A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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A PROPOSAL OF AN AGGREGATE INDEX OF LABOUR PRODUCTIVITY

Abstract. In this paper we propose some constructions of the aggregate index of labour productivity. The proposed systems of weights are based on theoretical considerations of the situation in which the number of observations – coming from the considered enterprises – is insufficient. First of all we consider the case in which we intend to measure the average, one-period dynamics of labour productivity having data from $T > 2$ periods.
Key words: aggregate index, arithmetic mean, harmonic mean, labour productivity.

I. INTRODUCTION

The contemporary economy makes use of lots of statistical indexes to calculate the dynamics of prices, quantities, and in particular, labour productivity. For example: Laspeyres and Paasche indexes have been known since 19-th century. Depending on the type of an economic problem we may also use one of the following indexes: Fisher ideal index (see Fisher (1972)), Törnqvist index (Törnqvist (1936)), Lexis index and other indexes (see Zając (1994), Domański (2001)). Indexes are often used in practise (see Moutlon, Seskin (1999), Seskin, Parker (1998)). Balk (1995) wrote about the axiomatic price index theory, Diewert (1978) showed that the Törnqvist index and Fisher ideal index approximate each other. But it is really hard to indicate the best of statistical indexes (see Dumagan (2002)).

The choice of index depends on the information we want to obtain. Unfortunately, most of indexes take into account no event from the inside of the considered time interval. Thus, if we want take into account the omitted time periods we should use a different formula.

A system of weights for the index of labour productivity (only this type of indexes is discussed further) should satisfy all the economic postulates. But the construction of index, based on economic postulates, have to take into

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consideration the accidental noise of partial indexes. The partial indexes of labour productivity, based on small number of observations, can lead to wrong conclusions about the global labour productivity. In this paper the proposed system of weights is based on theoretical considerations over the situation in which the number of observations – coming from some of the considered enterprises – is insufficient. We are going to construct the aggregate index, which strongly limits the influence of partial indexes of labour productivity connected with the small number of observations. In the first part of this paper we consider a group of N – enterprises and two periods of their activity. Next, we consider the case in which we want to measure the average, one-period dynamics of the labour productivity, having data from T >2 periods.

II. CONSTRUCTION OF INDEX IN THE CASE OF TWO PERIODS

Let us consider a group of $N-$enterprises observed in discrete moments: s (base period) and t (testing period). Let us denote:

W_i^s – labour productivity of $i-$th enterprise at time s, $i \in \{1,2,...,N\}$,

$W_i^t -$ labour productivity of $i-$th enterprise at time t, $i \in \{1,2,...,N\}$,

$I_j(s,t) \equiv I_j = \frac{W_j^t}{W_j^s}$ - partial index of labour productivity of $i-$th enterprise comparing periods s and t, where $j \in \{1,2,...,N\}$,

n_i^s – number of employees of $i-$th enterprise at time s,

$n_i^t -$ number of employees of $i-$th enterprise at time t.

We are going to find the proper vector of weights $\{g_1, g_2,...,g_N\}$. Using the above significations we can write the index of labour productivity as follows:

$$\bar{I} = \frac{\sum_{i=1}^{N} g_i I_i}{\sum_{i=1}^{N} g_i}. \quad (1)$$

When we treat each I_i as a random variable on some probabilistic space (Ω, F, P) and each g_i as a real number, we must treat also $\bar{I}$ as a random variable. We are interested in the differences among the calculated, noised by small number of observations from some enterprises index $\bar{I}$ and its theoretical, expected value $\bar{I}_0$ [2]. We will calculate the influence of accidental noise of partial indexes on the global index $\bar{I}$ as:

$$d\bar{I} = \bar{I} - \bar{I}_0 = \bar{I} - E\bar{I} . \quad (2)$$

so we are going to minimize the value of dispersion of the random variable described in (1):

$$\sigma_{\bar{I}}^2 = E(d\bar{I})^2 . \quad (3)$$

Let us denote

$$\gamma_i = \frac{g_i}{\sum_{i=1}^{N} g_i} \quad (4)$$

We get from (1) and (4) that

$$\bar{I} = \sum_{i=1}^{N} \gamma_i I_i , \quad (5)$$

$$I_0 = E\bar{I} = \sum_{i=1}^{N} \gamma_i I_{i0} , \quad (6)$$

where

$$I_{i0} = EI_i , \ i \in \{1,2,...,N\} , \quad (7)$$

$$\sum_{i=1}^{N} \gamma_i = 1 . \quad (8)$$

Let us assume that dI_i and dI_j are independent random variables for each i and j. Hence, we get the following consequence of this fact:

$$\sigma_{\bar{I}}^2 = E(d\bar{I})^2 = E(\bar{I} - I_0)^2 = E[\sum_{i=1}^{N} \gamma_i (I_i - I_0)^2] = \sum_{i=1}^{N} \gamma_i^2 \sigma_{I_i}^2 . \quad (9)$$

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[2] We assume $E\bar{I} < \infty$, $EI_i < \infty$, $Var\bar{I} < \infty$, $VarI_i < \infty$.

Now we have the optimization task where the aim function is

$$F = \sum_{i=1}^{N} \gamma_i^2 \sigma_{I_i}^2, \tag{10}$$

with the constraints specified in (8).

In the papers: Białek (2007) and Białek, Czajkowski (2008) we proved that the optimization task leads to the following solution:

$$g_i = \frac{2}{\frac{1}{n_i^s} + \frac{1}{n_i^t}}. \tag{11}$$

From (16) we can come to the following conclusion: each weight g_i is a harmonic mean of number of employees of *i*-th enterprise at considered moments s and t. It is easy to verify that if we assume weights as an arithmetic mean of these numbers:

$$\overline{g}_i = \frac{n_i^s + n_i^t}{2}, \tag{12}$$

we will not solve the optimization task for function F under the constraints specified in (8). In our opinion, this fact recommends the definition g_i.

III. CONSTRUCTION OF INDEX IN CASE OF MORE THAN TWO PERIODS

Let us consider a group of *N*-enterprises observed in discrete moments: $\{1, 2, .., T\}$. We are going to measure the average, one-period labour productivity. As in the previous case we expect that the new index $\hat{I}$ strongly limits the influence of partial indexes of labour productivity connected with the small number of observations. We propose a list of postulates for this aggregate index:

Postulate 1

$\forall i, i \;\; I_i(t-1, t) = 1 \;\Rightarrow\; \hat{I} = 1$.

This postulate says that in case in which partial indexes show no change in labour productivity, then the global index must absolutely inform us about no change in labour productivity of the group.

Postulate 2
The influence of enterprises with relatively small number of employees on the average one-period labour productivity is negligible.

Postulate 3
If all partial indexes of labour productivity grow by about the same $m\%$ then the value of global index $\hat{I}$ will increase by about the same $m\%$.

Postulate 4
If we increase the number of employees in each enterprise by about the same $m\%$, the index $\hat{I}$ will not change.

Under the above significations in the paper of Białek (2006) we propose the following index of the average labour productivity on the interval:

$$\hat{I}_A = \sum_{i=1}^{N}\left[\frac{\sum_{u=1}^{T} n_i^u}{\sum_{k=1}^{N}\sum_{u=1}^{T} n_k^u}\sum_{u=2}^{T}\frac{1}{2}\cdot\left(\frac{n_i^{u-1}}{\sum_{y=2}^{T} n_i^{y-1}}+\frac{n_i^u}{\sum_{y=2}^{T} n_i^y}\right)\cdot I_i(u-1,u)\right]. \quad (13)$$

Using the significations

$$\alpha_i^u = \frac{1}{2}\left(\frac{n_i^{u-1}}{\sum_{y=2}^{T} n_i^{y-1}}+\frac{n_i^u}{\sum_{y=2}^{T} n_i^y}\right),\ \beta_i = \frac{\sum_{u=1}^{T} n_i^u}{\sum_{k=1}^{N}\sum_{u=1}^{T} n_k^u},$$

for $i = 1,2,..., N,\ u = 2,3,...,T,$ (19) we get from (18)

$$\hat{I}_A = \sum_{i=1}^{N}\beta_i\sum_{u=2}^{T}\alpha_i^u I_i(u-1,u), \quad (20)$$

where $\sum_{u=2}^{T}\alpha_i^u = 1,\ \alpha_i^u \geq 0$ and $\sum_{i=1}^{N}\beta_i = 1,\ \beta_i \geq 0$.

We have the following interpretation of the above coefficients: β_i informs the producer how important a share of *i*-th enterprise is taking into consideration the number of employees, and α_i^u informs the producer how important *u*-th moment is in the case of *i*-th enterprise.

It is easy to prove that the index described in (20) satisfies postulates 1–4. Nevertheless, it is quite natural to formulate the following, additional postulates:

Postulate 5

In the case of $T = 2$ and two moments of observations s and t we should have $\hat{I} = \overline{I}$.

Certainly the $\hat{I}_A$ index does not satisfy the postulate no. 5 (see example 1).

Example 1

Let us consider two enterprises (1 and 2) and two moments of their observations: $s,\ t$. Let us assume that the labour productivity and the number of employees of these enterprises were as follows:

$$I_1(s,t) = 1.2;\ n_1^s = 4500;\ n_1^t = 4000$$

and

$$I_2(s,t) = 0.8;\ n_2^s = 4500;\ n_2^t = 4000 \cdot a,$$

where a is a parameter from the interval $(0\%, 100\%)$.

After calculations we get:

Tab. 1. Indexes $\overline{I}$ and $\hat{I}_A$ and their coefficients depending on the *a*-parameter.

Parameter *a*	g_1	g_2	β_1	β_2	$\overline{I}$	$\hat{I}_A$
100%	0.5	0.5	0.5	0.5	1	1
80%	0.531	0.469	0.525	0.475	1.012	1.009
50%	0.605	0.395	0.570	0.430	1.041	1.026
10%	0.852	0.148	0.635	0.365	1.140	1.054

Source: own calculations.

We can notice that the smaller value of a-parameter we assume, the bigger differences between values of presented indexes we get. For small value of a-parameter the influence of the labour productivity of the second enterprise on the global index should be scanty. In fact, the construction of the $\overline{I}$ index seems to be proper, because for $a = 10\%$ we have $g_1 << g_2$.

In the case of the $\hat{I}_A$ index the differences between β_1 and β_2 are not so significant.

It is possible to construct price indexes satisfying all the five postulates. In this paper we present two definitions of these indexes. The first formula is as follows:

$$\hat{I}_B = \sum_{i=1}^{N} \hat{\beta}_i \sum_{u=2}^{T} \hat{\alpha}_i^u I_i(u-1,u), \tag{21}$$

where (see Białek (2007))

$$\hat{\beta}_i = \frac{\dfrac{T}{\sum_{t=1}^{T}\dfrac{1}{n_i^t}}}{\sum_{j=1}^{N}\dfrac{T}{\sum_{t=1}^{T}\dfrac{1}{n_j^t}}} = \frac{\dfrac{1}{\sum_{t=1}^{T}\dfrac{1}{n_i^t}}}{\sum_{j=1}^{N}\dfrac{1}{\sum_{t=1}^{T}\dfrac{1}{n_j^t}}}, \tag{22}$$

$$\hat{\alpha}_i^u = \frac{\dfrac{2}{\dfrac{1}{n_i^{u-1}}+\dfrac{1}{n_i^u}}}{\sum_{t=2}^{T}\dfrac{2}{\dfrac{1}{n_i^{t-1}}+\dfrac{1}{n_i^t}}} = \frac{\dfrac{1}{\dfrac{1}{n_i^{t-1}}+\dfrac{1}{n_i^t}}}{\sum_{u=2}^{T}\dfrac{1}{\dfrac{1}{n_i^{u-1}}+\dfrac{1}{n_i^u}}}. \tag{23}$$

The second proposition is some modification of the previous one, namely:

$$\hat{I}_C = \sum_{i=1}^{N} \tilde{\beta}_i \sum_{u=2}^{T} \tilde{\alpha}_i^u I_i(u-1,u), \tag{24}$$

where $\tilde{\alpha}_i^u = \hat{\alpha}_i^u$ and

$$\tilde{\beta}_i = \frac{\dfrac{1}{\sum_{t=2}^{T}(\frac{1}{n_i^{t-1}} + \frac{1}{n_i^t})}}{\sum_{j=1}^{N} \dfrac{1}{\sum_{t=2}^{T}(\frac{1}{n_j^{t-1}} + \frac{1}{n_j^t})}} . \tag{25}$$

The following theorem is true:
Theorem: indexes $\hat{I}_B$ and $\hat{I}_C$ satisfiy all the postulates 1–5.

Remark
Certainly, for $T = 2$ we have:

$$\hat{\alpha}_i^u = \tilde{\alpha}_i^u = 1$$

and

$$\hat{\beta}_i = \tilde{\beta}_i = \frac{\dfrac{2}{\frac{1}{n_i^s} + \frac{1}{n_i^t}}}{\sum_{j=1}^{n} \dfrac{2}{\frac{1}{n_i^s} + \frac{1}{n_i^t}}} = \gamma_i .$$

Thus we come to the following conclusion: in the case of $T = 2$ we get $\hat{I}_B = \hat{I}_C = \overline{I}$ (see postulate no. 5). But for $T > 3$ formulas $\hat{\beta}_i$ and $\tilde{\beta}_i$ are different. For example for $T = 3$ we have:

$$\hat{\beta}_i = \frac{\dfrac{1}{\frac{1}{n_i^1} + \frac{1}{n_i^2} + \frac{1}{n_i^3}}}{\sum_{j=1}^{N} \dfrac{1}{\frac{1}{n_j^1} + \frac{1}{n_j^2} + \frac{1}{n_j^3}}} \neq \tilde{\beta}_i = \frac{\dfrac{1}{\frac{1}{n_i^1} + \frac{2}{n_i^2} + \frac{1}{n_i^3}}}{\sum_{j=1}^{N} \dfrac{1}{\frac{1}{n_i^1} + \frac{2}{n_i^2} + \frac{1}{n_i^3}}} .$$

We can notice that the share of the first and last moment of observations is smaller in case of $\tilde{\beta}_i$ definition. Thus, the use of the indexes $\hat{I}_B$ and $\hat{I}_C$ can depend on the type of time series which we have.

Example 2

Let us consider two enterprises (I, II) and $T = 3$ moments of their activity. Let us assume that we observe in case of the first enterprise: $n_1^1 = n_1^2 = n_1^3 = 400$, $I_1(1,2) = I_1(2,3) = 1.2$, and in case of the second enterprise: $n_2^1 = 200$, $n_2^2 = 220$, $n_2^3 = 10$, $I_2(1,2) = I_2(2,3) = 0.8$. After calculations we get:

Tab. 2. Coefficients of indexes $\hat{I}_A$, $\hat{I}_B$, $\hat{I}_C$ for the considered enterprises.

Enterprise I	Enterprise II
$\beta_1 = 0.736$	$\beta_2 = 0.264$
$\hat{\beta}_1 = 0.935$	$\hat{\beta}_2 = 0.065$
$\tilde{\beta}_1 = 0.92$	$\tilde{\beta}_2 = 0.08$
$\alpha_1^2 = 0.5,\ \alpha_1^3 = 0.5$	$\alpha_2^2 = 0.71,\ \alpha_2^3 = 0.29$
$\hat{\alpha}_1^2 = 0.5,\ \hat{\alpha}_1^3 = 0.5$	$\hat{\alpha}_2^2 = 0.916,\ \hat{\alpha}_2^3 = 0.084$
$\tilde{\alpha}_1^2 = 0.5,\ \tilde{\alpha}_1^3 = 0.5$	$\tilde{\alpha}_2^2 = 0.916,\ \tilde{\alpha}_2^3 = 0.084$

Source: own calculations.

Finally, we also get $\hat{I}_A = 1.094$, $\hat{I}_B = 1.174$, $\hat{I}_C = 1.167$.

CONCLUSIONS

The presented indexes of labour productivity ($\hat{I}_B$ and $\hat{I}_C$) seem to be well constructed. These indexes satisfy all the given postulates and we can use them when the number of observations – coming from the considered enterprises – is insufficient. Both the indexes $\hat{I}_B$ and $\hat{I}_C$ (unlike the index $\hat{I}_A$) limit the influence of the partial indexes (I_i) based on small number of observations on the global index of labour productivity.

REFERENCES

Balk M. (1995), *Axiomatic price index theory: a survey*, International Statistical Review 63, 69–93.

Białek J. (2006), *The Average Price Dynamics and Indexes of Price Dynamics – Discrete Time Stochastic Model,* Acta Universitatis Lodziensis, Folia Oeconomica 196, s. 155–172, WUŁ, Łódź.

Białek J. (2007), *Agregatowy indeks przeciętnej wydajności pracy*, Wiadomości Statystyczne 8, s. 1–13, Wydawnictwo GUS, Warszawa.

Białek J., Czajkowski A. (2008), *A proposition of the system of weights for aggregate indexes on the example of the index of labour productivity,* Acta Universitatis Lodziensis, Folia Oeconomica 216, Łódź

Domański Cz. (2001), *Metody statystyczne. Teoria i zadania*, Wydawnictwo Uniwersytetu Łódzkiego, Łódź.

Dumagan J. (2002), *Comparing the superlative Törnqvist and Fisher ideal indexes*, Economic Letters 76, 251–258.

Fisher F.M.. (1972), *The Economic Theory of Price Indices*, Academic Press, New York

Moutlon B., Seskin E. (1999)., *A preview of the 1999 comprehensive revision of the national income and product accounts* , Survey of Current Business 79 ,6–17.

Pielichaty E., Poszwa M. (1999), *Rachunek opłacalności inwestowania*, PWE, Warszawa

Seskin E., Parker P. (1998), *A guide to the NIPA'S* , Survey of Current Business 78, 26–68

Törnqvist L. (1936), *The Bank of Finland's consumption price index*, Bank of Finland Monthly Bulletin 10,1–8.

Zając K. (1994) , *Zarys metod statystycznych*, PWN, Warszawa

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PROPOZYCJA AGREGATOWEGO INDEKSU WYDAJNOŚCI PRACY

W pracy zaproponowano konstrukcje agregatowego indeksu wydajności pracy. Proponowane systemy wag wynikają z teoretycznych rozważań nad sytuacją, gdy liczba obserwacji pochodzących od któregoś z analizowanych przedsiębiorstw jest niewystarczająca.

Rozważania koncentrujemy na przypadku, gdy chcemy zmierzyć przeciętną, jednookresową dynamikę wydajności pracy posiadając dane pochodzące z $T > 2$ okresów.