in (6.2). We focus on two types of tests examined by Dickey and Fuller, viz. the test based directly on the least-squares estimate of λ or δ (which differ by one and are distributed according to (2.4)), and the test based on the studentized test statistic of either λ or δ (note that these two are equivalent) given in (2.5). [F u l l e r (1976)] has considered only two particular situations concerning the DGP (data generating process), viz. (A) the situation where the data are generated according to (6.1a) with $y_0 = 0$, and (B) the situation where the DGP is represented by (6.1b) with $y_0 = 0$ and β_1 arbitrary. Apart from limiting distributions he presents critical values (percentiles for α, $1 - \alpha = 0.01$, 0.025, 0.05, 0.10 at

T = 25, 50, 100, 250, 500) which have been found from Monte Carlo simulations. His Table 8.5.1. see [F u l l e r (1976), p. 371] refers to the least-squares estimate of δ (multiplied by T), and his Table 8.5.2 (p. 373) refers to the t-ratio of δ.

The upper parts of Fuller's tables contain critical values for the unit root test for DGP (A) whereas the test statistics are calculated from the consonant regression equation (6.2a). Since there are no nuisance parameters, it is quite obvious that these particular DF tests are similar. See also [E v a n s and S a v i n (1981, p. 762)], who indicate various methods to calculate numerically the exact critical values for the test statistic based on the estimate of λ.

The middle parts of both Fuller's tables refer to the case where the DGP is again (A), but the test statistics are calculated from the extended regression (6.2b). Hence, under the null both y_{t-1} and the constant are redundant. We shall show now that this conforms exactly to our general test procedure.

From (5.3) we find for $\lambda = 1$

$$\iota = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 0 & & & \\ 1 & 1 & 0 & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \\ 1 & \cdot & \cdot & 1 & 0 \end{bmatrix}, \; \text{and } \tau = C\iota \begin{bmatrix} 0 \\ 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ T-1 \end{bmatrix} \qquad (6.3)$$

where ι and τ are T x 1 vectors and C is T x T (lagged) cumulating matrix, and

$$y_{-1} = y_0\iota + Cu \quad \text{(for case (6.1a) with } \lambda = 1) \qquad (6.4)$$

Hence, if δ is estimated using auxiliary regression (6.2b), we find that

$$\hat{\delta} = \delta + \frac{y_{-1}'M(\iota)u}{y_{-1}'M(\iota)y_{-1}} = \frac{u'C'M(\iota)u}{u'C'M(\iota)Cu}, \quad \text{for } \lambda = 1 \qquad (6.5)$$

Since σ cancels, (6.5) does not depend on any nuisance parameters, nor on y_0, and so we see that the middle part of Fuller's Table 8.5.1 provides estimates of the exact critical values for the similar test for the unit root hypothesis in model (6.1a) with arbitrary y_0. The same holds for the t-ratio test since the ratio of $\hat{\delta}$ and its estimated standard deviation does not depend on nuisance parameters either. This is seen from (2.5) upon noting that due to (6.4) we have $u'M(y_{-1},\iota)u = u'M(\frac{1}{\sigma}Cu,\iota)u$, and so σ cancels again.

If the DGP is given by (6.1b) with $\lambda = 1$ and y_0 arbitrary we have

$$y_t = y_0 + \beta_1 t + \Sigma_{i=1}^{t} u_i \qquad t = 1, \ldots, T \qquad (6.6)$$

so, instead of (6.4) we then find

$$y_{-1} = y_0\iota + \beta_1\tau + Cu \qquad \text{(for case (6.1b) with } \lambda = 1) \qquad (6.7)$$

where τ is defined in (6.3). If δ is estimated now from auxiliary regression (6.2c), we then obtain

$$\hat{\delta} = \frac{y_{-1}'M(\iota:\tau)u}{y_{-1}'M(\iota:\tau)y_{-1}} = \frac{u'C'M(\iota:\tau)u}{u'C'M(\iota:\tau)Cu}, \quad \text{for } \lambda = 1 \qquad (6.8)$$

The lower parts of both Fuller's tables contain the percentiles for the situation where the DGP conforms to situation (B), whereas the test statistics are calculated from regression (6.2c); so both y_{t-1} and t are redundant. Since $\hat{\delta}$ of (6.8) and its t-ratio are distributed independently from y_0, β_1 and σ, they yield similar tests. Note that due to the independence with respect to y_0 the middle and lower parts of the Fuller tables have not been affected by taking $y_0 = 0$ in the Monte Carlo experiments.

[H y l l e b e r g and M i z o n (1989)] examine the situation where the DGP is represented by (B) and the unit root test is based on the estimate of δ in the parsimoneous specification (6.2b). They present critical values (found through Monte Carlo simulation where they fix $\sigma = 1$ and $y_0 = 0$) which depend on the value of β_1 illustrating the nonsimilarity of this test. It is easily seen that here again the results are not affected by taking $y_0 = 0$, since adoption of DGP (6.6) with arbitrary y_0 leads to

$$\hat{\delta} = \delta + \frac{y_{-1}'M(\iota)u}{y_{-1}'M(\iota)y_{-1}} = \frac{(u'C' + \beta_1\tau')M(\iota)u}{(u'C' + \beta_1\tau')M(\iota)(Cu + \beta_1\tau)}, \quad \text{for } \lambda = 1 \qquad (6.9)$$

This estimate of δ obtained from (6.2b) is not determined by the nature of y_0, but it does depend on the ratio β_1/σ.

Note that the simulated critical values for the tests based on estimates of δ given in Table 8.5.1 in [F u l l e r (1976)] and in Tab. 1 in [H y l l e b e r g and M i z o n (1989)] could also be calculated directly by numerical methods for the evaluation of the cumulative distribution function of a ratio of quadratic forms in normal variables. The relevant ratios are given by $u'C'u/u'C'Cu$ and by the right-hand sides of (6.5), (6.8) and (6.9) respectively. In the next section we present various of these tables. From (2.5) it is obvious that the finite sample distribution of the t ratio for λ or δ is much more complex.

Summarizing we note that our similar test for a unit root boils down to running an auxiliary regression where y is regressed on y_{-1}, and on regressors which span the space spanned by: (i) the original regressors X; (ii) the constant (possibly already contained in X), which induces invariance with respect to y_0; and (iii) the 'lagged cumulated regressors'. The latter involve a linear trend when X contains the constant, and they involve the augmentation of the regression by the trend squared if X contains the linear trend, etc.

In [D i c k e y, H a s z a and F u l l e r (1984)] unit root tests are developed for seasonal time series models for the special case where the lag polynomial for the dependent variable may be of order 2, 4 or 12, but is such that it only has one unknown coefficient which is tested for being unity. Obviously such tests can be inbedded in our general procedure and can be extended easily to the multiple dynamic seasonal regression model.

7. EMPLOYING THE TESTS IN PRACTICE

Although we made the calculation of the test statistic considerably more simple by indicating how it can be obtained from a properly extended auxiliary ordinary least-squares regression, the calculation of its critical values or of its Prob-values remains a bit cumbersome. To produce tables of critical values for once and for all is impossible, because these values depend on λ_0 and on the matrix X. Only for specific cases it is worthwhile to present tables. Since in general first-order autoregressive models are relevant in econometrics only when annual data are used (the dynamics of quarterly relationships is usually more complex) and since samples on annual data have a size well below 50 in general, we will produce critical values for T = 11, ..., 50 for H_0: $\lambda = 0$ and H_0: $\lambda = 1$ when X is either empty, or contains a constant, and possibly a linear trend. In our computations we used the algorithm given in Davies (1980). The tables in [F u l l e r (1976)] only concern the test for unit roots in data generating processes which have no deterministic trend. Moreover, these tables contain estimated critical values and they lack detail with respect to the sample size; they only cover T = 25, 50, 100, 250, 500, ∞. (In fact a sample described by Fuller as being of size T is actually a sample of size T - 1, see [E v a n s and S a v i n (1984), p. 1256]).

For the special case $\lambda_0 = 0$ our test procedure boils down to the regression of y on y_{-1} and the regressors X, X_{-1} and the dummy variable (1, 0, 0, ..., 0)'. The effect of the latter variable is that in fact the first observation is annihilated. If, apart from y_0, also x_0' is available (which is the case if X simply contains the constant and/or the linear trend), then the full set of T observations can be used in this regression.

The Tables 1A, 1B and 1C deal with the test of $\lambda = 0$, i.e. infinite roots or uncorrelatedness of the observations on the dependent variable in very simple models. In Table 1A the null hypothesis of normal zero-mean white-noise is tested against the alternative of a first-order (possibly non-stationary) autoregressive scheme. The test statistic (λ estimate) is symmetric around zero. In Table 1B a constant is allowed for, and in Table 1C a trend is added too. In the latter two tables the λ estimate is clearly biased, especially for small sample sizes.

The Tables 2A, 2B and 2C concern unit root tests. In Table 2A the random walk hypothesis is tested in the simple first-order autoregressive model without drift. The table corresponds to the middle part of Fuller's Table 8.5.1 where the critical values for $T(\hat{\lambda}_z - 1)$ are presented. From our table the strong negative bias in the $\hat{\lambda}_z$ estimate is apparent. In Table 2B a drift is included; these percentiles correspond with the lower part of

Table 1A

Percentiles for the test for H_0: $\lambda = 0$ in: $y_t = \lambda y_{t-1} + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	-0.486	-0.386	-0.207	0.000	0.207	0.386	0.486
12	-0.465	-0.369	-0.198	0.000	0.198	0.369	0.465
13	-0.447	-0.354	-0.190	0.000	0.190	0.354	0.447
14	-0.431	-0.341	-0.183	0.000	0.183	0.341	0.431
15	-0.417	-0.330	-0.176	0.000	0.176	0.329	0.417
16	-0.404	-0.319	-0.171	0.000	0.171	0.319	0.404
17	-0.392	-0.309	-0.165	0.000	0.165	0.309	0.392
18	-0.381	-0.301	-0.161	0.000	0.161	0.301	0.381
19	-0.371	-0.293	-0.156	0.000	0.156	0.293	0.371
20	-0.362	-0.285	-0.152	0.000	0.152	0.285	0.362
21	-0.353	-0.278	-0.148	0.000	0.148	0.278	0.353
22	-0.345	-0.272	-0.145	0.000	0.145	0.272	0.345
23	-0.338	-0.266	-0.142	0.000	0.142	0.266	0.338
24	-0.331	-0.260	-0.139	0.000	0.139	0.260	0.331
25	-0.324	-0.255	-0.136	0.000	0.136	0.255	0.324
26	-0.318	-0.250	-0.133	0.000	0.133	0.250	0.318
27	-0.312	-0.246	-0.131	0.000	0.131	0.246	0.312
28	-0.307	-0.241	-0.128	0.000	0.128	0.241	0.307
29	-0.302	-0.237	-0.126	0.000	0.126	0.237	0.302
30	-0.297	-0.233	-0.124	0.000	0.124	0.233	0.297
31	-0.292	-0.229	-0.122	0.000	0.122	0.229	0.292
32	-0.287	-0.226	-0.120	0.000	0.120	0.226	0.287
33	-0.283	-0.222	-0.118	0.000	0.118	0.222	0.283
34	-0.279	-0.219	-0.116	0.000	0.116	0.219	0.279
35	-0.275	-0.216	-0.115	0.000	0.115	0.216	0.275
36	-0.271	-0.213	-0.113	0.000	0.113	0.213	0.271
37	-0.268	-0.210	-0.111	0.000	0.111	0.210	0.268
38	-0.264	-0.207	-0.110	0.000	0.110	0.207	0.264
39	-0.261	-0.205	-0.108	0.000	0.108	0.205	0.261
40	-0.258	-0.202	-0.107	0.000	0.107	0.202	0.258
41	-0.254	-0.200	-0.106	0.000	0.106	0.200	0.254
42	-0.251	-0.197	-0.104	0.000	0.104	0.197	0.251
43	-0.249	-0.195	-0.103	0.000	0.103	0.195	0.249
44	-0.246	-0.193	-0.102	0.000	0.102	0.193	0.246
45	-0.243	-0.190	-0.101	0.000	0.101	0.190	0.243
46	-0.240	-0.188	-0.100	0.000	0.100	0.188	0.240
47	-0.238	-0.186	-0.099	0.000	0.099	0.186	0.238
48	-0.235	-0.184	-0.098	0.000	0.098	0.184	0.235
49	-0.233	-0.183	-0.097	0.000	0.097	0.183	0.233
50	-0.231	-0.181	-0.096	0.000	0.096	0.181	0.231

Table 1B

Percentiles for the test for H_0: $\lambda = 0$ in: $y_t = \lambda y_{t-1} + \beta_1 + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + \beta_1 + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	-0.572	-0.477	-0.306	-0.105	0.102	0.283	0.387
12	-0.544	-0.452	-0.288	-0.095	0.103	0.276	0.376
13	-0.520	-0.431	-0.273	-0.087	0.103	0.269	0.365
14	-0.499	-0.413	-0.259	-0.080	0.103	0.263	0.356
15	-0.480	-0.397	-0.248	-0.074	0.102	0.257	0.347
16	-0.464	-0.382	-0.237	-0.069	0.101	0.252	0.338
17	-0.448	-0.369	-0.228	-0.065	0.101	0.246	0.331
18	-0.434	-0.357	-0.219	-0.061	0.100	0.241	0.323
19	-0.422	-0.346	-0.212	-0.057	0.099	0.237	0.316
20	-0.410	-0.336	-0.205	-0.054	0.098	0.232	0.310
21	-0.399	-0.326	-0.199	-0.051	0.097	0.228	0.304
22	-0.389	-0.318	-0.193	-0.049	0.096	0.224	0.299
23	-0.380	-0.310	-0.187	-0.047	0.095	0.220	0.293
24	-0.371	-0.302	-0.182	-0.045	0.094	0.217	0.288
25	-0.363	-0.295	-0.178	-0.043	0.093	0.213	0.283
26	-0.356	-0.289	-0.173	-0.041	0.092	0.210	0.279
27	-0.349	-0.283	-0.169	-0.039	0.091	0.207	0.275
28	-0.342	-0.277	-0.165	-0.038	0.090	0.204	0.270
29	-0.335	-0.272	-0.162	-0.036	0.090	0.201	0.267
30	-0.329	-0.267	-0.158	-0.035	0.089	0.199	0.263
31	-0.324	-0.262	-0.155	-0.034	0.088	0.196	0.259
32	-0.318	-0.257	-0.152	-0.033	0.087	0.193	0.256
33	-0.313	-0.253	-0.149	-0.032	0.086	0.191	0.252
34	-0.308	-0.249	-0.147	-0.031	0.085	0.189	0.249
35	-0.303	-0.245	-0.144	-0.030	0.085	0.186	0.246
36	-0.299	-0.241	-0.142	-0.029	0.084	0.184	0.243
37	-0.294	-0.237	-0.139	-0.028	0.083	0.182	0.240
38	-0.290	-0.234	-0.137	-0.027	0.082	0.180	0.238
39	-0.286	-0.230	-0.135	-0.027	0.082	0.178	0.235
40	-0.282	-0.227	-0.133	-0.026	0.081	0.176	0.232
41	-0.278	-0.224	-0.131	-0.025	0.080	0.175	0.230
42	-0.275	-0.221	-0.129	-0.025	0.080	0.173	0.227
43	-0.271	-0.218	-0.127	-0.024	0.079	0.171	0.225
44	-0.268	-0.215	-0.125	-0.024	0.079	0.169	0.223
45	-0.265	-0.213	-0.124	-0.023	0.078	0.168	0.221
46	-0.262	-0.210	-0.122	-0.023	0.077	0.166	0.219
47	-0.259	-0.208	-0.121	-0.022	0.077	0.165	0.217
48	-0.256	-0.205	-0.119	-0.022	0.076	0.163	0.215
49	-0.253	-0.203	-0.118	-0.021	0.076	0.162	0.213
50	-0.251	-0.201	-0.116	-0.021	0.075	0.160	0.211

Table 1C

Percentiles for the test for H_0: $\lambda = 0$ in: $y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	-0.647	-0.558	-0.398	-0.205	-0.003	0.178	0.284
12	-0.615	-0.529	-0.373	-0.186	0.007	0.181	0.282
13	-0.586	-0.502	-0.351	-0.171	0.016	0.183	0.280
14	-0.562	-0.480	-0.332	-0.158	0.022	0.183	0.277
15	-0.539	-0.459	-0.316	-0.146	0.028	0.183	0.274
16	-0.519	-0.441	-0.301	-0.137	0.032	0.183	0.270
17	-0.501	-0.425	-0.288	-0.128	0.036	0.182	0.267
18	-0.485	-0.410	-0.276	-0.120	0.039	0.181	0.264
19	-0.470	-0.396	-0.266	-0.114	0.041	0.180	0.260
20	-0.456	-0.384	-0.256	-0.108	0.044	0.178	0.257
21	-0.444	-0.372	-0.247	-0.102	0.045	0.177	0.254
22	-0.432	-0.362	-0.239	-0.097	0.047	0.176	0.251
23	-0.421	-0.352	-0.232	-0.093	0.048	0.174	0.248
24	-0.411	-0.343	-0.225	-0.089	0.049	0.173	0.245
25	-0.401	-0.335	-0.219	-0.085	0.050	0.171	0.242
26	-0.392	-0.327	-0.213	-0.082	0.051	0.170	0.239
27	-0.384	-0.319	-0.207	-0.078	0.052	0.168	0.236
28	-0.376	-0.312	-0.202	-0.075	0.053	0.167	0.233
29	-0.368	-0.306	-0.197	-0.073	0.053	0.165	0.231
30	-0.361	-0.299	-0.193	-0.070	0.053	0.164	0.228
31	-0.354	-0.294	-0.188	-0.068	0.054	0.162	0.226
32	-0.348	-0.288	-0.184	-0.066	0.054	0.161	0.224
33	-0.342	-0.283	-0.180	-0.064	0.054	0.159	0.221
34	-0.336	-0.278	-0.177	-0.062	0.055	0.158	0.219
35	-0.331	-0.273	-0.173	-0.060	0.055	0.157	0.217
36	-0.325	-0.268	-0.170	-0.058	0.055	0.155	0.215
37	-0.320	-0.264	-0.167	-0.056	0.055	0.154	0.213
38	-0.316	-0.260	-0.164	-0.055	0.055	0.153	0.211
39	-0.311	-0.256	-0.161	-0.053	0.055	0.152	0.209
40	-0.306	-0.252	-0.158	-0.052	0.055	0.151	0.207
41	-0.302	-0.248	-0.156	-0.051	0.055	0.149	0.205
42	-0.298	-0.245	-0.153	-0.049	0.055	0.148	0.203
43	-0.294	-0.241	-0.151	-0.048	0.055	0.147	0.201
44	-0.290	-0.238	-0.149	-0.047	0.055	0.146	0.200
45	-0.287	-0.235	-0.146	-0.046	0.055	0.145	0.198
46	-0.283	-0.232	-0.144	-0.045	0.055	0.144	0.196
47	-0.280	-0.229	-0.142	-0.044	0.055	0.143	0.195
48	-0.276	-0.226	-0.140	-0.043	0.055	0.142	0.193
49	-0.273	-0.223	-0.138	-0.042	0.054	0.141	0.192
50	-0.270	-0.221	-0.137	-0.041	0.054	0.140	0.190

Table 2A

Percentiles for the test for H_0: $\lambda = 1$ in: $y_t = \lambda y_{t-1} + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + \beta_1 + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	0.078	0.224	0.448	0.656	0.822	0.949	1.019
12	0.134	0.274	0.486	0.681	0.835	0.951	1.015
13	0.184	0.318	0.520	0.703	0.846	0.953	1.012
14	0.228	0.357	0.549	0.722	0.856	0.955	1.009
15	0.268	0.391	0.575	0.738	0.864	0.957	1.007
16	0.304	0.423	0.598	0.753	0.872	0.959	1.006
17	0.337	0.451	0.618	0.767	0.879	0.961	1.004
18	0.367	0.476	0.637	0.778	0.885	0.962	1.003
19	0.394	0.499	0.654	0.789	0.891	0.964	1.003
20	0.419	0.521	0.669	0.799	0.896	0.965	1.002
21	0.442	0.540	0.683	0.808	0.900	0.967	1.001
22	0.463	0.558	0.696	0.816	0.905	0.968	1.001
23	0.482	0.575	0.708	0.823	0.908	0.969	1.001
24	0.501	0.590	0.719	0.830	0.912	0.970	1.000
25	0.518	0.605	0.730	0.837	0.915	0.971	1.000
26	0.534	0.618	0.739	0.843	0.918	0.972	1.000
27	0.548	0.631	0.748	0.848	0.921	0.973	1.000
28	0.562	0.642	0.756	0.853	0.924	0.974	0.999
29	0.576	0.653	0.764	0.858	0.926	0.974	0.999
30	0.588	0.664	0.771	0.862	0.929	0.975	0.999
31	0.600	0.673	0.778	0.867	0.931	0.976	0.999
32	0.611	0.683	0.785	0.871	0.933	0.976	0.999
33	0.621	0.691	0.791	0.874	0.935	0.977	0.999
34	0.631	0.700	0.796	0.878	0.937	0.978	0.999
35	0.640	0.707	0.802	0.881	0.938	0.978	0.999
36	0.649	0.715	0.807	0.884	0.940	0.979	0.999
37	0.658	0.722	0.812	0.887	0.942	0.979	0.999
38	0.666	0.728	0.816	0.890	0.943	0.980	0.999
39	0.674	0.735	0.821	0.893	0.945	0.980	0.999
40	0.681	0.741	0.825	0.895	0.946	0.981	0.999
41	0.688	0.747	0.829	0.898	0.947	0.981	0.999
42	0.695	0.752	0.833	0.900	0.948	0.982	0.999
43	0.701	0.758	0.837	0.902	0.950	0.982	0.999
44	0.708	0.763	0.840	0.905	0.951	0.982	0.999
45	0.713	0.768	0.844	0.907	0.952	0.983	0.999
46	0.719	0.772	0.847	0.909	0.953	0.983	0.999
47	0.725	0.777	0.850	0.910	0.954	0.983	0.999
48	0.730	0.781	0.853	0.912	0.955	0.984	0.999
49	0.735	0.785	0.856	0.914	0.956	0.984	0.999
50	0.740	0.789	0.858	0.916	0.956	0.984	0.999

Table 2B

Percentiles for the test for H_0: $\lambda = 1$ in: $y_t = \lambda y_{t-1} + \beta_1 + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	-0.252	-0.116	0.108	0.338	0.540	0.712	0.816
12	-0.185	-0.052	0.164	0.383	0.573	0.731	0.826
13	-0.124	0.005	0.213	0.422	0.602	0.748	0.835
14	-0.069	0.056	0.257	0.457	0.627	0.763	0.844
15	-0.020	0.102	0.296	0.487	0.649	0.777	0.852
16	0.025	0.144	0.331	0.515	0.669	0.789	0.859
17	0.066	0.182	0.363	0.539	0.686	0.800	0.866
18	0.105	0.217	0.392	0.561	0.702	0.810	0.872
19	0.140	0.249	0.418	0.581	0.716	0.819	0.877
20	0.173	0.279	0.442	0.600	0.729	0.827	0.882
21	0.203	0.306	0.465	0.617	0.741	0.835	0.887
22	0.231	0.331	0.485	0.632	0.752	0.842	0.892
23	0.257	0.355	0.504	0.646	0.762	0.848	0.896
24	0.281	0.377	0.522	0.660	0.771	0.854	0.900
25	0.304	0.397	0.538	0.672	0.780	0.859	0.903
26	0.326	0.416	0.554	0.683	0.788	0.864	0.906
27	0.346	0.434	0.568	0.694	0.795	0.869	0.909
28	0.365	0.451	0.582	0.704	0.802	0.873	0.912
29	0.383	0.467	0.594	0.713	0.808	0.878	0.915
30	0.400	0.482	0.606	0.722	0.814	0.881	0.918
31	0.416	0.497	0.618	0.730	0.820	0.885	0.920
32	0.431	0.510	0.628	0.738	0.825	0.888	0.922
33	0.446	0.523	0.638	0.745	0.830	0.892	0.925
34	0.460	0.535	0.648	0.752	0.835	0.895	0.927
35	0.473	0.546	0.657	0.759	0.839	0.897	0.929
36	0.485	0.557	0.665	0.765	0.844	0.900	0.930
37	0.497	0.568	0.674	0.771	0.848	0.903	0.932
38	0.508	0.578	0.681	0.777	0.851	0.905	0.934
39	0.519	0.587	0.689	0.782	0.855	0.907	0.935
40	0.530	0.596	0.696	0.787	0.858	0.910	0.937
41	0.540	0.605	0.703	0.792	0.862	0.912	0.938
42	0.549	0.614	0.709	0.797	0.865	0.914	0.940
43	0.558	0.622	0.715	0.801	0.868	0.916	0.941
44	0.567	0.629	0.721	0.805	0.871	0.918	0.942
45	0.575	0.637	0.727	0.809	0.874	0.919	0.944
46	0.584	0.644	0.732	0.813	0.876	0.921	0.945
47	0.591	0.650	0.738	0.817	0.879	0.923	0.946
48	0.599	0.657	0.743	0.821	0.881	0.924	0.947
49	0.606	0.663	0.748	0.824	0.884	0.926	0.948
50	0.613	0.669	0.752	0.827	0.886	0.927	0.949

Table 2C

Percentiles for the test for H_0: $\lambda = 1$ in: $y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + u_t$, with y_0 arbitrary and $u_t \sim NID(0, \sigma^2)$, performed by the OLS estimator of λ in: $y_t = \lambda y_{t-1} + \beta_1 + \beta_2 t + \beta_3 t^2 + u_t$, $t = 1, \ldots, T$

T	Probabilities of a smaller value						
	0.05	0.10	0.25	0.50	0.75	0.90	0.95
11	-0.478	-0.351	-0.136	0.095	0.309	0.496	0.617
12	-0.405	-0.280	-0.071	0.152	0.356	0.529	0.638
13	-0.338	-0.216	-0.012	0.203	0.397	0.559	0.659
14	-0.279	-0.159	0.040	0.248	0.433	0.586	0.678
15	-0.224	-0.106	0.087	0.287	0.465	0.610	0.695
16	-0.173	-0.058	0.130	0.323	0.494	0.631	0.712
17	-0.127	-0.014	0.169	0.356	0.520	0.650	0.726
18	-0.084	0.027	0.204	0.385	0.543	0.667	0.740
19	-0.045	0.064	0.237	0.412	0.564	0.684	0.752
20	-0.008	0.099	0.267	0.437	0.583	0.698	0.763
21	0.027	0.131	0.295	0.459	0.601	0.711	0.773
22	0.060	0.161	0.321	0.480	0.617	0.723	0.782
23	0.090	0.189	0.345	0.499	0.632	0.734	0.791
24	0.118	0.215	0.367	0.517	0.646	0.745	0.799
25	0.145	0.240	0.388	0.534	0.659	0.754	0.806
26	0.170	0.263	0.407	0.550	0.671	0.763	0.813
27	0.194	0.284	0.425	0.564	0.682	0.771	0.819
28	0.216	0.305	0.443	0.578	0.692	0.779	0.825
29	0.237	0.324	0.459	0.591	0.702	0.786	0.831
30	0.257	0.342	0.474	0.603	0.711	0.793	0.836
31	0.276	0.360	0.489	0.614	0.720	0.799	0.841
32	0.294	0.376	0.502	0.625	0.728	0.805	0.846
33	0.312	0.392	0.515	0.635	0.736	0.810	0.850
34	0.328	0.407	0.528	0.645	0.743	0.815	0.854
35	0.344	0.421	0.539	0.654	0.750	0.820	0.858
36	0.359	0.434	0.551	0.663	0.756	0.825	0.862
37	0.373	0.447	0.561	0.671	0.762	0.830	0.865
38	0.386	0.459	0.571	0.679	0.768	0.834	0.869
39	0.400	0.471	0.581	0.686	0.774	0.838	0.872
40	0.412	0.482	0.590	0.694	0.779	0.842	0.875
41	0.424	0.493	0.599	0.701	0.784	0.845	0.878
42	0.435	0.504	0.608	0.707	0.789	0.849	0.880
43	0.447	0.514	0.616	0.713	0.794	0.852	0.883
44	0.457	0.523	0.624	0.719	0.798	0.856	0.886
45	0.467	0.532	0.631	0.725	0.802	0.859	0.888
46	0.477	0.541	0.638	0.731	0.806	0.862	0.890
47	0.487	0.550	0.645	0.736	0.810	0.864	0.893
48	0.496	0.558	0.652	0.741	0.814	0.867	0.895
49	0.505	0.566	0.658	0.746	0.817	0.870	0.897
50	0.513	0.574	0.664	0.751	0.821	0.872	0.899

Fuller's table. Now the negative bias is much stronger; even under the null a λ estimate exceeding unity is found to be most unlikely. Table 2C concerns a test which is advised against in [D i c k e y et al. (1984), p. 16] for non-obvious reasons. This table (which does not correspond to any of the Fuller tables) enables to test exactly whether a series appears to be trend stationary or difference stationary; therefore it is in our opinion also relevant in the context of the (asymptotic) problems indicated in [D u r l a u f and P h i l l i p s (1988), section 4]. Yet, in the presence of (polynomial) trends the similar unit root tests suggested by [S c h m i d t and P h i l l i p s (1989)] may be more powerful, since these do not involve a redundant regressor.

However, models with a lagged dependent variable and merely polynomial trends are rather sterile, in our opinion. We shall show now how our results can be used also in real empirical econometric modelling. Just for illustrative purposes we consider a model for aggregate UK consumption of non-durables which originates from [H e n d r y (1983)]. This model is based on annual data published in *Economic Trends Annual Supplement* (1983). The sample ranges from 1954 through 1983 and concerns:

- C = consumer's expenditure on non-durables and services in constant prices;
- I = real personal disposable income;
- P = implicit deflator of C.

Lower case letters denote natural logarithms. Employing OLS (like Hendry did) we find (asymptotic standard errors in parentheses):

$$\underset{}{c_t} = \underset{(0.120)}{0.861\, c_{t-1}} + \underset{(0.048)}{0.501\, i_t} - \underset{(0.091)}{0.383\, i_{t-1}} - \underset{(0.035)}{0.115\, \Delta p_t} + \underset{(0.288)}{0.231} \tag{7.1}$$

$$T = 27 \qquad R^2 = 0.9993 \qquad s = 0.00499$$

$$SC(4,\ 18) = 0.529 \quad PF(3,\ 22) = 0.937 \quad H(4) = 1.016 \quad N(2) = 1.652$$

No form of misspecification is detected by the four mentioned asymptotic diagnostics. The SC statistic is the Lagrange multiplier test for serial correlation which under the null of white-noise disturbances is compared with the F(4, 18) distribution. The PF statistic is Chow's F-test for predictive failure. The H statistic is the Breusch-Pagan Lagrange multiplier test for heteroscedasticity; here this test is used against the alternative that the disturbance variance is functionally dependent on the four regressors, so under the null it is asymptotically $\chi^2(4)$. The N(2) statistic tests the normality of the disturbances by checking the skewness and the excess curtosis of the residuals; under the null it is asymptotically $\chi^2(2)$.

Relying on blunt standard asymptotic procedures a 95% confidence interval for λ, the coefficient of the lagged dependent variable c_{t-1} in (7.1), is given by $0.861 \pm t^{0.025}_{(27-5)} * 0.120$, yielding [0.61, 1.11]. This seems pretty wide, and it is not known how accurate this result is for this particular small sample on these particular data.

Table 3A

Percentiles of $\hat{\lambda}_z$ and test findings for H_0: $\lambda = \lambda_0$ in model (7.1)

λ_0	Probabilities of a smaller value							Actual estimate	
	0.05	0.10	0.25	0.50	0.75	0.90	0.95	$\bar{\lambda}_z$	P-value
0.00	-0.463	-0.394	-0.275	-0.136	0.007	0.135	0.211	0.886	1.000
0.50	-0.188	-0.101	0.044	0.200	0.347	0.470	0.539	0.915	1.000
0.75	-0.100	-0.005	0.149	0.310	0.459	0.582	0.652	0.688	0.966
1.00	-0.076	0.024	0.183	0.346	0.495	0.620	0.692	0.303	0.426
1.05	-0.078	0.022	0.182	0.346	0.494	0.618	0.690	0.306	0.432
1.10	-0.080	0.019	0.179	0.343	0.491	0.615	0.687	0.327	0.473
1.20	-0.094	0.006	0.169	0.333	0.480	0.602	0.673	0.391	0.601
1.40	-0.131	-0.026	0.137	0.304	0.452	0.566	0.631	0.502	0.826

The findings of this paper enable to produce exact inference on λ assuming that the disturbances are normal and that we can condition on the other regressors (which is rash since the consumption function is usually seen as a structural equation of a simultaneous system). In Table 3A results are given on statistic $\hat{\lambda}_z$ for various values of λ_0; percentiles of the exact distribution of $\hat{\lambda}_z$ are presented, but also the actual estimate $\bar{\lambda}_z$ of the estimator $\hat{\lambda}_z$ is given. This is obtained in the regression of y on y_{-1} and the space spanned by $[X: (\lambda_0):C(\lambda_0)X]$. Also the Prob--value, i.e. the value of the cumulative distribution function of $\hat{\lambda}_z$ for the actual estimate $\bar{\lambda}_z$ is mentioned.

From the first two lines of the table we see that H_0: $\lambda = 0$ and H_0: $\lambda = 0.5$ are rejected in favour of $\lambda > 0$ and $\lambda > 0.5$ respectively. From the third line we see that H_0: $\lambda = 0.75$ is rejected against $\lambda > 0.75$ at the 5% level, but not at the 2.5% level. It follows from the fourth line that the unit root hypothesis $\lambda = 1$ cannot be rejected at any reasonable significance level. The inaccuracy and the discontinuity (at $\lambda = 1$) of the classical asymptotic approach stem Sims (1988) skeptical about unit root econometrics; we see here that a much more satisfying analytic and exact classical approach is feasible. So, from a statistical point of view we find that it seems acceptable to model the consumption function in first differences. However, from the bottom lines of the table we see that neither $\lambda = 1.05$ nor $\lambda = 1.20$ have to be rejected. Since these values are not acceptable at all from any reasonable economic point of view we tentatively conclude that the test procedure - although exact - seems to have a very moderate power with respect to explosive alternatives. With respect to one-sided stable alternatives we can establish that the intervals $[\lambda > 0.738]$, $[\lambda > 0.766]$ and $[\lambda > 0.797]$ have confidence coefficients of 0.975, 0.95 and 0.90 respectively. Hence, with respect to the left-hand side of the confidence interval for λ we can improve considerably in comparison to the standard asymptotic result. It seems noteworthy that the asymptotic procedure apparently leads to conservative inference here; the actual confidence coefficient of the interval $[\lambda > 0.61]$ must be well above 97.5%.

However, we must realize that the above findings are based on only one actual sample of this particular model. We also have to recognise that this illustration is hampered because income will in fact be jointly dependent with consumption, and hence strong-exogeneity and even weak-exogeneity is not the case here. Purely for illustrative purposes we shall therefore also construct a case which does meet all the requirements exactly. This is achieved by employing the real income and inflation data, and by generating artificial consumption data according to the following (stylized) scheme:

$$c_t = 0.9\, c_{t-1} + 0.5\, i_t - 0.4\, i_{t-1} - 0.1\, \Delta p_t + 0.0 + \sigma\varepsilon_t, \qquad (7.2)$$

with $c_0 = 11.112$, $\sigma = 0.005$, and $\varepsilon_t \sim IIN(0,1)$ for $t = 1, \ldots, 27$.

The starting value c_0 is the actual 1953 figure. We produced only one random ε_t series, and used the resulting (exactly normal) c_t series in a regression (where i_t, i_{t-1} and Δp_t are genuinly fixed now), which yielded:

$$\begin{aligned} c_t = \underset{(0.084)}{0.853}\, c_{t-1} + \underset{(0,051)}{0.529}\, i_t - \underset{(0.076)}{0.388}\, i_{t-1} - \underset{(0.039)}{0.066}\, \Delta p_t + \underset{(0.155)}{0.068} \\ T = 27 \qquad R^2 = 0.9994 \qquad s = 0.00547 \\ SC(4, 18) = 0.764 \qquad H(4) = 3.625 \qquad N(2) = 1.199 \end{aligned} \tag{7.3}$$

Next we applied the $\hat{\lambda}_z$ based procedures again in order to test hypotheses and to construct confidence intervals for λ. Application of the crude asymptotic procedure leads now to a confidence interval $0.853 \pm t^{0.025}_{(27-5)} * 0.084$ which amounts to [0.68, 1.03], whereas we find that the intervals [$\lambda > 0.621$], [$\lambda > 0.668$] and [$\lambda > 0.719$] have confidence coefficients of 0.975, 0.95 and 0.90 respectively. Hence, for this single realisation of the stochastic generating mechanism we find that the exact left-hand confidence boundary is smaller then the asymptotic value, and thus the asymptotic interval seems too liberal now. Note that the percentiles of $\hat{\lambda}_z$ as given in Table 3A are also valid for model (7.3) since the distribution of $\hat{\lambda}_z$ is invariant with respect to β and σ; it is determined merely by X and λ_0. The actual estimates $\overline{\lambda}_z$ and the confidence intervals are of course different for (7.3), since these are based on the c_t series as well. In Table 3B some results are given. For values of λ_0 above unity the same anomalous P-values are found which indicate that these high values of λ_0 cannot be rejected. From these

Table 3B

Findings in model (7.3)

λ_0	Actual estimate	
	λ_z	P-value
0.00	0.818	1.000
0.50	0.728	0.996
0.75	0.534	0.851
1.00	0.214	0.291
1.05	0.261	0.362
1.20	0.584	0.883
1.40	0.819	0.995

(pseudo-)empirical results we have to conclude that an investigation into the power of the exact and similar test procedure is needful.

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DOKŁADNE PODOBNE TESTY DLA PIERWIASTKA PIERWSZEGO RZĘDU AUTOREGRESYJNEGO MODELU REGRESJI

Opisano procedurę testu dla zbadania czy współczynnik zmiennej zależnej opóźnionej w autoregresyjnym modelu regresji wielokrotnej pierwszego rzędu równa się pewnej konkretnej wartości, np. zeru lub jedności lub innej dowolnej

stabilnej lub niestabilnej wartości. Przy hipotezie zerowej estymator tego współczynnika ma rozkład jak rozkład ilorazu dwóch kwadratowych form w standardowych zmiennych normalnych, gdy prócz regresorów egzogenicznych, włączono także niektóre zbędne zmienne objaśniające. Rozkład związany z hipotezą zerową jest wolny od jakichkolwiek kłopotliwych parametrów. Zatem estymatory te są łatwo policzalne i mogą być użyte jako statystyka testu; jej błędy typu I mogą być dokładnie kontrolowane, podczas gdy test ten jest podobny a także niezmienniczy. Poszczególne testy pierwiastków jednostkowych stworzone przez Dickeya i Fullera wydają się być prostymi przykładami naszego testu dla bardzo specyficznych macierzy planu. Podane zostają rozszerzone tablice dokładnych wartości krytycznych dla tych i niektórych innych form. W końcu ilustrujemy przydatność naszej ogólnej procedury testu przy dynamicznej specyfikacji ekonometrycznych modeli szeregów czasowych.