A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
FOLIA PHILOSOPHICA 7, 1990

Janusz Maciaszek

ON DIFFERENT NOTIONS OF QUANTIFIERS

1. Introduction

Among the most central notions of logic seem to be quantifier and quantification. Although these terms were introduced in the 19th century, the very idea of quantification is older and it can be traced to antiquity. The present paper will attempt an explication of that notion. Aristotle's ideas of quantification have close correspondence with modern notion of generalized quantifier formulated by Lindström in opposition to classical quantifiers of predicate calculus. Modern notion of a generalized quantifier enabled the development of practical applications in linguistics and logic of induction. It is obvious that the two c l a s s i c a l quantifiers are useless in these subjects.

2. The notion of quantification in Aristotle's syllogistic

The earliest attempt to create a system of formal logic was Aristotle's syllogistic. Aristotle recognizes four categorical statements and specifies the rules of inference for them. Categorical statements represent relations between the interpretations of subjects and predicative words.

$$S\ a\ P \qquad \|S\| - \|P\| = \emptyset$$
$$S\ e\ P \qquad \|S\| \cap \|P\| = \emptyset$$
$$S\ i\ P \qquad \|S\| \cap \|P\| \neq \emptyset$$
$$S\ o\ P \qquad \|S\| - \|P\| \neq \emptyset$$

In set-theoretical terms copulas "a", "e", "i", "o" can be interpreted as binary relations between the subsets of a domain of interpretation E, (i.e. the interpretation of "a" in E is the relation of inclusion: for any X, Y ⊆ E holds (X, Y) ∈ ‖a‖ iff X ⊆ Y, where ‖a‖ denotes the symbol "a"). Making use of Lindström's generalized quantifier the above symbols can be treated as generalized quantifier symbols characteristic for syllogistic language. These symbols have set-theoretical denotations in the modified model-theory.

On account of the development of mathematics, modern predicate logic with the classical notion of quantifier has been formulated. However, in spite of its usefulness in mathematics and metamathematics, Aristotle's intuition of quantification has been lost.

3. Classical predicate calculus and its extensions

First-order predicate calculus is not a mere extension of syllogistic. It has different and reacher dictionary and syntax. There is a significant difference of syntactical roles between copulas in sylogistic and c l a s s i c a l quantifiers in predicate calculus. Symbols ∀ and ∃ are operators that convert propositional functions to propositions; in the classical theory of models they have no explicit interpretation, they are usually interpretated in the context of a definition of satisfying a formula beginning with quantifier symbol by a valuational function.

From the point of view of linguistics, predicate calculus seems artificial and unsufficient, because of its poor possibility to express "quantitative words" of natural language.

The first step towards the extension of "expressive power" of predicate calculus consisted in the reformulating of syllogistic in terms of predicate logic. It was possible by introduction of the symbols of quantifiers of limited range[1].

$$S\ a\ P \text{ iff } (\forall s_{\langle x\rangle})\ P(x) \equiv (\forall x)(S(x) \longrightarrow P(x))$$

$$S\ i\ P \text{ iff } (\exists s_{\langle x\rangle})\ P(x) \equiv (\forall x)(S(x) \wedge \sim P(x)$$

$$S\ e\ P \text{ iff } (\forall s_{\langle x\rangle}) \sim P(x) \equiv (\forall x)(S(x) \longrightarrow P(x))$$

$$S\ o\ P \text{ iff } (\exists s_{\langle x\rangle}) \sim P(x) \equiv (\forall x)(S(x) \wedge \sim P(x))$$

---

[1] See: Z. S ł u p e c k i, L. B o r k o w s k i, Elementy logiki matematycznej i teorii mnogości, Warszawa 1984.

Thus the formulas of predicate logic become shorter and their syntax more adequate for natural language description.

The next extention was the introduction of numerical quantifiers[2].

- For exactly m holds... $-\sum\langle m\rangle$, $(m \geq 0)$

$$(\sum\langle o\rangle)xA(x) \equiv \sim \exists xA(X) \text{ for } m = 0$$

$$(\sum\langle m\rangle)xA(x) \equiv \exists x_1 \ldots x_m\,(A(x_1)) \wedge \ldots \wedge A\,(xm) \wedge x_1 \neq x_2 \wedge \ldots \wedge x_1 \neq x_m \wedge \ldots \wedge x_{m-1} \neq x_m \wedge \forall x_{m+1}\,(A(x_{m+1}) \longrightarrow (x_{m+1} = x_1 \vee \ldots \vee x_{m+1} = x_m)) \text{ for } m > 0$$

- For exactly n doesn t'hold... $-\pi\langle n\rangle$, $(n \geq 0)$

$$\pi\langle n\rangle xA(x) \equiv \sum\langle n\rangle x(\sim A(x)) \text{ for } n \geq 0$$

- For at most m holds... $-\sum[m]$, $m \geq 0$

$$\sum[m]xA(x) = \sum\langle o\rangle xA(x) \vee \ldots \vee \sum\langle m\rangle xA(x)$$

- For at most n doesn't hold... $-\pi[n]$, $n \geq 0$

$$\pi[n]xA(x) = \pi\langle o\rangle xA(x) \vee \ldots \vee \pi\langle m\rangle xA(x)$$

It is clear that both extentions are abbreviations of complex formulas of predicate calculus.

The symbols belong to the class of *first-order quantifier* symbols and can be defined in terms of *classical* quantifiers $\forall$ and $\exists$, within first-order predicate logic. The first real extension of predicate calculus is due to Mostowski[3], who introduced quantifiers of other classes.

4. Generalized quantifiers of Mostowski

One of the most important steps towards modern notion of quantifier was made by Mostowski[4]. Mostowski presents an explicite model-theoretic definition of quantifier as an interpretation of a quantifier symbol.

---

[2] See e.g. A. Mostowski, On a Generalization of Quantifiers, "Fundamenta Mathematica" 1957, vol. 44, pp. 12-36.

[3] Ibidem.

[4] Ibidem.

Definition 1[5]:

A q u a n t i f i e r limited to the domain of interpretation E is a function Q_E which assignes one of truth values 1, 0: to each interpretation of one-argument propositional function F on E, which satisfies the i n v a r i a n c e c o n d i c t i o n: $Q(F) = Q(F\phi)$, where ϕ is a p e r m u t a t i o n of E, such that for $a \in E$ holds $F\phi(\phi(a)) = F(a)$.

By this definition, for every quantifier symbol there is a family of subsets of E that "satisfies" its interpretation (a quantifier limited to E).

The i n v a r i a n c e c o n d i c t i o n enables to formulate a very interesting n u m b e r - t h e o r e t i c r e p r e s e n t a t i o n of a quatifier. Let (m, n) be the sequence of pairs of cardinal numbers satisfying the equation $m + n = \mathrm{card}(E)$. For each function T which assignes 1 or 0 to each pair we put:

$$Q_T(F) = T(\mathrm{card}\{a \in E: F(a) = 1\}, \mathrm{card}\{a \in E: F(A) = 0\})$$

Theorem 1[6]:

a) Q_T is a quantifier limited to E.

b) For each quantifier Q_E limited to E there is a function T such that $Q_T = Q_E$.

If $Q_T = Q_E$, then T is a s s o c i a t e d f u n c t i o n of Q_E This notion is very useful in defining the specific quantifiers by means of the conditions under which the a s s o c i a t e d f u n c t i o n of a quantifier posesses value 1.

Examples:

1. Classical quantifiers

$$\exists_E: m \neq 0$$
$$\forall_E: n = 0$$

2. Numerical quantifiers

$$\Sigma\langle m\rangle_E: m = m$$
$$\Sigma[m]_E: m \leq m$$
$$\Pi\langle n\rangle_E: n = n$$
$$\Pi[n]_E: n \leq n$$

---

5 Ibidem.

6 Ibidem.

3. Non-first-order quantifiers

S_E : $m \leq \aleph_0$ - for denumerably many holds

$S_{\sim E}$: $m < \aleph_0 \vee n < \aleph_0$ - for finitely many holds or for finitely many doesn't hold

4. Trivial quantifiers (defined in set-theoretical terms)

P_E : for every F holds $P_E(F) = 1$

$\emptyset_E$: for every F holds $\emptyset_E(E) = 0$

It is also possible to define *duals* and *boolean combinations* of quantifiers.

Definition 2[7]:

Let $T(m, n)$ be *associated function* of Q_E. A quantifier Q^*_E with *associated function* $T^*(m, n) = \sim T(n, m)$ is a *dual* of Q_E (e.g. $\exists_E = \forall^*_E$).

Definition 3[8]:

$Q_E = \sim Q'_E$ iff $Q_E(F) = \sim Q'_E(F)$

$Q_E = Q'_E \vee Q''_E$ iff $Q_E(F) = Q'(F) \vee Q''(F)$

$Q_E = Q'_E \wedge Q''_E$ iff $Q_E(F) = Q'(F) \wedge Q''(F)$

for every F.

The present theoretical background enables to examine the *boolean algebra* $\mathfrak{A}$ of quantifiers limited to *denumerable* set E. $\mathfrak{A}$ is isomorfic to the *product* of three *boolean algebras*:

$$\circledcirc = \boldsymbol{\pi} \times \boldsymbol{\pi} \times (\{1, 0\}, \sim, \vee, \wedge)$$

where $\boldsymbol{\pi} = (P(N), ', \cup, \cap)$ and $P(N)$ is a powerset of all real numbers (including 0). It occurs that quantifiers of $\mathfrak{A}$ belong to one of the four classes: $\| \emptyset_E \|$, $\| S_E \|$, $\| S_{*E} \|$ and $\| S_{\sim E} \|$ represented by the triples of $\circledcirc$ respectively:

$$P(N)fin \times P(N)fin \times \{0\}$$
$$P(N)infin \times P(N)fin \times \{0\}$$
$$P(N)infin \times P(N)fin \times \{1\}$$

---

7 Ibidem.

8 Ibidem.

$$P(N)infin \times P(N)fin \times \{1\}$$
$$P(N)infin \times P(N)infin \times \{0\}$$

The class $\| \emptyset_E \|$ is the class of f i r s t - o r d e r quantifiers. Other quantifiers can't be defined in terms of $\forall_E$ and $\exists_E$ only.

Clearly, a s s o c i a t e d f u n c t i o n s of Mostowski's g e n e r a l i z e d q u a n t i f i e r s of $\mathfrak{U}$ are of the form mRm, nRn and their boolean combinations, where R is a relation on numbers: "=", "<", ">", "≤", "≥" and n, m < card(E). It is possible to express other interesting quantifiers not considered here, e. g. for more than half holds...: $m > n$.

Mostowski managed to give an e x p l i c i t definition of a quantifier as an interpretation of a q u a n t i f i e r s y m b o l and pointed out how to define specific quantifiers by means of set theory or a s s o c i a t e d f u n c t i o n s. Mostowski didn't use terms of t h e o r y o f m o d e l s but his ideas were continued by Lindström who found their model-theoretical form.

5. A formal model-theoretical interpretation of quantifiers

A final generalization of a notion of a quantifier was introduced by Lindström. Lindström[9] has given a formal model-theoretical definition of a g e n e r a l i z e d q u a n t i f i e r as an interpretation of a q u a n t i f i e r s y m b o l.

Definition 4[10]:

A quantifier Q is a class of r e l a t i o n a l s t r u c t u r e s of type $t \in N^n$ such that Q is closed under isomorphism.

Choosing a special domain of interpretation we get a definition of a g e n e r a l i z e d q u a n t i f i e r l i m i t e d t o E as a relational structure of type $t \in N^n$.

$$Q_E = \langle E, R_1 \subseteq P(E^{t1}), \ldots, R_n \subseteq P(E^{tn}) \rangle$$

---

[9] P. L i n d s t r ö m, First Order Predicate Logic with Generalized Quantifiers, "Theoria" 1966, pp. 186-195.

[10] Ibidem.

closed under permutations of E. The type $t = \langle t_1, \ldots, t_n \rangle (t_i \geqslant 0)$ is a sequence of arities of relations $R_1, \ldots, R_n$.

It is obvious that Mostowski's *generalized quantifiers* are a special case of Lindström's, as they are *relational structures* of type $\langle 1 \rangle$.

$$Q_E = \langle E, R \subseteq P(E) \rangle$$

The copulas of Aristotle's syllogistics are quantifier symbols of type $\langle 1. 1 \rangle$, e.g.

$$\| a \|_E = \langle E, R_1 \subseteq P(E), R_2 \subseteq P(E): R_1 - R_2 = \emptyset \rangle$$

Lindström's idea of quantification seems to be a generalization of ancient intuitions of Aristotle, it will be obvious when examining *logic of induction* based on *generalized quantifiers*.

The notion of a *type of a quantifier* is bound to syntactical role of a respective *quantifier symbol* Q of a *generalized first-order predicate logic*. If $t = \langle t_1, \ldots, t_n \rangle$, then n expresses number of formulas and t_i number of variables in each formula, bound by Q, e.g.

$$\Phi = Qx_{11} \ldots x_{1t_1} \ldots\ldots x_{n1} \ldots x_{nt_n} (\phi_1, \ldots, \phi_n)$$

Let M_{IE} be a *valuational function* (under interpretation I in a domain E). Every function M_{IE} and formula Φ fix a *relational structure* of a type t.

$$[M_{IE}, \Phi] = \langle E, \{(a_{11}, \ldots, a_{1t1}) \in E^{t1}: \mathrm{stsf}[M_{IE}(a_{11}/x_{11}, \ldots, a_{1t1}/x_{1t1}), \phi_1]\}, \ldots, \{(a_{n1}, \ldots, a_{ntn}) \in E^{tn}: \mathrm{stsf}[M_{IE}(a_{n1}/x_{1n}, \ldots, a_{ntn}/x_{ntn}, \phi_n]\} \rangle$$

where a_{ij} is an object of E, such that $M_{IE}(x_{ij}) = a_{ij}$ and $\mathrm{stsf}[M_{IE}(\ldots), \phi_i]$ means that M_{IE} satisfies the sub-formula ϕ_i.

Les P_{jE} and Q_E be interpretations I in E of a predicate symbol P_i of arity m and a quantifier Q. *Generalized first-order predicate logic* is an extension of first-order predicate logic (with identity), by adding *generalized quantifiers* of various types.

The definition below is an inductive definition of satysfying a formula by a valuational function M_{IE}.

Definition 5[11]:

1. $stsf[M_{IE}, P_j(x_{j1}, \ldots, x_{jm})]$ iff $\langle M_{IE}(x_{j1}), \ldots, M_{IE}(x_{jm})\rangle \in P_{jE}$.

2. $stsf[M_{IE}, x_k = x_l]$ iff $M_{IE}(x_k) = M_{IE}(x_l)$.

3. $stsf[M_{IE}, \Phi]$ where Φ begins with quantifier symbol Q iff $[M_{IE}, \Phi] \in Q_E$

Examples:

1. Well known sentential connectives: $\sim$, $\vee$, $\rightarrow$, $\wedge$, $\leftrightarrow$, are in terms of Lindström quantifier symbols of type $\langle 0\rangle$ and $\langle 0, 0\rangle$, as their interpretations are relational structures of arities 0. This resembles Łukasiewicz's notation of sententional connectives.

2. Let W_E be a quantifier of type $\langle 1, 1\rangle$, $\langle E, R_1, R_2\rangle \in W_E$ iff card $(R_1) \geqslant$ card(R_2). The quantifier denotes: There are no less... then...

The examples show the utility of Lindström concept of a quantifier. The next chapter deals with the practical application of this idea.

6. Practical applications of a modern notion of a generalized quantifier

Accordingly to Aristotle's intuition, the most important quantifiers in the natural language investigations are of the type $\langle 1, 1\rangle$ since they fit for the specific syntax of a natural language. They were thoroughly examined during last ten years.

The studies were started by Barwise and Cooper[12], who treat a quantifier as a noun phrase -NP:

---

[11] Ibidem.

[12] J. Barwise, R. Cooper, Generalized Quantifiers and Natural Language, "Linguistics and Philosophy" 1981, No. 4, pp. 159-219.

Q (quantifier) = NP

D (determiner) Set expresion

Examples:

$\|$ Every man $\|$ $= \{X: \|man\| \subseteq X\}$

$\|$ Two boys $\|$ $= \{X: card(\|boy\| \cap X) = 2\}$

$\|$ most women $\|$ $= \{X: card(\|woman\| \cap X) > card(\|woman\| - X)\}$.

$\|$ Every (thing) $\|$ $= \{X: X = E\}$.

where every, two, most etc. are d e t e r m i n e r s. In terms of Barwise and Cooper interpretation every quantifier is of type ⟨1⟩ and is closely related to the quantifier of l i m i t e d r a n g e.

The main goal of Barwise and Cooper's paper was to formulate a formal language LGQ capable of formalizing a fragment of English. The LGQ syntax is closely related to the English syntax, e.g. the special function in LGQ is assigned to a term "thing" which will manifold compound expressions like "everything", "nothing", "something".

Other papers concerning quantifiers in natural language[13] deal with a quantifier identified with a determiner (it is of type ⟨1, 1⟩. As quantifiers are binary relations on subsets of E it is possible to impose some conditions on them. First condition is a u n i v e r s a l one, it is satisfied by all quantifiers.

CONSERV. (conservativity)

For every E and every A, $B \subseteq E$ holds: $Q_E AB$ iff $Q_E A(A \cap B)$.

This condition implies for example that e.g. some men are running iff some men are running men. Second condition is not a universal one. It is not satisfied by some special quantifiers e.g. many, some, few, a few.

QUANT. (quantity)

For every E, E', every bijection $\phi: E \longrightarrow E'$ and every A, $B \subseteq E$ holds: $Q_E AB$ iff $Q_{E'} \phi(A)\phi(B)$.

---

[13] See: J. van B e n t h e m, Questions about Quantifiers, "The Journal of Symbolic Logic" 1984, vol. 49, pp. 443-446; i d e m, Esseys in Logical Semantics, Amsterdam 1985; D. W e s t e r s t å h l, Some Remarks on Quantifiers, Göteborg 1982.

This condition is closely related to Mostowski's and Lindström's condition of invariance.

It is also possible to formulate another condition satisfied only by special groups of quantifiers. The monotonicity conditions of quantifiers are fundamental in defining the class of first-order quantifiers of type <1, 1>. For quantifiers that satisfy CONSERV. and QUANT. there exists possibility to introduce number-teoretical representation[14], that enables to examin their relational properties.

Logic of induction is another domain of application of generalized quantifiers. A quantifier as a "relation on relations" is very useful to express various correlations searched by empirical sciencies. Let us look at Aristotle's scheme of induction:

S_1 is P

S_2 is P

....

S_n is P

S a P (every S is P)

All other schemes of induction have the same form but as other quantifiers are used it is possible to express various correlations of interest.

The practical utility of generalized induction was possible thanks to computers. The group of Czech scientists have invented special computer methods of obtaining induction hypothesis, called GUHA methods - General Unary Hypothesis Automaton[15]. The GUHA methods are based on number-theoretic representation of generalized quantifiers. Whith support of associated functions of quantifiers it is possible for a computer to search interesting correlations in empirical data. Very important sort of generalized quantifiers used in GUHA methods are statistical quantifiers. Their associated functions ex-

---

[14] See: A. Mostowski, op. cit.

[15] See: P. Hájek, T. Havránek, Mechanizing Hypcthesis Formation, Springer Verlag, Heidelberg 1977.

press well known statistical tests. Statistical quantifiers were applicated in sociology, medicine, linguistics and even industry.

The given examples prove that modern notion of a quantifier due to Lindström is a very important issue of same branches of contemporary logic.

University of Łódź
Poland

Janusz Maciaszek

O RÓŻNYCH POJĘCIACH KWANTYFIKATORA

Jednym z podstawowych pojęć logiki jest pojęcie kwantyfikacji i związane z nim pojęcie kwantyfikatora. Pierwsze problemy związane z kwantyfikacją pojawiły się wraz z powstaniem sylogistyki Arystotelesa, chociaż nie istniało wówczas pojęcie kwantyfikatora. Powstanie rachunku predykatów I rzędu oraz wprowadzenie klasycznych kwantyfikatorów ∀ i ∃ zmieniło zupełnie sens pojęcia kwantyfikacji.

Prace Mostowskiego i Lindströma poświęcone kwantyfikatorom uogólnionym wprowadziły zupełnie nowe rozumienie kwantyfikacji, które zaowocowało wieloma praktycznymi zastosowaniami w lingwistyce logicznej i logice indukcji.

Głównym celem pracy jest wykazanie, że koncepcja kwantyfikatorów uogólnionych Lindströma nawiązuje bezpośrednio do intuicji Arytotelesa, zaś spójki "a", "e", "i", "o" występujące w języku sylogistyki mogą być uznane za symbole kwantyfikatorów uogólnionych typu ⟨1, 1⟩.