A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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# THREE ASPECTS OF DECIDABILITY

Metamathematical problems of decidability recur in logic in the shape of the question: is a given logical theory decidable or undecidable. Basic results concerning different mathematical and logical theories were achieved mainly in the thirties and the fourties of our century. No wonder that the notion of decidability is usually combined with the terminology of that time. However, I would suggest that perceiving at least three different periods in the studies on decidability is fully justified.

The first period covers the years till 1930. At that time the problem of decidability acquired some importance, mainly on account of the studies connected with Hilbert's programme. The second period falls on the thirties and the fourties. Then, on the basis of various approaches, a narrow mathematical concept of decidability was built up, which rendered possible an equally precise definition of decidability. Finally, in the third period that started about the early sixties, there appeared purely mechanical procedures for solving various problems, thus stimulating the studies on practical decidability. The notion of decidable theory in these three periods will be the basic theme of this paper[1].

[1] There is a great variety of interesting works concerned with recursive functions and the problems of decidability and undecidability. The most valuable are, for example, A. G r z e g o r c z y k, Zagadnienia rozstrzygalności, Warszawa 1957; A. T a r s k i, A. M o s t o w s k i, R. M. R o b i n s o n, Undecidable Theories, North-Holland, Amsterdam 1953; D. V a n D a l e n, Algorithmus and Decision Problems: a Crash Course in Recursion Theory, [in:] Handbook of Philosophical Logic, eds D. M. Gabbay,

1. Period 1. The decidability of a theory was defined in terms of the informal intuitive notion of an effective method. What is an effective method? Its paradigm can be found in simple and well-known mathematical algorithms as, for example, Euclid's algorithm for finding the greatest common divisor of two positive integers not-relatively prime; effective method is a method which allows to solve a problem or a class of problems in a finite number of steps. Accordingly, a general problem (a class of specific questions) is decidable if there exists an effective method of solving every question of that class, e.g. general question: "is a given number n prime?" is decidable. Consequently, a theory T is decidable if its general problem: "is a formula valid in it" is decidable[2].

Metamathematical studies advanced by Hilbert and his school before 1930 aimed at proving the consistency and decidability of the main logical and mathematical theories. At that time classical mathematics was generally claimed to be not only consistent but also decidable. That thesis became even more convincing after Russell and Whitehead "Principia Mathematica". Such attitude made the studies on decidability focus on new effective methods and new decidable theories; their results being referred only to mathematical theories or their fragments, e.g., L. Löwenheim[3] proved the decidability of first-order monadic predicate logic (1915), E. Post[4], the decidability of standard truth-functional propositional logic (1921), M. Presburger[5]; the

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F. Guenthner, 1983, vol. 1, pp. 409-478. But all of them, however, deal only with the first two periods of studies on decidability. As far as the author knows, some problems of practical decidability are discussed only in: M. O. R a b i n, Decidable Theories, [in:] Handbook of Mathematical logic, ed. J. Barwise, North-Holland, Amsterdam 1977.

[2] See, e.g. A. G r z e g o r c z y k, op. cit.; A. T a r s k i, A. M o s t o w s k i, R. M. R o b i n s o n, op. cit.

[3] L. L ö w e n h e i m, Über Möglichkeiten im Relativkalkül, "Mathematische Annalen" 1915, vol. 76, pp. 447-470.

[4] E. P o s t, Introduction to a General Theory of Elementary Propositions, "Americal Journal of Mathematic" 1921, vol. 43, pp. 163-185.

[5] M. P r e s b u r g e r, Über die Vollständigkeit eines gewissen Systems der Arithmetik genzer Zahlen, in welchem die Addition als einzige Operation hervortritt; [in:] Sprawozdanie z I kongresu matematyków krajów słowiańskich, Warszawa 1929, pp. 92-101.

decidability of first-order number theory with addition but without multiplication (1929), T. Skolem[6]: the decidability of first-order number theory with multiplication but without addition (1935). Moreover some effective methods were invented as, for example, the reduction to conjunctive normal form, truth-table method (independently - E. Post[7] and L. Wittgenstein[8]), and the methods of proving the decidability for some specified theories. Beyond the limits of the basic research work were, houever, undecidable problems and theories, namely those with no effective methods for solving their general problem. At least two reasons can be given for that: 1) after Hilbert had realized his programme, undecidable problems were assumed not to belong to mathematics; 2) the failure in the field of undecidable theories was to some extent due to unprecise definition of decidability of a theory. As has been stated above, that definition based on the intuitive concept of effective method. In the second period the effective method was more precisely defined, which was connected with the introduction of its mathematical counterpart: the concept of recursiveness.

2. Period II. The second period belongs to the most prolific in the history of decidability studies, and thus it can be treated as a whole. When Gödel's work[9] was published in 1931, it became evident that the deductive power of formalized systems is limited and, consequently, not always the right procedure (i.e. the one that solves the problem in every situation), for a given problem of a theory can be settled. Based, on the one hand, on the intuitive notion of what is calculable, and, on the other hand, on the existing algorithms, several different mathematical definitions of recursiveness were formulated (that is, definitions of recursive function, Gödel in [4] for the first time, and of recursive set), which in consequence made precise definition of decidability

[6] T. S k o l e m, Über einige Satzfunktionen in der Arithmetik, Skrifter Utgitt av Det Norske Videnskaps-akademi i Oslo, I. Mat.-Naturv. Klasse 1930, No. 7, Oslo 1931.

[7] E. P o s t, op. cit.

[8] L. W i t t g e n s t e i n, Tractatus logico-philosophicus, Routledge and Kegan Paul, London 1922.

[9] K. G ö d e l, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, "Monatschrift für Mathematik und Physik" 1931, Bd. 38, pp. 173-198.

possible. The most notable results originated with A. Church, K. Gödel, S. C. Kleene, J. B. Rosser, E. Post, A. Turing, and later A. Markov. The most perspicuous definition of recursive functions (which at the same time characterizes their construction) is founded on the notions of effective minimum, primitive recursion[10] and composition operations[11]. Namely, if we consider easily calculable functions:

$$(*)\ Z(x) = 0,\quad S(x) = x + 1,\quad U_n^i(x_1, \ldots, x_n) = x_i \quad \text{for } i \leq n,\ n = 1, 2, \ldots$$

then a class of recursive functions can be defined as the minimal class of functions containing (*) and closed under the operations of composition, primitive recursion and effective minimum. Making use of this observation, recursive set was defined as a set for which there exists a recursive function characteristic for this set; and decidable theory - as a theory with recursive set of theorems.

In 1936 Turing[12] proposed to define computable function in terms of computing machines he had invented. Function f is computable if there can be designed Turing machine for it, such that it may be capable of printing succeeding values of that function on the tape. Independently, E. Post made a similar analysis. Next proposition concerning λ-definable functions was introduced by Church and Kleene; it was based on Church's λ-calculi[13], where

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[10] The operation of effective minimum. The operation of minimum leads from a function $g(y, x_1, \ldots, x_n)$ of $n + 1$ arguments to a function $f(x_1, \ldots, x_n)$ of $n$ arguments, whose value for given $x_1, \ldots, x_n$ is the least value of $y$, if one such exists, for which $g(y, x_1, \ldots, x_n) = 0$ and which is undefined if no such $y$ exists. If for given $x_1, \ldots, x_n$ there always exists the least value of $y$, then the operation of minimum is called effective. The operation of primitive recursion. This operation associates with the given total functions $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_{n+2})$ the function $h(x_1, \ldots, x_{n+1})$, where: $h(0, x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$, $h(k + 1, x_1, \ldots, x_n) = g(k, h(k, x_1, \ldots, x_n), \ldots, x_n)$.

[11] See, e.g. A. Grzegorczyk, op. cit.

[12] A. Turing, On Computable Numbers, with an application to the Entscheidungsproblem, "Proceedings London Mathematical Society" 1936/1937, ser. 2, vol. 42, pp. 230-265.

[13] See: A. Church, An Unsolvable Problem of Elementary Number Theory, "American Journal of Mathematic" 1936, vol. 58, pp. 345-363.

due to λ-operator producing functional symbols, the consequent notation is unified[14]. Also other different aproaches were tried later, bringing altogether various, more precise definitions of the intuitive notion of calculability. The important fact, however, is that in all those aproaches the same class of recursive functions was defined since:

1) The class of recursive functions is equivalent to the class of λ-definable functions[15].

2) The class of computable functions is equivalent to the class of λ-definable functions[16].

This fact made Church and Turing put forward a hypothesis that the intuitive notion of what is calculable has its precise mathematical counterpart. That hypothesis called Church's Thesis, Turing's Hypothesis, or the most rightly, Church-Turing Hypothesis, can have manifold formes. Let us present two of them:

H1) Every recursive function is effectively calculable.

H2) Algorithmic processes in the intuitive sense can be realized by means of Turing machines.

Church-Turing Hypothesis is not proved. Various approaches to what is calculable, however, make us belive it is plausible; when its validity is assumed, the undecidability of, for instance, predicate calculi or of the halting problem can be showed.

Not only defining decidability in terms of mathematics but also the fact of surmounting the difficulties connected with the possible realization of Hilbert programme resulted in various conclusions concerning the undecidability of the main mathematical and logical theories. The best known results (proved in 1936) are as follows:

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[14] λ-definable function can be determined as follows: function F is called λ-definable if there exists a formula $\underline{F}$ such that, if F(m) = r and $\underline{m}$ and $\underline{r}$ the formulas corresponding in this calculi to the positive integers m and r, then $\underline{F}(\underline{m})$ conv $\underline{r}$, where "conv" is a relation between formulas $\underline{F}(\underline{m})$ and $\underline{r}$ iff the formula $\underline{r}$ can be derived from $\underline{F}(\underline{m})$ by means of appropriate operations introduced in λ-calculi and called conversion (most exactly: any finite sequence of these operations is called a conversion).

[15] A. Church, op. cit.

[16] A. Turing, op. cit.

3) First-order predicate logic is undecidable[17].

4) Peano's arithmetic is undecidable[18].

and so are first-order theories of groups, rings, fields and lattices[19]. Each of those cases employs that there is no effective method to decide whether any given formula of a theory is valid or not. In terms of decidability the set of theorems of a given theory is said to be recursive. It becomes evident that if we additionally introduce the notion of recursively enumerable set as a set of values of a recursive function, then Gödel's First Theorem can be formulated as follows:

5) The set of Gödel numbers of arithmetical theorems is recursively enumerable but is not recursive.

The thorough studies on the decidability problems based on the notion of recursiveness (this brief presentation does not even mention all the significant results) disclose their similarity to computer practice. On the one hand, the models of mathematical machines created by Turing and Post (Turing machine and Post machine are equivalent) are not in fact mechanisms but mathematical concepts. Speaking precisely, they are deductive systems: by means of formal transformation made in accordance with the rules given in advance, they form new sequences of symbols out of the symbol sequences given in the input. On the other hand, however, it is evident that today every really existing computer can be reduced to Turing machine. From the theoretical point of view, Turing machine, and hence also every computer can solve any computable problem. Consequently, the problems of decidable theories can be reduced to some appropriate computer procedures. Are the procedures that solve the problems of decidable theories feasible for computer? This question was of special attention in the third period.

3. Period III. From what has been stated above too hasty a conclusion can be inferred that, as regards the decidable theory,

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[17] A. Church, A Note on the Entscheidungsproblem, "The Journal of Symbolic Logic" 1936, vol. 1, pp. 40-41; Correction, ibidem, pp. 101-102.

[18] Ibidem. Church obtained this result having assumed Church-Turing hypothesis and consistency of arithmetical system.

[19] See, e.g. A. Tarski, A. Mostowski, R. M. Robinson, op. cit.; D. Van Dalen, op. cit.

the question whether a formula is valid creates a trivial problem. However, this is not the case since decidable procedures manifest evident computational complexity as, for example, decidable theory of arithmetics with addition but without multiplication. In 1974 Fischer and Rabin[20] showed that in the case of elementary theory of addition, for every decidable procedure, there can be singled out a sentence of a length n for which given procedure needs $2^{2^n}$ steps in order to provide the answer. Let us notice that computation in more than $2^{2^{30}}$ steps is impossible (i.e., we say: it is not practical decidability). Similar results were achieved in the theory of linear-order and the second-order weak theory with one succesor[21]. Thus, what should be inevitably defined at this moment, is such a notion of practical decidability, that would embrace the computational complexity of the procedures. Therefore, it is accepted that if a function is practically calculable then its computational complexity is determined by the fact that the number of steps necessary for producing the answer grows polinomially depending on to the length of a input word. Cleary, if this relation is exponential the due function is not computable.

The studies of the third period are not completed, therefore, it is difficult to foretell what results will be obtained. However, it is possible to specify its objective: to construct new, interesting, practically decidable theories.

4. Conclusions. Each of the three periods has essentially the same notion of decidable theory: a theory is decidable if a method which is claimed to be effective settles whether a formula is valid. Whereas what makes them vary is caused by the fact, that they concentrate on different problems concerning decidability; in the first, "intuitive" period the stress was put on the decidability of various theories: the second mathematical one managed to prove many elementary theories to be undecidable and built up a hierarchy of degrees of undecidability; the third period, on the other hand, was marked by computable practices and restored decidable theories but from the point of view of practically realized procedures.

[20] J. M. F i s c h e r, M. O. R a b i n, Super Exponential Complexity of Presburger's Arithmetic, "SIAM AMS Proceedings" 1974, No. 7, pp. 27-41.

[21] See: M. O. R a b i n, Decidable Theories.

In spite of the fact that the studies on decidable theories are still making progress, I think two facts are worth mentioning:

$1^o$. The range of the problems connected with decidability was successively reduced by every next period. If the denotation of the concept of decidable theory (problem) is refered to us $D_I$, $D_{II}$, $D_{III}$, the following inclusion is true:

$$D_I \supseteq D_{II} \supseteq D_{III}$$

Les us, however, stress the pecularity of that inclusion: Church-Turing Thesis states the equivalence between intuitive calculability and mathematical recursiveness, which cannot be entirely proved (although it is obvious that every recursive function is intuitively calculable). In order to reject the thesis, there must be pointed out an intuitively calculable function which is at the same time not recursive. The above reasoning deals naturally with the concept of decidability, and this fact is the source of an unexpected problem: we are not able to single out such an element a, that $a \in D_I$ and $a \notin D_{II}$, which would prove that $D_I$ is in fact a greater class (according to Church-Turing Thesis no such an element should be formed). Class $D_{II}$, on the other hand, is essentially greater then class $D_{III}$, which is obvious since the studies were limited only to practically decidable theories. Here, however, a sensible practically decidable theory can be hardly appointed.

$2^o$. The problems of decidability and the practice are (entirely) analogous; the former stem from the studies on calculating algorithms: they were combined with computer practice in the second period, and dealt with the concrete decidable theories and solving procedures in Period III. The question can be put if this analogy is really close? For the time being the answer is negative. The results of the studies on practical decidability show that algorithms can be hardly expected to be practically realized since solving procedures are of too high computational complexity. However, some algorithms for data from a finite set can be realized in practice. This is an important fact in the studies of automatic theorem proving since the possibilities to form theorem, proving procedures (for a formula of limited length) are not excluded.

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## TRZY POJĘCIA ROZSTRZYGALNOŚCI

W artykule dokonano krótkiej analizy pojęcia rozstrzygalności w trzech kolejnych okresach jej rozwoju.

Pierwszy okres obejmuje lata przed rokiem 1930, w których problemy rozstrzygalności uzyskały właściwe znaczenie głównie za sprawą badań związanych z programem Hilberta. Okres drugi obejmuje lata trzydzieste i czterdzieste. Wykorzystując rozmaite podejścia, wypracowano wówczas ścisłe, matematyczne pojęcie obliczalności, co umożliwiło przyjęcie równie precyzyjnej definicji rozstrzygalności. Z kolei okres trzeci rozpoczyna się mniej więcej od początku lat sześćdziesiątych, kiedy to w związku z procedurami możliwymi do realnego urzeczywistnienia pojawiła się refleksja nad praktyczną rozstrzygalnością.