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ON MONITORING THE AVERAGE LEVEL OF THE PROCESS BASED ON THE SEQUENCE OF PERMUTATION TESTS

Abstract. The classical methods for monitoring the average level of the process in quality control procedures are based on the normality assumption. The Shewhart's control charts are based on the sequence of parametric tests. The main assumptions in these tests are that the population is normally distributed and observations are independent of each other.

Permutation tests could be used even if the normality assumption is not fulfilled. The control chart based on the sequence of permutation tests is presented in the paper. The properties of the proposed control chart are considered in the Monte Carlo study.

Key words: process mean, control charts, permutation test, Monte Carlo.

I. INTRODUCTION

The statistical control charts concept was developed in 1924 by Walter A. Shewhart. This fact is often considered as a formal beginning of statistical quality control. The control chart is a graphical display of a quality characteristic which has bean measured or computed (for example sample mean or standard deviation) from a sample (D.C. Montgomery, 1996). There is a close connection between control charts and hypothesis testing. The control charts can be treatment as a graphical view of a sequence of statistical parametric tests. The main assumptions in Shewhart's control charts for variables are independence and normality distribution of a quality characteristic. In many situations we may have reason to doubt the validity of normality assumption. The control chart based on the permutation tests is presented in the paper. This control chart could be used even if the distribution is not normal.

II. THE NORMALITY ASSUMPTION

To establish control limits in classical control charts we have to know the process parameters μ and σ. In practice μ and σ are usually unknown. In these

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cases we have to estimate the parameters from the preliminary samples taken when process is thought to be in control. The assumption of normality should be tested and parameters should be estimated.

In the computer simulation the results of testing normality for three non-normal distributions were considered. The following distributions were analyzed in this study:

- log-normal distribution with parameters μ and σ – LN(0, 1), LN(0.5, 1), LN(1,1),
- beta distribution with the shape and the scale parameters – B(2, 5), B(2, 4), B(2, 3),
- gamma distribution with the shape parameter – Γ(2), Γ(3), Γ(4).

The following normality tests were used in this study: Pearson's chi-square test (P), Cramer – von Mises test (CvM), Lilliefors test (L) and Shapiro-Wilk's test (S-W).

The study was performed for sample sizes of n = 5, 15 and 30 except for Pearson's chi square test (only for n = 30) and Cramer von Mises test (for n = 15 and n = 30) . Tables 1–3 show the estimated probabilities of failing to reject the null hypothesis (normality) for these non-normal distributions. The estimated probabilities are based on the 10 000 simulations in each case. The simulation study was performed using R (http://www.r-project.org) procedures.

Table 1. The estimated probabilities of failing to reject the normality hypothesis (sample of size n = 5)

Test	Log-normal distribution			Beta distribution			Gamma distribution		
	LN(0;1)	LN(0.5;1)	LN(1;1)	B(2;5)	B(2;4)	B(2;3)	Γ (2)	Γ (3)	Γ (4)
L	0.8104	0.8075	0.8070	0.9463	0.9535	0.9541	0.9118	0.9273	0.9398
S-W	0.7667	0.7652	0.7657	0.9378	0.9491	0.9521	0.9062	0.9225	0.9340

Source: Monte Carlo study.

Table 2. The estimated probabilities of failing to reject the normality hypothesis (sample of size n = 15)

Test	Log-normal distribution			Beta distribution			Gamma distribution		
	LN(0;1)	LN(0.5;1)	LN(1;1)	B(2;5)	B(2;4)	B(2;3)	Γ (2)	Γ (3)	Γ (4)
CvM	0.2385	0.2387	0.2377	0.8891	0.9179	0.9432	0.6876	0.7791	0.8272
L	0.3391	0.3409	0.3501	0.9073	0.9241	0.9429	0.7578	0.8258	0.8645
S-W	0.1730	0.1785	0.1793	0.8771	0.9083	0.9423	0.6172	0.7140	0.7806

Source: Monte Carlo study.

Table 3. The estimated probabilities of failing to reject the normality hypothesis (sample of size n = 30)

Test	Log-normal distribution			Beta distribution			Gamma distribution		
	LN(0;1)	LN(0.5;1)	LN(1;1)	B(2;5)	B(2;4)	B(2;3)	Γ (2)	Γ (3)	Γ (4)
P	0.0514	0.0520	0.0535	0.8800	0.9035	0.9332	0.5864	0.7583	0.8277
CvM	0.0231	0.0271	0.0242	0.7983	0.8535	0.9055	0.4010	0.5815	0.6745
L	0.0661	0.0698	0.0718	0.8369	0.8801	0.9148	0.5288	0.6689	0.7411
S-W	0.0067	0.0094	0.0085	0.7240	0.8154	0.8988	0.2505	0.4459	0.5643

Source: Monte Carlo study.

The decision is very often “fail to reject” the null hypothesis (normality) for considered non-normal distributions. It is especially in the small sample cases (n = 5). It could be said that the experimental data does not decisively reject the null hypothesis. In such cases the classical control charts, designed for use in normality assumption are often used, but the normality assumption is not fulfilled.

III. CONTROL CHART BASED ON THE PERMUTATION TEST

Permutation tests

Permutation tests were introduced by R.A. Fisher in 1930’s (P. Good, 2005). These tests are a computer-intensive statistical technique complementary to bootstrap (W.J. Welch, 1990). Within the context of such tests, the word permutation is used to represent a specific configuration/arrangement of scores (D.J. Sheskin, 2004). The main application of these tests is two sample problem (B. Efron, R. Tibshirani, 1993). The Shewhart’s control charts are based on the sequence of parametric tests. The sequence of permutation tests will be used for the construction of permutation control chart.

The main idea of permutation tests is attractively simple and free of mathematical assumptions. Let us assume that the samples $S_1 = \{x_{11}, x_{12}, ..., x_{1n_1}\}$ of size n_1 and $S_2 = \{x_{21}, x_{22}, ..., x_{2n_2}\}$ of size n_2 were taken from possibly different continuous distributions F and G. The null hypothesis that the samples were taken from identically distribution will be tested. This hypothesis can be written as

$$H_0 : F = G \tag{1}$$

against the alternative hypothesis

$$H_A : F \neq G \tag{2}$$

Let α be the significance level (α usually in hypothesis testing is equal to 0.05, 0.01 or 0.1 and in control charts is equal to 0.001). Let $\overline{X}_1$ and $\overline{X}_2$ are the samples means

$$\overline{X}_k = \frac{1}{n_k}\sum_{i}^{n_k} x_{ki} \text{ , for } k = 1, 2. \tag{3}$$

Let us consider the statistic:

$$T = \overline{X}_1 - \overline{X}_2 \tag{4}$$

Let us denote by T_0 the value of this statistic calculated from the samples S_1 and S_2. Having observed T_0, the achieved significance level (*ASL*, see B. Efron, R. Tibshirani, 1993) of the test is defined to be the probability of observing at least that large or at most that small value when the null hypothesis is true. *ASL* could be written as

$$ASL = P_{H_0}(|T^*| \geq |T_0|) \tag{5}$$

where the random variable T^* has the null hypothesis distribution, the distribution of T if H_0 is true. The small value of *ASL*, the stronger the evidence against H_0. If $ASL < \alpha$ then the null hypothesis is rejected.

The distribution of the statistic T^* should be estimated under the assumption null hypothesis. The set $S = S_1 \cup S_2$ should be randomly divided N times (N – should be at least 1000) into two samples of sizes n_1 and n_2. Next for each case the values T_i (i = 1, 2, …, N) are calculated using formula (4). Then the value of *ASL* can be estimated

$$ASL \approx ASL_{perm} = \frac{card\{T_i : |T_i| \geq |T_0|\}}{N} \tag{6}$$

Control chart based on the permutation tests

Let us assume that we have k samples $S_{11}, S_{12}, ..., S_{1k}$ each of size n from process which is thought to be in control (the sample from the distribution F). In time t (t = 1, 2, …, N) the sample S_{2t} of size n is taken from the distribution G.

The null hypothesis that there is no difference between distributions F and G will be tested.

Let $S_1 = S_{11} \cup S_{12} \cup ... \cup S_{1k}$ and $S_2 = S_{2t}$. The permutation test instead of the parametric test will be used to test the H_0 hypothesis.

The scheme of the Shewhart's control chart is presented in the Figure 1 (D.C. Montgomery, 1996). There are three horizontal control lines in the charts: upper control line, central line and lower control line. The levels of upper and lower control lines are related to the critical values in hypothesis testing. A point plotting outside the control limits is equivalent to reject the null hypothesis and the point plotting between control lines is equivalent to failing to reject this hypothesis in parametric test. On the permutation control chart there is only one control line on the significance level α. (see Fig. 1). The control chart shows the *ASL* values versus time t (t = 1, 2, …., N). There are plotted calculated ASL_t values for a t-th sample versus the sample number. A point plotting bellow the *ASL* line (*ASL-L*) is equivalent to reject the null hypothesis in the permutation test.

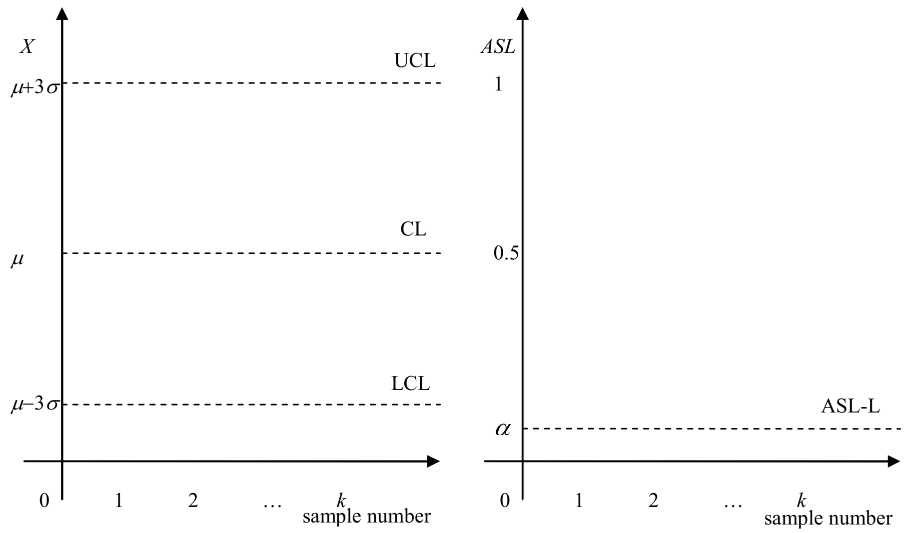

Figure 1. The graphical scheme of the Shewhart's X control chart (left) and permutation control chart (right)

The rejection area on the chart is outside the area between upper control line (*UCL*) and lower control line (*LCL*) in the Shewhart's control chart case. The rejection area is below the *ASL* line (*ASL-L*) in the permutation control chart.

IV. MONTE CARLO STUDY

Monte Carlo simulations were used to compare the properties of the permutation control chart and the classic control chart. There were used R procedures (http://www.r-project.org) in the Monte Carlo study.

Description of random variables in Monte Carlo study

There were analyzed four types of distributions in the Monte Carlo study. For each of them there were used the random variables *X*, *Y* and *Z* with different values of parameters. The details of random variables are presented in table 4.

Table 4. The random variables details in Monte Carlo study

The distribution	R function	The expected value	The parameters of random variables	
			X	*Y* / *Z*
Normal $N(\mu,\sigma)$	rnorm(n,μ,σ)	μ	$\mu = 10$, $\sigma = 1$ EX=10	$\mu = 10.5$, $\sigma = 1$, EY = 10.5
				$\mu = 11$, $\sigma = 1$, EZ = 11
Lognormal $LN(\mu,\sigma)$	rlnorm(n, μ, σ)	$\exp\left(\mu + \frac{\sigma^2}{2}\right)$	$\mu = 0$, $\sigma = 1$ $EX = \sqrt{e}$	$\mu = 0.5$, $\sigma = 1$, EY = e
				$\mu = 1$, $\sigma = 1$, $EZ = \sqrt{e^3}$
Beta $B(s_1, s_2)$	rbeta(n,s_1,s_2)	$\frac{s_1}{s_1 + s_2}$	$s_1 = 2$, $s_2 = 5$ $EX = \frac{2}{7}$	$s_1 = 2$, $s_2 = 4$, $EY = \frac{1}{3}$
				$s_1 = 2$, $s_2 = 3$, $EZ = \frac{2}{5}$
Gamma $\Gamma(s)$	rgamma(n,s)	s	$s = 2$ EX = 2	$s = 3$, EY = 3
				$s = 4$, EZ = 4

Source: Monte Carlo study

The simulation study was performed for $k = 3$ samples of size $n = 5$ from in and out of control process. There four in control processes were considered (one normal and three non-normal).

The simulation procedure

The Monte Carlo study was done for random variables described in table 4. There was assumed the significance level $\alpha = 0.05$ in the Monte Carlo study. The steps of this study were as follows

1. k samples of size n (k = 3, n = 5) from distribution F (as random variable X in control process) were generated. These samples formed a sample S_1.

2. The sample S_2 of size n (n = 5) from distribution as random variable X (in control process), Y and Z (out of control processes) was generated.

3. Two tests were performed

a) one sample t test where the tested value was estimated from the first sample

b) permutation test for two samples

4. Steps 1–3 were repeated N = 10 000 times.

5. The empirical probabilities of rejection H_0 for t test and permutation test were calculated, the standard errors were estimated using jackknife methods.

The results of Monte Carlo study

The estimated probabilities of rejection of the null hypothesis for classical Shewhart's control charts and for the permutation control chart are presented in the table 5.

Table 5. The estimated probabilities of rejection the normality hypothesis (t test and permutation test) and the standard errors of estimate

The distributions parameters		*t* test		Permutation test	
The in control process sample	Process sample	probability	standard error	probability	standard error
Normal N(10, 1)	N(10,1)	0.0685	0.0020	0.0529	0.0021
	N(10,5,1)	0.1615	0.0037	0.1517	0.0025
	N(11,1)	0.4184	0.0041	0.4525	0.0058
Log-normal LN(0, 1)	LN(0,1)	0.1924	0.0054	0.0472	0.0015
	LN(0,5;1)	0.0667	0.0027	0.2075	0.0040
	LN(1,1)	0.0889	0.0033	0.5149	0.0031
Beta B(2, 5)	B(2,5)	0.0877	0.0021	0.0546	0.0027
	B(2,4)	0.0739	0.0030	0.1004	0.0026
	B(2,3)	0.1476	0.0030	0.2616	0.0037
Gamma Γ(2)	Γ(2)	0.1065	0.0043	0.0512	0.0022
	Γ(3)	0.1259	0.0027	0.2754	0.0040
	Γ(4)	0.3731	0.0045	0.6506	0.0053

Source: Monte Carlo study

The estimated probabilities of rejection of the null hypothesis are presented in the Figure 2. The dotted line denotes the assumed significance level α in the Monte Carlo study. It can be noticed, that performing t test is acceptable in the first case (normal distribution) only. For non-normal distribution t test shouldn't be performed. In these cases the control chart based on the permutation test could be recommended.

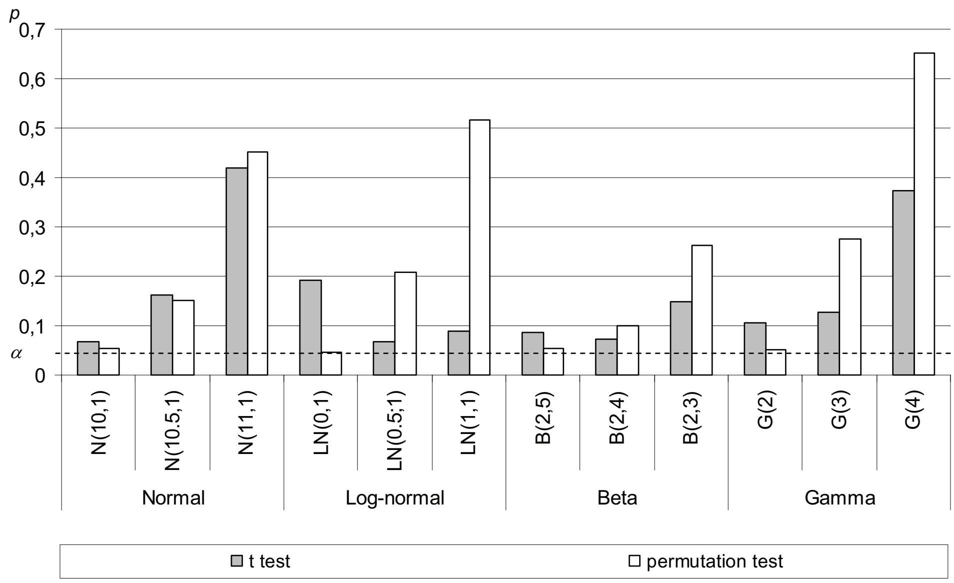

Figure 2. The estimated probabilities of rejection the normality hypothesis and the standard errors of estimate

Source: table 5.

It can be noticed that for analyzed samples of sizes n = 5 the permutation test is more accuracy than t test. If H_0 is true, permutation tests gives almost exactly 100α % chance that ASL_{perm} will be less than α even if the distribution of random variable is skewed (for example log-normal distribution). If the null hypothesis is true (there is no difference between distributions) there for each type of distribution (normal and non-normal distributions) the probability of rejection the null hypothesis is almost α. If the null hypothesis is false, then the permutation test more often leads to rejection of this hypothesis, especially in non-normal cases.

V. CONCLUDING REMARKS

The classical control charts can be used under the normality assumption. The construction of the control chart based on the sequence of permutation tests is presented in the paper. Permutation tests are free of mathematical assumptions, especially completely removes the normality condition. The properties of the proposed control chart are considered in the Monte Carlo study.

The simulation study has shown that the permutation tests could be used for monitoring process mean. The permutation tests are useful for monitoring non-normal processes. Permutation tests give almost exact α probability of rejection the true null hypothesis even for non-normal distributions.

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O MONITOROWANIU POZIOMU PRZECIĘTNEGO PROCESU Z WYKORZYSTANIEM SEKWENCJI TESTÓW PERMUTACYJNYCH

Klasyczne metody pozwalające na monitorowanie poziomu przeciętnego procesów produkcyjnych odwołują się zwykle do założenia normalności rozkładu badanej zmiennej. Wynika to z faktu, że w konstrukcji kart kontrolnych Shewharta wykorzystuje się sekwencje testów parametrycznych, które wymagają spełnienia wspomnianego założenia. Stosowanie testów permutacyjnych nie wymaga spełnienia tak ostrych założeń. W artykule zaproponowano zastosowanie zamiast sekwencji testów parametrycznych sekwencji testów permutacyjnych. Zaproponowano konstrukcję karty kontrolnej wykorzystującej sekwencje testów permutacyjnych. Rozważania teoretyczne zostały uzupełnieniowe analizami symulacyjnymi. Analizy symulacyjne wykazały, że stosowanie proponowanej karty kontrolnej może być szczególnie przydatne dla prób o małych liczebnościach pochodzących z rozkładów o silnej asymetrii.