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SINGLE FUNCTIONAL INDEX QUANTILE REGRESSION FOR INDEPENDENT FUNCTIONAL DATA UNDER RIGHT-CENSORING

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Abstract: The main objective of this paper was to estimate non-parametrically the quantiles of a conditional distribution based on the single-index model in the censorship model when the sample is considered as independent and identically distributed (i.i.d.) random variables. First of all, a kernel type estimator for the conditional cumulative distribution function (*cond-cdf*) is introduced. Then the paper gives an estimation of the quantiles by inverting this estimated *cond-cdf*, the asymptotic properties are stated when the observations are linked with a single-index structure. Finally, a simulation study was carried out to evaluate the performance of this estimate.

Keywords: censored data, functional data, kernel estimator, normality, non-parametric estimation, small ball probability.

1. Introduction

The estimation of a conditional model, because of the variety of its application possibilities, is an important question in statistics. This subject can (and must) be approached from several angles depending on the complexity of the problem posed: the possible presence of censorship in the observed sample (a common phenomenon in medical applications for example), the possible presence of dependence between the observed variables (e.g. a common phenomenon in seismological, and econometric applications), and the presence of explanatory variables. Many techniques have been studied in the literature to deal with these different situations, but they all only deal with real or multi-dimensional explanatory random variables.

The technical progress made in the collection and storage of data make it possible to have more and more often functional statistical data: curves, images, tables, etc. These data are modeled as being the realizations of a random variable taking its values in an abstract space of infinite dimension, and the scientific community has naturally been interested in recent years in the development of statistical tools capable of processing this type of sample.

Thus, the estimation of conditional models in the presence of a functional explanatory variable from a simple index regression model is a topical question to which this article proposes to provide a first element of answer. After a brief bibliographic overview presented in Section 1, the conditional model for a functional explanatory variable is presented in Section 2, to which this work proposes an extension of the simple index model, when considering an explanatory random variable with values in an infinite dimensional space. Such a model was designated generically by a simple functional index model. Naturally, these methods have some drawbacks, and to overcome these, an alternative approach is naturally provided by semi-parametric modelling which supposes the introduction of a parameter on the regressors – these models are known in the literature as simple index models, which have two major advantages. First it is possible to generalise existing models, and then to remedy the problems of the scourge of dimension. These are single revealing direction models (or simple functional index models), and have the advantage of specifying the model to a minimum. The authors used a non-parametric link function having previously determined linear combinations of explanatory variables which contain the maximum information, thus alleviating the scourge of dimension. The idea of these models, in the case of conditional density estimation or regression, consists in reducing to covariates of a dimension smaller than the dimension of the space of variables, thus making it possible to overcome the problem of the scourge of dimension. These models make it possible to obtain a compromise between a parametric model, generally too restrictive, and a non-parametric model where the speed of convergence of the estimators deteriorates quickly in the presence of a large number of explanatory variables, for example, in the partially linear model one decomposes the quantity which one seeks to estimate, in a linear part and a functional part. This last quantity does not pose an estimation problem since it is expressed as a function of the explanatory variables of the defined dimension, thus avoiding the problems linked to the scourge of dimension.

This work proposes an extension of the simple index model when considering an explanatory random variable with values in an infinite dimensional space. Such a model is generically referred to as a model with a simple functional index. The main contribution of this paper lies in a double generalisation of the simple index model. On the one hand, the authors place themselves in a framework of functional random variables, and on the other, introduce hypotheses on the law of the explanatory random variable that are less restrictive than those usually used in the vector framework. First point convergence results were established. The non-parametric method only considers regularity assumptions. Naturally, these methods have some drawbacks, therefore an alternative approach was provided by semi-parametric modelling which supposes the introduction of a parameter on the regressors, these models are known in the literature as simple index models, with two major advantages which, firstly makes it possible to generalise the already existing models, and then to remedy the problems of the scourge of the dimension.

Non-parametric methods based on convolution kernel ideas, which are known to perform well in model estimation problems (conditional or not), are thus widely used in non-parametric estimation of conditional models. A wide range of literature in this area is provided by the bibliographic reviews of Singpurwallam and Wong (1983), Hassani, Sarda and Vieu (1986), Izenman (1991), Gefeller and Michels (1992) and Pascu and Vaduva (2003). The immediate consequence of progress in data collection processes is to offer statisticians the opportunity to increasingly have observations of functional variables. Ramsay and Silverman (2005), and Ferraty and Vieu (2006) proposed a wide range of statistical methods, parametric or non-parametric, recently developed to deal with various estimation problems involving functional random variables (i.e. with values in a space of infinite dimension). To date, such statistical developments for directionally revealing functional variables occurred rarely in this context, despite the obvious potential for their application. In practice, in medical applications in particular, one may be in the presence of censored variables. This problem is usually modelled by considering positive variable C called (censorship), and the observed random variables. Such censoring models have been extensively studied in the literature on real and multidimensional random variables, and in non-parametric frameworks, particularly in kernel techniques (see van Keilegom and

Veraverbeke, 2001; Lecoutre and Ould-Saïd, 1995; Padgett, 1988; Tanner and Wong, 1983), for not a necessarily exhaustive sample of the literature in this field).

Other authors have been interested in the estimation of conditional models from censored or truncated observations (see, e.g. Khardani et al., 2010, 2011, 2012; Lemdani, Ould-Saïd, and Poulin, 2009; Liang and Una-Álvarez, 2010; Ould Saïd and Tatachak, 2011; Ould Saïd and Djabrane, 2011). Many statistical applications had to involve a variable of duration denoted T, designating the time elapsed until the occurrence of the event of interest. These types of variables are observed in various fields such as in reliability (first failure for a machine, lifespan of a material, etc.), in medicine (death or remission for a patient, etc.) in economics and insurance (duration of unemployment, time between two successive breakdowns of a device, etc.). A specificity of these models is the existence of incomplete observations, for which the variable of interest is not completely observed for all the data in the sample. This work studied models where the duration is likely to be right-censored by then calling on techniques adapted to this type of context to take into account the censored observations without losing too much information on T. This study was only interested in the case of right-censored random data. This corresponds to the model frequently used in practice. For example, during a therapeutic trial this can be caused by a loss of sight (the patient leaves the study in progress), the stopping or the change of a treatment, in which case the patients are excluded from the study, or the study ends when some individuals have not experienced the event.

The well-known functional regression model with scalar response postulates a relation between real random variable Y and functional random variable X.

A large class of flexible and useful tools for modelling regression operator r is presented by the simple functional index model. This consists in putting a semi--parametric dimension reduction approach on the model by introducing functional parameter θ. The main idea was to find the direction of θ on which the projection of covariate X captures the most information about answer Y. The considered model was a single revealing direction model (or simple functional index model). This approach arouses various interests. Firstly, to avoid the problems due to dimensionality that can be encountered in the purely non-parametric approach (Ferraty and Vieu, 2002)). The non-parametric estimation of the regression would no longer be affected by the scourge of dimension since it is a dependent function of θ which is of dimension 1. Finally, the estimation of functional parameter θ provides an easily interpretable tool. The simple index approach is well-known in the standard multivariate context for its interest in its predictive abilities, and for its interpretability attested by various works that appeared over the past two decades (Härdle, Hall, and Ichumira, 1993; Xia and Härdle, 2006). Extensions to the functional framework of such functional semi-parametric methodology have been the subject of extensive study in the literature. The first work linking the single index model and the non-parametric regression model for functional variables is made (Ferraty, Peuch, and Vieu, 2003) in the case of independent observations, and they established almost complete convergence. Their results were extended to dependent cases by (Aït-Saidi, Ferraty, and Kassa, 2005). Aït-Saidi, Ferraty, Kassa, and Vieu (2008) studied the case where the simple functional index is unknown; they proposed an estimator of this parameter based on the cross-validation technique. These results were extended to the multiple functional index models by Bouraine, Aït-Saidi, Ferraty and Vieu (2010). Ferraty, Park and Vieu (2011) proposed a new estimator of this parameter based on the idea of functional derivative estimation; the problem of the single index model to the functional data where the observations are censored does not seem to have been considered much in the literature, which makes this paper one of the more recent research work on the subject.

Moreover, the analysis of functional data being a branch of statistics that has been the subject of several recent studies and developments, this paper makes it possible to adapt the functional conditional models to censored data based on a single functional index structure.

The rest of the paper is arranged as follows: Section 2 presents the non-parametric estimator of the functional conditional model when the data are censored. Section 3 poses useful assumptions for the theoretical study, after which the point-wise almost complete convergence and the uniform almost complete convergence of the kernel estimator for these models (with rates) are established. Section 5 presents a simulation study in order to illustrate some properties of the resulting estimator.

In the censoring case, instead of lifetime T, the authors observed the censored lifetime of the items under study. That is, assuming that $(T_k)_{k\geq 1}$ is a stationary sequence of lifetimes which satisfy some kind of dependency and $(C_k)_{k\geq 1}$ is a sequence of i.i.d censoring rv with common unknown continuous G, where $Y_k = \min\{T_k, C_k\}$ and $\delta_k = 1_{T_k \leq C_k}$.

To ensure the identifiability of the model, it was supposed that $(T_k)_k$ are independent of $(C_k)_k$. Let $\left\{\left(Y_k, \delta_{k,} X_k\right)_k\right\}$ be a sequence of strictly stationary random vectors where $X_{k_{k\geq 1}}$is valued in infinite dimensional semi-metric vector space, and Y_k is real valued. To follow the convention in biomedical studies and as indicated before, the study assumed that $(C_k)_{k\geq 1}$ and $\{(X_k, T_k)_{k\geq 1}\}$ are independent; this condition is plausible whenever the censoring is independent of the patient's modality. Furthermore, this condition permits to get an unbiased Kernel estimator.

2. The model and the estimates

2.1. The functional nonparametric framework

Consider a random pair (X, T) where T is valued in $\mathbb{R}$ and X is valued in some infinite dimensional Hilbertian space $\mathcal{H}$with scalar product $<\cdot, \cdot>$, and consider that, given $(X_k, T_k)_{k=1,\ldots,n}$ is the statistical sample of pairs which are identically distributed like (X, T), but not necessarily independent. Hence for thg, X is called a functional random variable *f.r.v*. Let x be fixed in $\mathcal{H}$ and let $\Psi(\theta, t, x)$ be the conditional cumulative distribution function (*cond-cdf*) of T given $<\theta, X> = <\theta, x>$ specifically:

$$\forall t \in \mathbb{R}, \Psi(\theta, t, x) = \mathbb{P}(T \leq t | <\theta, X> = <\theta, x>).$$

By saying that, one is implicitly assuming the existence of a regular version of the conditional distribution T given $<\theta, X> = <\theta, x>$.

In this infinite dimensional purpose, the term *functional nonparametric* was used, where the word *functional* refers to the infinite dimensionality of the data and where *non-parametric* refers to the infinite dimensionality of the model. Such *functional non-parametric* statistics can also be called *doubly infinite dimensional* (see Ferraty and Vieu, 2003). The authors also used the term *operational statistics* since the target object to be estimated (the *cond-cdf* $\Psi(\theta, ., x)$) can be viewed as a non-linear operator.

2.2. The estimators

The kernel estimator $\Psi_n(\theta, ., x)$ of $\Psi(\theta, ., x)$ is presented as follows:

$$\Psi_n(\theta, t, x) = \frac{\sum_{k=1}^{n} \Gamma\left(a_n^{-1}(< x - X_k, \theta >)\right)\Omega\left(b_n^{-1}(t - T_k)\right)}{\sum_{k=1}^{n} \Gamma\left(a_n^{-1}(< x - X_k, \theta >)\right)}, \tag{2.1}$$

where Γ is a kernel function, Ωa cumulative distribution function and a_n(resp. b_n) a sequence of positive real numbers. Note that using similar ideas, Roussas (1969) introduced some related estimates but in a special case when X is real, while Samanta (1989) produced previously an asymptotic study.

Such an estimator is unique as soon as Ω is an increasing continuous function. This approach was largely used in cases where variable X is of a finite dimension (see e.g. Whang and Zhao (1999), Cai (2002), Zhou and Liang (2003), and Gannoun, Saracco and Yu (2003)).

In practice, particularly in medical applications, one can be deal with censored variables. This problem is usually modelled by considering positive C variable–censorship, and the observed random variables are not couples (T_k, X_k), but rather $\left(Y_k, \delta_{k,} X_k\right)$, where $Y_k = \min\{T_k, C_k\}$ and $\delta_k = 1_{T_k \leq C_k}$. The following calculations use the notations Ψ_1^X and $\psi_1^X$to describe the conditional distribution function and conditional density C, knowing covariate X.

Such censorship models have been amply studied in the literature on real and multi-dimensional random variables, and in non-parametric frameworks the kernel techniques are particularly used (see Tanner and Wong (1983), Padgett (1988), Lecoutre and Ould-Saïd (1995) and Van-Keilegom and Veraverbeke (2001), for a not necessarily exhaustive sample of literature in this field).

The objective of this section is to adapt these ideas under functional random variable X, and build a kernel type estimator of conditional distribution

$\Psi(\theta,.,x)$ adapted for censored samples. Thus one can reformulate the expression (2.1) as follows:

$$\widetilde{\Psi}(\theta,t,x)=\frac{\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}(Y_k)}\,\Gamma\big(a_n^{-1}(<x-X_k,\theta>)\big)\Omega\big(b_n^{-1}(t-T_k)\big)}{\sum_{k=1}^{n}\Gamma\big(a_n^{-1}(<x-X_k,\theta>)\big)}. \tag{2.2}$$

In practice $\bar{G}(.)=1-G(.)$ is unknown, hence it is impossible to use the estimator (2.2). Next, the authors replaced $\bar{G}(.)$ by its Kaplan and Meier (1958) estimate $\bar{G}_n(.)$ given by

$$\bar{G}_n(t)=1-G_n(t)=\begin{cases}\prod_{i=1}^{n}\left(1-\frac{1-\delta_{(k)}}{n-k+1}\right)^{1_{\{Y_{(k)}\le t\}}}, & if\ t\le Y_{(n)},\\ 0, & if\ t\ge Y_{(n)},\end{cases}$$

where $Y_{(1)}<Y_{(2)}<\cdots<Y_{(n)}$ are the order statistics of Y_k and $\delta_{(k)}$ is the non--censoring indicator corresponding to $Y_{(k)}$.

Therfore the feasible estimator of conditional distribution function $\Psi(\theta,.,x)$ is given by

$$\widehat{\Psi}(\theta,t,x)=\frac{\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}(Y_k)}\,\Gamma\big(a_n^{-1}(<x-X_k,\theta>)\big)\Omega\big(b_n^{-1}(t-T_k)\big)}{\sum_{k=1}^{n}\Gamma\big(a_n^{-1}(<x-X_k,\theta>)\big)}. \tag{2.3}$$

2.3. Assumptions on the functional variable

Let N_x be a fixed neighbourhood of *x* in $\mathcal{H}$ and let $B_\theta(x,h)$ be the sphere of center *x* and radius *h*, namely $B_\theta(x,h)=\{f\in\mathcal{H}:0<|<x-f,\theta>|<h\}$ and $S_{\mathbb{R}}$ is a fixed compact of $\mathbb{R}^+$.

For any df Λ, let $\tau_\Lambda := sup\{t, such\ that\ \Lambda(t)<1\}$ and $S_{\mathbb{R}}$ be its support's right endpoint. Let S be a compact set such that $\vartheta_\theta(\gamma,x)\in S\cup(-\infty,\tau]$, where $\tau<min(\tau_G,\tau_F)$. Assume that $(C_k)_{k\ge1}$ are independent and letus consider the following hypotheses:

(H1): $\mathbb{P}\big(X\in B_\theta(x,h_K)\big)=:\varphi_{\theta,x}(h_K)>0;\ \varphi_{\theta,x}(h_K)\longrightarrow 0$ as $h_K\to 0$.

2.4. The non-parametric model

As usual in non-parametric estimation, it issupposed that the *cond-cdf* $\Psi(\theta,\dots,x)$ verifies some smoothness constraints. Let α_1 and α_2 be two positive numbers; such that:

(H2): $\forall(x_1,x_2)\in N_x\times N_x, \forall(t_1,t_2)\in S_{\mathbb{R}}\times S_{\mathbb{R}}$

(i) $|\Psi(\theta,t_1,x_1)-\Psi(\theta,t_2,x_2)|\le C_{\theta,x}(\|x_1-x_2\|^{\alpha_1}+|t_1-t_2|^{\alpha_2})$,
(ii) $\int t\varphi(\theta,t,x)dt<\infty$ for all $\theta,x\in\mathcal{H}$.

(H3): $\Psi(\theta,.,x)$ is l-times continuously differentiable in some neighbourhood of $\vartheta_\theta(\gamma,x)$.
(H4): $\forall(x_1,x_2)\in N_x\times N_x, \forall(t_1,t_2)\in S_{\mathbb{R}}\times S_{\mathbb{R}}$

$$\left|\Psi^{(l)}(\theta,t_1,x_1)-\Psi^{(l)}(\theta,t_2,x_2)\right|\le C_{\theta,x}(\|x_1-x_2\|^{\alpha_1}+|t_1-t_2|^{\alpha_2}),$$

where for any positive integer l, $\Psi^{(l)}(\theta,t_1,x_1)$ denotes its l-th derivative ($i.e\ .\frac{\partial^l\Psi(\theta,t,x)}{\partial t^l}\Big|_{t=z}$).

3. Asymptotic study

The objective of this section was to adapt these ideas to the framework of a functional explanatory variable, and to construct a kernel-type estimator of conditional distribution function $\Psi(\theta,t,x)$ suitable for censored samples. The objective was to establish almost complete convergence[1] of kernel estimator $\widehat{\Psi}(\theta,t,x)$ when the observed sample is censored. The results presented are accompanied by the data on the rate of convergence. In what follows C and C' denote generic strictly positive real constants, a_n (resp. b_n) is a sequence which tends to 0 with n.

3.1. Point-wise almost complete convergence

In addition to the assumptions introduced in Section 2.4, additional conditions are needed. The assumptions concern the parameters of the estimator, i.e. Ω,Γ, a_n and b_n, which are not very restrictive. In fact, on the one hand they are rather inherent in the estimation problem of $\Psi(\theta,t,x)$, and on the other, they correspond to the assumptions usually made in the context of non-functional variables. More precisely, the authors introduce the following conditions which guarantee the good behaviour of estimator $\widehat{\Psi}(\theta,.,x)$:

(H5): (i) $\forall(t_1,t_2)\in\mathbb{R}^2, |\Omega(t_1)-\Omega(t_2)|\le C|t_1-t_2|$ and $\int|t|^{\alpha_2}\Omega^{(1)}(t)dt<\infty$, where, for all $l\in\mathbb{N}^\star, \Omega^{(l)}(t)=\frac{\partial^l\Omega(z)}{\partial z^l}\Big|_{z=t}$ and $\lim_{n\to\infty} n^\varsigma b_n=\infty$, for some $\varsigma>0$.

(ii) The support of $\Omega^{(1)}$ is compact and $\forall l\ge j$, $\Omega^{(l)}$ exists and is bounded.

(H6): The restriction of Ω to the set $\{u\in\mathbb{R},\Omega(u)\in(0,1)\}$ is a strictly increasing function.
(H7): Γ is a positive bounded function with support [0,1]: $\forall u\in(0,1)$, 0 < $\Gamma(u)$.

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[1] Remember that a sequence $(S_n)_{n\in\mathbb{N}}$ of random variables is said to converge almost completely to some variable S, $\epsilon>0$, where $\sum_n\mathbb{P}(|S_n-S|>\epsilon)<\infty$. This mode of convergence implies both almost certain and probabile convergence (see e.g. Bosq and Lecoutre, 1987).

Remark 3.1

1. (H6) ensuresthe existence of $\hat{\vartheta}_\theta(\gamma,x)$, while (H5) ensures its uniqueness.
2. (H1) to (H4) and (H7) are standard assumptions for the distribution conditional estimation in a single functional index model, adopted by Bouchentouf, Djebbouri, Rabhi and Sabri (2014) for i.i.d case.

First observe that (2.3) can be rewritten as:

$$\widehat{\Psi}(\theta,t,x)=\frac{\widehat{\Psi}_N(\theta,t,x)}{\widehat{\Psi}_D(\theta,x)}.$$

Theorem 3.1. Suppose that hypotheses (H1) to (H3), (H5)-(i), (H6) are satisfied and if

$$\frac{log\, n}{n\phi_{\theta,x}(a_n)}\xrightarrow[n\to\infty]{}0,$$

then

$$\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}(\theta,t,x)-\Psi(\theta,t,x)\right|=\mathcal{O}\left(a_n^{\alpha_1}+b_n^{\alpha_2}\right)+\mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, n}{n\phi_{\theta,x}(a_n)}}\right).$$

Proof. Consider now, for $k=1,\ldots,n$, and in what follows, denote:

$$\Gamma_k(\theta,x)=\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right),\Omega_k(t)=\Omega(b_n^{-1},(t-Y_k)),\bar{G}_k=\bar{G}(Y_k),$$

$$\widehat{\Psi}_N(\theta,t,x)=\frac{1}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}_n(Y_k)}\Gamma_k(\theta,x)\Omega_k(t),$$

$$\widetilde{\Psi}_N(\theta,t,x)=\frac{1}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}(Y_k)}\Gamma_k(\theta,x)\Omega_k(t),$$

$$\widehat{\Psi}_D(\theta,x)=\frac{1}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sum_{k=1}^{n}\Gamma_k(\theta,x),\Delta_k(\theta,x)=\frac{\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right)}{\mathbb{E}\left(\Gamma_1(\theta,x)\right)}.$$

The proof is based on the following decomposition, valid for any $t\in S_{\mathbb{R}}$:

$$\begin{aligned}\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}(\theta,t,x)-\Psi(\theta,t,x)\right| &\leq\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}_N(\theta,t,x)-\widetilde{\Psi}_N(\theta,t,x)\right|\\ &+\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| \qquad (3.1)\\ &+\frac{1}{\widehat{\Psi}_{\mathrm{D}}(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\mathbb{E}\widetilde{\Psi}_N(\theta,t,x)-\Psi(\theta,t,x)\right|+\frac{\Psi(\theta,t,x)}{\widehat{\Psi}_{\mathrm{D}}(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|1-\widehat{\Psi}_{\mathrm{D}}(\theta,x)\right|.\end{aligned}$$

Finally, the proof of this theorem is a direct consequence of the following intermediate results.

Lemma 3.1. Assume that either (H5)-(i) is satisfied together with under conditions (H6)-(H7) and if

$$\left(\frac{log\, log\, n}{n}\right)^{1/2} \underset{n\to\infty}{\longrightarrow} o\big(\phi_{\theta,x}(a_n)\big),$$

then

$$\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}_N^{(l)}(\theta,t,x)-\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right| = \mathcal{O}_{a.s.}\left(\frac{log\, log\, n}{n}\right), \text{ for } l\geq 0\,.$$

Proof. Let

$$\left|\widehat{\Psi}_N^{(l)}(\theta,t,x)-\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right| \leq$$

$$\frac{b_n^{-l}}{n\mathbb{E}\big(\Gamma_1(\theta,x)\big)}\sum_{k=1}^{n}\left|\frac{\delta_k}{\bar{G}_n(Y_k)}\Gamma_k(\theta,x)\Omega_k{}^{(l)}(t)-\frac{\delta_k}{\bar{G}(Y_k)}\Gamma_k(\theta,x)\Omega^{(l)}(t)\right|$$

$$\leq \frac{b_n^{-l}}{n\mathbb{E}\big(\Gamma_1(\theta,x)\big)}\sum_{k=1}^{n}\left|\delta_k\Gamma_k(\theta,x)\Omega_k{}^{(l)}(t)\right|\left|\frac{1}{\bar{G}_n(Y_k)}-\frac{1}{\bar{G}(Y_k)}\right|$$

$$\leq \frac{b_n^{-l}}{n\phi_{\theta,x}(a_n)}\frac{C}{\bar{G}_n(\tau_F)\bar{G}(\tau_F)}\sup_{t\in S_{\mathbb{R}}}\left|\bar{G}_n(t)-\bar{G}(t)\right|\frac{1}{n}\sum_{k=1}^{n}\delta_k\left|\Gamma_k(\theta,x)\Omega_k{}^{(l)}(t)\right|.$$

In conjunction with the SLLN and the LIL on the censoring law (see Deheuvels and Einmahl, 2000), hypotheses (H1), (H6) and $\left(\frac{log\, log\, n}{n}\right)^{1/2} = o\big(\phi_{\theta,x}(a_n)\big)$ complete the proof.

The following lemma shows the asymptotic bias term of $\widetilde{\Psi}_N(\theta,t,x)$ and $\widehat{\Psi}_{\mathrm{D}}(\theta,x)$ as n tends to infinity.

Lemma 3.2. Under hypotheses (H1), (H3) and (H5)-(i), and $n\to\infty$

$$\sup_{t\in S_{\mathbb{R}}}\left|\mathbb{E}\big[\widetilde{\Psi}_N(\theta,t,x)\big]-\Psi(\theta,t,x)\right| = \mathcal{O}\big(a_n^{\alpha_1}+b_n^{\alpha_2}\big).$$

Proof. The asymptotic behaviour of bias term is standard; hypotheses (H1), (H6) and $\left(\frac{log\, log\, n}{n}\right)^{1/2} = o\big(\phi_{\theta,x}(a_n)\big)$ complete the proof.

$$\begin{aligned}\mathbb{E}\big[\widetilde{\Psi}_N(\theta,t,x)\big]-\Psi(\theta,t,x) &= \frac{1}{\mathbb{E}\big(\Gamma_1(\theta,x)\big)}\mathbb{E}\left(\frac{\delta_k}{\bar{\mathrm{G}}(Y_k)}\Gamma_{\mathrm{k}}(\theta,x)\Omega_k(t)\right)-\Psi(\theta,t,x)\\ &= \frac{1}{\mathbb{E}\big(\Gamma_1(\theta,x)\big)}\mathbb{E}\left(\frac{\delta_k}{\bar{G}(Y_k)}\Gamma_k(\theta,x)[\mathbb{E}(\Omega_k(t)|<\theta,X_1>)]\right)-\Psi(\theta,t,x),\end{aligned} \qquad (3.2)$$

integrating by parts, and using the fact that Ω is *cdf* and the use a double conditioning with respect to T_1, one can easily obtain

$$I = \mathbb{E}\left(\frac{\delta_k}{\overline{\mathrm{G}}(Y_k)}\Omega_k(t)|< \theta, X_1 >\right) = \mathbb{E}\left(\mathbb{E}\left[\frac{1_{T_1\leq C_1}}{\overline{\mathrm{G}}(T_1)}\Omega\left(\frac{t-T_1}{b_n}\right)|< \theta, X_1 >, T_1\right]\right)$$
$$= \mathbb{E}\left(\frac{1}{\overline{\mathrm{G}}(T_1)}\Omega\left(\frac{t-T_1}{b_n}\right)\mathbb{E}\left[1_{T_1\leq C_1}|T_1\right]|< \theta, X_1 >\right)$$
$$= \mathbb{E}\left(\Omega\left(\frac{t-T_1}{b_n}\right)|< \theta, X_1 >\right) = \int_{\mathbb{R}}\Omega\left(\frac{t-u}{b_n}\right)\psi(\theta, u, X_1)du$$
$$= \int_{\mathbb{R}}\Omega\left(\frac{t-u}{b_n}\right)\mathrm{d}\Psi(\theta, u, X_1) = \int_{\mathbb{R}}\Omega^{(1)}\left(\frac{t-u}{b_n}\right)\Psi(\theta, u, X_1)du$$
$$= \int_{\mathbb{R}}\Omega^{(1)}(v)\Psi(\theta, t-vb_n, X_1)dv$$
$$= \int_{\mathbb{R}}\Omega^{(1)}(v)\big(\Psi(\theta, t-vb_n, X_1) - \Psi(\theta, t, x)\big)dv + \Psi(\theta, t, x)\int_{\mathbb{R}}\Omega^{(1)}(v)dv,$$

hence, because of (H2) and (H5)-(i):

$$I \leq C_{\theta,x}\int_{\mathbb{R}}\Omega^{(1)}(v)\big(a_n^{\alpha_1} + |v|^{\alpha_2}b_n^{\alpha_2}\big)dv + \Psi(\theta, t, x) = \mathcal{O}\big(a_n^{\alpha_1} + b_n^{\alpha_2}\big) + \Psi(\theta, t, x).$$

Combining this last result with (3.2) allows achieving the proof. The following result deals with the variance term of the right-hand side of (3.1) which is expressed by $\sup_{t\in S_{\mathbb{R}}}\left\{\left|\widetilde{\Psi}_N(\theta, t, x) - \mathbb{E}\widetilde{\Psi}_N(\theta, t, x)\right|\right\}$. For $\widehat{\Psi}_D(\theta, x) - \mathbb{E}\left[\widehat{\Psi}_D(\theta, x)\right]$ the same arguments are used with a slight difference.

Lemma 3.3. Under assumptions of Theorem 3.1 and if

$$\left(\frac{\log\log n}{n}\right)^{1/2} \underset{n\to\infty}{\longrightarrow} o\big(\phi_{\theta,x}(a_n)\big),$$

then

(i) $\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}_D(\theta, x) - \mathbb{E}\widehat{\Psi}_D(\theta, x)\right| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(a_n)}}\right)$.

(ii) $\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N(\theta, t, x) - \mathbb{E}\widetilde{\Psi}_N(\theta, t, x)\right| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(a_n)}}\right)$.

Proof. Using the compactness of $S_{\mathbb{R}}$, one can write that $S_{\mathbb{R}} \subset \bigcup_{j=1}^{\tau_n}\big(z_j - l_n,\ z_j + l_n\big)$ with l_n and τ_n can be chosen such that $l_n = C\tau_n^{-1} \sim Cn^{-\varsigma-1/2}$. Taking

$$j_t = \arg\min_{\{z_1,\cdots,z_{\tau_n}\}}\left|t - z_j\right|.$$

Hence the following decomposition:

$$\begin{aligned}
\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| &\leq \frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N(\theta,t,x)-\widetilde{\Psi}_N\left(\theta,t_j,x\right)\right| \\
&+\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N\left(\theta,t_j,x\right)-\mathbb{E}\widetilde{\Psi}_N\left(\theta,t_j,x\right)\right| \\
&+\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\mathbb{E}\widetilde{\Psi}_N\left(\theta,t_j,x\right)-\mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| \\
&\leq B_1+B_2+B_3.
\end{aligned}$$

As the first and the third terms can be treated in the same manner, this deals only with first term. Making use of (H5)-(i) one obtains

$$\begin{aligned}
&\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N(\theta,t,x)-\widetilde{\Psi}_N\left(\theta,t_j,x\right)\right| \leq \\
&\frac{1}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sup_{t\in S_{\mathbb{R}}}\sum_{k=1}^{n}\left|\frac{\delta_k}{\bar{G}(Y_k)}\Omega_k(t)-\frac{\delta_k}{\bar{G}(Y_k)}\Omega_k(t_j)\right|\Gamma_k(\theta,x) \\
&\leq \frac{1}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sup_{t\in S_{\mathbb{R}}}\sum_{k=1}^{n}\left|\frac{\delta_k}{\bar{G}(Y_k)}\Omega_k(t)-\frac{\delta_k}{\bar{G}_n(Y_k)}\Omega_k(t_j)\right|\Gamma_k(\theta,x) \\
&\leq \frac{C}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\sup_{t\in S_{\mathbb{R}}}\frac{\left|t-t_j\right|}{b_n}\times\left(\sum_{k=1}^{n}\Gamma_k(\theta,x)\left(\frac{1}{\bar{G}(Y_k)}-\frac{1}{\bar{G}_n(Y_k)}\right)\right) \\
&\leq \frac{Cl_n}{b_n\bar{G}_n(\tau_G)\bar{G}(\tau_G)}\sup_{t\in S_{\mathbb{R}}}\left|\bar{G}_n(t)-\bar{G}(t)\right|\widehat{\Psi}_D(\theta,x).
\end{aligned}$$

Using $l_n=n^{-\varsigma-1/2}$ we obtain

$$B_1\leq\frac{Cn^{-\varsigma-1/2}}{b_n\bar{G}_n(\tau_G)\bar{G}(\tau_G)}\left(\frac{\log\log n}{n}\right)^{1/2},$$

and note that, because of (H6)-(i),

$$\frac{l_n}{b_n}=o\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(a_n)}}\right).$$

Thus, for n large enough,

$$B_1=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{n\phi_{\theta,x}(a_n)}}\right).$$

Following similar arguments, one can state that

$$B_3 \leq B_1.$$

Concerning B_2, consider $\varepsilon = \varepsilon_0 \sqrt{\frac{log\, n}{n\phi_{\theta,x}(a_n)}}$. Since for all $\varepsilon_0 > 0$, one has that

$$\mathbb{P}\left(\sup_{t\in S_{\mathbb{R}}}\left|\mathbb{E}\widetilde{\Psi}_N(\theta,t_j,x) - \mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| > \varepsilon\right) \leq$$
$$\mathbb{P}\left(\max_{j\in\{1,\cdots,\tau_n\}}\left|\mathbb{E}\widetilde{\Psi}_N(\theta,t_j,x) - \mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| > \varepsilon\right)$$
$$\leq \tau_n\, \mathbb{P}\left(\left|\mathbb{E}\widetilde{\Psi}_N(\theta,t_j,x) - \mathbb{E}\widetilde{\Psi}_N(\theta,t,x)\right| > \varepsilon\right).$$

Applying Berstain's exponential inequality to:

$$\Pi_k = \frac{1}{\mathbb{E}\left(\Gamma_1(\theta,x)\right)}\left[\frac{\delta_k}{\bar{G}(Y_k)}\Gamma_k(\theta,x)\Omega_k(t_j) - \mathbb{E}\left(\frac{\delta_k}{\bar{G}(Y_k)}\Gamma_k(\theta,x)\Omega_k(t_j)\right)\right].$$

Firstly, it follows from the fact that the Kernel Γ is bonded and $\Omega \leq 1$, then

$$\mathbb{P}\left(\left|\widetilde{\Psi}_N(\theta,t_j,x) - \mathbb{E}\widetilde{\Psi}_N(\theta,t_j,x)\right| > \varepsilon\right) \leq \mathbb{P}\left(\frac{1}{n}\left|\sum_{k=1}^{n}\Pi_k\right| > \varepsilon\right) \leq 2n^{-C\varepsilon_0^2}.$$

Finally, by choosing ε_0 large enough, the proof can be concluded by the use of the Borel-Cantelli lemma. The result can be easily deduced.

The proof of Theorem 3.1 was concluded by making use inequality (3.1), in conjunction with Lemma 3.1, Lemma 3.2 and Lemma 3.3.

The proof of these is presented in Section 5.

3.2. Almost complete point-wise convergencerate rate

This part studies the rate of convergence of estimator $\widehat{\Psi}(\theta,t,x)$. The main objective is the estimation of the conditional cumulative distribution of *T* given *<θ,X>=<θ,x>*, denoted by $\Psi(\theta,.,x)$.

Theorem 3.2. Suppose that hypotheses (H1), (H3)-(H7) are satisfied and if

$$\lim_{n\to\infty}\frac{log\, n}{nb_n^{2l-1}\phi_{\theta,x}(a_n)} = 0,$$

then

$$\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}^{(l)}(\theta,t,x) - \Psi^{(l)}(\theta,t,x)\right| = \mathcal{O}\left(a_n^{\alpha_1} + b_n^{\alpha_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, n}{nb_n^{2l-1}\phi_{\theta,x}(a_n)}}\right).$$

Proof. Thus in a classical way the announced result follows the same path as Theorem 3.1, and follows directly from decomposition (3.3):

$$\begin{aligned} L(\theta,t,x) &= \sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}^{(l)}(\theta,t,x)-\Psi^{(l)}(\theta,t,x)\right| \\ &\leq \frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}_N^{(l)}(\theta,t,x)-\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right| \\ &+\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right| \\ &+\frac{1}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\Psi^{(l)}(\theta,t,x)\right| \\ &+\frac{\Psi^{(l)}(\theta,t,x)}{\widehat{\Psi}_D(\theta,x)}\sup_{t\in S_{\mathbb{R}}}\left|1-\widehat{\Psi}_D(\theta,x)\right|. \end{aligned} \tag{3.3}$$

As before, in view of decomposition (3.3), it suffices to prove that the results of Lemma 3.4 and Lemma 3.5 below, in conjunction with the first part of Lemma 3.3 and Lemma 3.1, to complete the result of Theorem 3.2.

Lemma 3.4. Under hypotheses (H1) and (H4)-(H7)

$$\sup_{t\in S_{\mathbb{R}}}\left|\Psi^{(l)}(\theta,t,x)-\mathbb{E}\left[\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right]\right| = \mathcal{O}\left(a_n^{\alpha_1}+b_n^{\alpha_2}\right).$$

Proof. To deal with this deterministic term, the calculations performed during the proof of Lemma 3.2 did not use successive derivatives i.e. replacing $\Psi(\theta,t,x)$ (respectively $\widetilde{\Psi}(\theta,t,x)$) with $\Psi^{(l)}(\theta,t,x)$ (respectively $\widetilde{\Psi}_N^{(l)}(\theta,t,x)$). The result of Lemma 3.4 therefore remains valid under the differentiability conditions (hypotheses (H4) and (H5)), while taking up the approach and the notations introduced during the proof of Lemma 3.2, one obtains:

$$\Psi^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x) = \mathcal{O}\left(a_n^{\alpha_1}+b_n^{\alpha_2}\right).$$

Lemma 3.5. Under the assumptions of Theorem 3.2

$$\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\left[\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right]\right| = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nb_n^{2l-1}\phi_{\theta,x}(a_n)}}\right).$$

Proof. To arrive at the asymptotic behavior of the quantity $\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x)$, the proof follows the same path as during the proof of Lemma 3.3, it suffices to replace $\Psi(\theta,t,x)$ (respectively $\widetilde{\Psi}_N(\theta,t,x)$) with $\Psi^{(l)}(\theta,t,x)$ (respectively $\widetilde{\Psi}_N^{(l)}(\theta,t,x)$).

Note that (H5)-(ii) and (H9) permit to show that

$$\mathbb{E}\left(\frac{\delta_k}{\bar{G}(Y_k)}\Omega^{(l)}\left(b_n^{-1}(t-T_k)\right)\frac{\delta_m}{\bar{G}(Y_m)}\Omega^{(l)}\left(b_n^{-1}(t-T_m)\right)\middle|(X_k,X_m)\right)=\mathcal{O}(b_n^2),$$

while (H4) imply that

$$\mathbb{E}\left(\frac{\delta_k}{\bar{G}(Y_k)}\Omega^{(l)}\left(b_n^{-1}(t-T_k)\right)\middle|X_k\right)=\mathcal{O}(b_n).$$

Indeed, it can be found that

$$\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x)=\frac{1}{n}\sum_{k=1}^{n}A_k(\theta,t,x),$$

where

$$A_k=b_n^{-l}\frac{\delta_k}{\bar{G}(Y_k)}\Omega_k^{(l)}(t)\Delta_k(\theta,x)-\mathbb{E}\left(b_n^{-l}\frac{\delta_k}{\bar{G}(Y_k)}\Omega_k^{(l)}(t)\Delta_k(\theta,x)\right),$$

has zero mean and satisfies $|A_k(\theta,t,x)|\leq Cb_n^{-l}\phi_{\theta,x}^{-1}(a_n)$.

Now, since $\Omega^{(l)}$ is bounded, it allows to use directly similar arguments of the second part by Lemma 3.3, thus

$$\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\widetilde{\Psi}_N^{(l)}(\theta,t,x)=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log n}{nb_n^{2l-1}\phi_{\theta,x}(a_n)}}\right),$$

which leads directly to the result of Lemma 3.5.

3.3. Conditional quantile estimation

This part studies the rate of convergence of conditional $\hat{\vartheta}_\theta(\gamma,x)$.

Now, let $\vartheta_\theta(\gamma,x)$ be the γ-order quantile of the distribution of T given $<\theta,X>=<\theta,x>$. From the *cond-cdf* $\Psi(\theta,.,x)$, the general definition of the γ-order quantile is given as:

$$\vartheta_\theta(\gamma,x)=\inf\{t\in\mathbb{R},\Psi(\theta,t,x)\geq\gamma\}.$$

In order to simplify the framework and focus on the main interest of this paper (the functional feature of $<\theta,X>$), it is assumed that $\Psi(\theta,.,x)$ is strictly increasing and continuous in the neighbourhood of $\vartheta_\theta(\gamma,x)$. Thisensures that conditional quantile $\vartheta_\theta(\gamma,x)$ is uniquely defined by:

$$\vartheta_\theta(\gamma,x)=\Psi^{-1}(\theta,\gamma,x),\forall\gamma\in(0,1). \quad (3.4)$$

As a by-product of (3.4) and (2.1), it is easy to derive estimator $\vartheta_{\theta,n}(\gamma,x)$ of $\vartheta_\theta(\gamma,x)$:

$$\vartheta_{\theta,n}(\gamma,x) = \Psi_n^{-1}(\theta,\gamma,x).$$

Then a natural estimator of $\vartheta_\theta(\gamma,x)$ is given by

$$\hat{\vartheta}_\theta(\gamma,x) = \widehat{\Psi}^{-1}(\theta,\gamma,x) \; = \inf\{t \in \mathbb{R}, \widehat{\Psi}(\theta,t,x) \geq \gamma\},$$

which satisfies

$$\widehat{\Psi}\left(\theta, \hat{\vartheta}_\theta(\gamma,x), x\right) = \gamma. \tag{3.5}$$

Corollary 3.1. Under hypotheses of Theorem 3.1

$$\hat{\vartheta}_\theta(\gamma,x) - \vartheta_\theta(\gamma,x) \underset{n\to\infty}{\longrightarrow} 0, a.co.$$

Proof. The proof is based on the point-wise convergence of $\widehat{\Psi}(\theta,.,x)$ and the Lipschitz property introduced in (H5)-(i) and hypothesis (H6), $\widehat{\Psi}(\theta,t,x)$ is a continuous and strictly increasing function. Hence

$$\begin{aligned}\forall\, \epsilon > 0, \exists \delta(\epsilon) > 0, \forall y, &\left|\widehat{\Psi}(\theta,t,x) - \widehat{\Psi}(\theta,\vartheta_\theta(\gamma,x),x)\right| \leq \delta(\epsilon)\\ &\Rightarrow |t - \vartheta_\theta(\gamma,x)| \leq \epsilon.\end{aligned}$$

This leads to write $\forall\, \epsilon > 0, \exists \delta(\epsilon) > 0,$

$$\begin{aligned}\mathbb{P}\left(\left|\hat{\vartheta}_\theta(\gamma,x) - \vartheta_\theta(\gamma,x)\right| > \epsilon\right) &\leq \mathbb{P}\left(\left|\widehat{\Psi}\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right) - \widehat{\Psi}(\theta,\vartheta_\theta(\gamma,x),x)\right| \geq \delta(\epsilon)\right)\\ &= \mathbb{P}\left(\left| \Psi(\theta,\vartheta_\theta(\gamma,x),x) - \widehat{\Psi}(\theta,\vartheta_\theta(\gamma,x),x)\right| \geq \delta(\epsilon)\right),\end{aligned}$$

Since (3.4) and (3.5), implying that

$$\widehat{\Psi}\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right) = \gamma = \Psi(\theta,\vartheta_\theta(\gamma,x),x).$$

Moreover

$$\begin{aligned}\left| \Psi\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right) - \Psi(\theta,\vartheta_\theta(\gamma,x),x)\right| &= \left| \Psi\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right) - \widehat{\Psi}\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right)\right|\\ &\leq \sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}(\theta,t,x) - \Psi(\theta,t,x)\right|.\end{aligned}$$

The consistency of $\hat{\vartheta}_\theta(\gamma,x)$ follows then immediately from Theorem 3.1 and the following inequality

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\vartheta_{n,\theta}(\gamma,x) - \vartheta_\theta(\gamma,x)\right| \geq \epsilon\right) \leq \sum_{n\geq 1} \mathbb{P}\left(\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}(\theta,t,x) - \Psi(\theta,t,x)\right| \geq \delta(\epsilon)\right).$$

Naturally, obtaining these results requires more sophisticated technical developments than those presented previously. To ensure a good read ability of this section, the authors introduced conditions related to the flatness of *cond-cdf* $\Psi(\theta,.,x)$ around conditional quantile $\vartheta_\theta(\gamma,x)$.

(H8): $\begin{cases} \Psi^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)=0, if; 1\le m<l \\ \Psi^{(l)}(\theta,.,x) \text{ is uniformly continuous on } S_{\mathbb{R}} \\ \text{such that } \Psi^{(l)}(\theta,\vartheta_\theta(\gamma,x),x)>C>0. \end{cases}$

The focus is on the local behaviour of $\Psi(\theta,.,x)$ around $\vartheta_\theta(\gamma,x)$ via its derivatives, prompting to consider the successive derivatives of $\widehat{\Psi}(\theta,.,x)$.

Corollary 3.2. Under hypotheses (H1) to (H8)

$$\hat{\vartheta}_\theta(\gamma,x)-\vartheta_\theta(\gamma,x)=\mathcal{O}\left(\left(a_n^{\alpha_1}+b_n^{\alpha_2}\right)^{\frac{1}{l}}\right)+\mathcal{O}_{a.co.}\left(\left(\frac{\log n}{n\phi_{\theta,x}(a_n)}\right)^{\frac{1}{2l}}\right).$$

Proof. The proof is based on the Taylor expansion of $\widehat{\Psi}(\theta,.,x)$ at $\vartheta_\theta(\gamma,x)$ and on the use of (H8):

$$\widehat{\Psi}(\theta,\vartheta_\theta(\gamma,x),x)-\widehat{\Psi}\left(\theta,\hat{\vartheta}_\theta(\gamma,x),x\right)=$$

$$\sum_{m=1}^{l}\frac{\left(\vartheta_\theta(\gamma,x)-\hat{\vartheta}_\theta(\gamma,x)\right)^{m-1}}{m!}\widehat{\Psi}^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)$$

$$+\frac{\left(\vartheta_\theta(\gamma,x)-\hat{\vartheta}_\theta(\gamma,x)\right)^{l}}{l!}\widehat{\Psi}^{(l)}(\theta,\vartheta_\theta^*(\gamma,x),x)$$

$$=\sum_{m=1}^{l-1}\frac{\left(\vartheta_\theta(\gamma,x)-\hat{\vartheta}_\theta(\gamma,x)\right)^{m-1}}{m!}(\widehat{\Psi}^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)$$

$$-\Psi^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)$$

$$+\frac{\left(\vartheta_\theta(\gamma,x)-\hat{\vartheta}_\theta(\gamma,x)\right)^{l}}{l!}\widehat{\Psi}^{(l)}(\theta,\vartheta_\theta^*(\gamma,x),x),$$

where, for all $t\in\mathbb{R}$,

$$\widetilde{\Psi}^{(l)}(\theta,t,x)=\frac{b_n^{-l}\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}(Y_k)}\,\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right)\Omega^{(l)}\left(b_n^{-1}(t-T_k)\right)}{\sum_{k=1}^{n}\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right)},$$

$$\widehat{\Psi}^{(l)}(\theta,t,x)=\frac{b_n^{-l}\sum_{k=1}^{n}\frac{\delta_k}{\bar{G}_n(Y_k)}\,\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right)\Omega^{(l)}\left(b_n^{-1}(t-T_k)\right)}{\sum_{k=1}^{n}\Gamma\left(a_n^{-1}(<x-X_k,\theta>)\right)},$$

and where $\min\left(\vartheta_\theta(\gamma,x),\hat{\vartheta}_\theta(\gamma,x)\right) < \vartheta_\theta^*(\gamma,x) < \max\left(\vartheta_\theta(\gamma,x),\hat{\vartheta}_\theta(\gamma,x)\right)$. Suppose now that the following result is obtained.

Due to Corollary 3.1, Theorem 3.2 and (H8)

$$\widehat{\Psi}^{(l)}(\theta,\vartheta_\theta^*(\gamma,x),x) \longrightarrow \Psi^{(l)}(\theta,\vartheta_\theta(\gamma,x),x) \neq 0, a.co.$$

Then one derives

$$\left|\vartheta_\theta(\gamma,x)) - \hat{\vartheta}_\theta(\gamma,x)\right|^l = \mathcal{O}\left(\widehat{\Psi}(\theta,\vartheta_\theta(\gamma,x),x) - \Psi(\theta,\vartheta_\theta(\gamma,x),x)\right)$$
$$+\mathcal{O}\left(\sum_{m=1}^{l-1}\left(\vartheta_\theta(\gamma,x) - \hat{\vartheta}_\theta(\gamma,x)\right)^m\left(\widehat{\Psi}^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)\right.\right.$$
$$\left.\left.-\Psi^{(m)}(\theta,\vartheta_\theta(\gamma,x),x)\right)\right), a.co.$$

Now, comparing the convergence rates given in Theorem 3.1 and Theorem 3.2, one obtains

$$\left|\vartheta_\theta(\gamma,x)) - \hat{\vartheta}_\theta(\gamma,x)\right|^l = \mathcal{O}\left(\widehat{\Psi}(\theta,t,x) - \Psi(\theta,t,x)\right), a.co.$$

Thus, the first part of Lemma 3.3 together with Lemma 3.4 and Lemma 3.5, allow for the claimed result.

4. Uniform almost complete convergence and rate of convergence

In this section the authors derive the uniform version of Theorem 3.1 and Theorem 3.2, which is a standard extension of the point-wise one. Undoubtedly, obtaining these results requires more sophisticated technical developments than those presented previously. To ensure a good readability of this, some additional tools and topological conditions are needed (see Ferraty, Laksaci, Tadj, and Vieu, 2010). Firstly, the compactness of sets $S_{\mathcal{H}}$ and $\Theta_{\mathcal{H}}$ respectively allows them to be covered by $d_n^{S_{\mathcal{H}}}$ intervals and $d_n^{\Theta_{\mathcal{H}}}$ intervals disjoint respectively so that $d_n^{S_{\mathcal{H}}}$, $d_n^{\Theta_{\mathcal{H}}}$ are the minimal numbers of open spheres with radius r_n in $\mathcal{H}$,x_p (respectively θ_q) $\in \mathcal{H}$.

$$S_{\mathcal{H}} \subset \bigcup_{p=1}^{d_n^{S_{\mathcal{H}}}} B_\theta\left(x_p,r_n\right) \; and \; \Theta_{\mathcal{H}} \subset \bigcup_{q=1}^{d_n^{\Theta_{\mathcal{H}}}} B_\theta\left(\theta_q,r_n\right).$$

4.1. Conditional distribution estimation

The aim of this section is to establish almost complete uniform convergence. To be able to extend the results obtained above, it is necessary to introduce a topological structure of the functional space of the observations and the functional character of the model. The asymptotic results of the paper exploit the topological structure of functional space for the observations. Note that all the rates of convergence are based on a hypothesis of the concentration of the measure of probability of the functional variable on the small spheres, and also on Kolmogorov's entropy which measures the number of spheres necessary to cover a certain set. A natural way to do this is to introduce the conditions below:

(A1): There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in S_{\mathcal{H}}$ and $\forall \theta \in \Theta_{\mathcal{H}}$,

$$0 < C\phi(h) \leq \phi_{\theta,x}(h) \leq C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0, \eta < \eta_0, \phi'(\eta) < C.$$

(A2): Kernel K satisfies (H3), and Lipschitz's condition holds

$$|\Gamma(x) - \Gamma(y)| \leq \|x - y\|.$$

(A3): $\forall (t_1, t_2) \in S_{\mathbb{R}} \times S_{\mathbb{R}}, \forall (x_1, x_2) \in N_x \times N_x, \forall \theta \in \Theta_{\mathcal{H}}$,

$$|\Psi(\theta, t_1, x_1) - \Psi(\theta, t_2, x_2)| \leq C(\|x_1 - x_2\|^{\alpha_1} + |t_1 - t_2|^{\alpha_2}).$$

(A4): For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^\nu b_n = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{S_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy:

$$\begin{cases} (i)\dfrac{(\log n)^2}{n\phi(a_n)} < \log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{n\phi(a_n)}{\log n}, \\ (ii)\displaystyle\sum_{n=1}^{\infty} n^{1/2\alpha_2}\left(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{1-\beta} < \infty \text{ for some } \beta > 1, \\ (iii) n\phi(a_n) = \mathcal{O}((\log n)^2). \end{cases}$$

(A5): $\forall (t_1, t_2) \in S_{\mathbb{R}} \times S_{\mathbb{R}}, \forall (x_1, x_2) \in N_x \times N_x, \forall \theta \in \Theta_{\mathcal{H}}$,

$$\left|\Psi^{(l)}(\theta, t_1, x_1) - \Psi^{(l)}(\theta, t_2, x_2)\right| \leq C(\|x_1 - x_2\|^{\alpha_1} + |t_1 - t_2|^{\alpha_2}).$$

(A6): For some $\nu \in (0,1)$, $\lim_{n\to\infty} n^\nu b_n = \infty$, and for $r_n = \mathcal{O}\left(\frac{\log n}{n}\right)$, the sequences $d_n^{S_{\mathcal{H}}}$ and $d_n^{\Theta_{\mathcal{H}}}$ satisfy:

$$\begin{cases} (i)\dfrac{(\log n)^2}{n b_n^{2l-1}\phi(a_n)} < \log d_n^{S_{\mathcal{H}}} + \log d_n^{\Theta_{\mathcal{H}}} < \dfrac{n b_n^{2l-1}\phi(a_n)}{\log n}, \\ (ii)\displaystyle\sum_{n=1}^{\infty} n^{(3\nu+1)/2}\left(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{1-\beta} < \infty \; for\; some\; \beta > 1, \\ (iii) n b_n^{2l-1}\phi(a_n) = \mathcal{O}((\log n)^2). \end{cases}$$

In what follows, denote

$$\Upsilon_k(\theta,x) = \frac{1}{n\phi(a_n)} 1_{B_\theta(x,h)\cup B_\theta(x_{p(x)},h)}(X_k),$$
$$\Xi_k(\theta,x) = \frac{1}{n\phi(a_n)} 1_{B_\theta(x_{p(x)},h)\cup B_{\theta_{q(\theta)}}(x_{p(x)},h)}(X_k),$$
$$\Delta_k\left(\theta_{q(\theta)},x_{p(x)}\right) = \frac{\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)}{\mathbb{E}\left(\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)\right)}.$$

$$\Sigma_k(\theta,x) = \frac{\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)}{\mathbb{E}\left(\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)\right)} \frac{\delta_k}{\bar{G}(Y_k)} \Omega\left(b_n^{-1}(v_{k_t} - T_k)\right)$$
$$-\mathbb{E}\left(\frac{\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)}{\mathbb{E}\left(\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)\right)} \frac{\delta_k}{\bar{G}(Y_k)} \Omega\left(b_n^{-1}(v_{k_t} - T_k)\right)\right),$$

and

$$\Sigma_k^{(l)}(\theta,x) = \frac{1}{b_n^l} \frac{\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)}{\mathbb{E}\left(\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)\right)} \frac{\delta_k}{\bar{G}(Y_k)} \Omega^{(l)}\left(b_n^{-1}(v_{k_t} - T_k)\right)$$
$$-\frac{1}{b_n^l}\mathbb{E}\left(\frac{\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)}{\mathbb{E}\left(\Gamma\left(a_n^{-1}(< x_{p(x)} - X_k, \theta_{q(\theta)} >)\right)\right)} \frac{\delta_k}{\bar{G}(Y_k)} \Omega^{(l)}\left(b_n^{-1}(v_{k_t} - T_k)\right)\right).$$

Theorem 4.1. Under hypotheses (H1) and (H2), (H5) to (H7) and (A1) to (A4)

$$\sup_{x\in S_{\mathcal{H}}}\sup_{t\in S_{\mathbb{R}}}\left|\widehat{\Psi}(\theta,t,x) - \Psi(\theta,t,x)\right| = \mathcal{O}\left(a_n^{\alpha_1} + b_n^{\alpha_2}\right) + \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi(a_n)}}\right).$$

Proof. Naturally, the proof of these results can be deduced from decomposition (3.3) and the following intermediate results which are only a uniform version of Theorem 3.2.

Lemma 4.1. Under conditions (H1) and (H2), and (H5) to (H7)

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{t\in S_{\mathbb{R}}}\left|\Psi^{(l)}(\theta,t,x) - \mathbb{E}\left[\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right]\right| = \mathcal{O}\left(a_n^{\alpha_1} + b_n^{\alpha_2}\right).$$

Proof. The proof follows the same path as during the proof of Lemma 3.4.

Lemma 4.2. Under assumptions of Theorem 4.1

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\left|\widehat{\Psi}_D(\theta,x)-\mathbb{E}\widehat{\Psi}_D(\theta,x)\right|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{n\phi(a_n)}}\right).$$

Proof. Similarly to the proof of Lemma 4.4 in (Bouchentouf, Djebbouri, Rabhi, and Sabri, 2014), it can be completed easily. Here the authorsomitted its proof.

Lemma 4.3. Under assumptions of Theorem 4.1

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\left[\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right]\right|=\mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}}d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right).$$

Proof. For all $x\in S_{\mathcal{H}}$ and $\forall\theta\in\Theta_{\mathcal{H}}$, it is set

$$p(x)=\arg\min_{\left\{1,\cdots,d_n^{S_{\mathcal{H}}}\right\}}\left\|x-x_p\right\|\ and\ q(\theta)=\arg\min_{\left\{1,\cdots,d_n^{\Theta_{\mathcal{H}}}\right\}}\left\|x-\theta_q\right\|,$$

and by the compact property of $S_{\mathbb{R}}\subset\mathbb{R}$, one obtains $S_{\mathbb{R}}\subset\bigcup_{p=1}^{\tau_n}\left(v_p-l_n,v_p+l_n\right)$ with l_n, and τ_n can be chosen such that $l_n=\mathcal{O}(\tau_n^{-1})=\mathcal{O}\left(n^{-(3\nu+1)/2}\right)$. Taking $p_t=\arg\min_{\{v_1,\cdots,v_{\tau_n}\}}\left|t-v_p\right|$.

Consider the following decomposition

$$\begin{aligned}\widetilde{\Lambda}_N^{(l)}(\theta,t,x)=\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{t\in S_{\mathbb{R}}}\Big\{&\left|\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\widetilde{\Psi}_N^{(l)}\left(\theta,t,x_{p(x)}\right)\right|\\&+\left|\widetilde{\Psi}_N^{(l)}\left(\theta,t,x_{p(x)}\right)-\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},t,x_{p(x)}\right)\right|\\&+\left|\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},t,x_{p(x)}\right)-\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},v_{p_t},x_{p(x)}\right)\right|\\&+\left|\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},v_{p_t},x_{p(x)}\right)-\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},v_{p_t},x_{p(x)}\right)\right)\right|\\&+\left|\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},v_{p_t},x_{p(x)}\right)\right)-\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},t,x_{p(x)}\right)\right)\right|\\&+\left|\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},t,x_{p(x)}\right)\right)-\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta,t,x_{p(x)}\right)\right)\right|\\&+\left|\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}\left(\theta,t,x_{p(x)}\right)\right)-\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right)\right|\Big\}\\&\le F_1+F_2+F_3+F_4+F_5+F_6+F_7,\end{aligned}$$

where $\widetilde{\Lambda}_N^{(l)}(\theta,t,x)=\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}}\sup_{t\in S_{\mathbb{R}}}\left|\widetilde{\Psi}_N^{(l)}(\theta,t,x)-\mathbb{E}\left(\widetilde{\Psi}_N^{(l)}(\theta,t,x)\right)\right|$.

- Concerning F_3 and F_5 by conditions (H5)-(ii) and (A6), boundness of Γ, one obtains

$$\left|\widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)}, t, x_{p(x)}\right) - \widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)}, v_{p_t}, x_{p(x)}\right)\right| \leq$$

$$\frac{1}{nb_n^l \mathbb{E}\left(\Gamma_1(\theta, x)\right)} \sup_{t \in S_{\mathbb{R}}} \sum_{k=1}^{n} \left|\frac{\delta_k}{\bar{\mathrm{G}}(Y_{\mathrm{k}})} \Gamma_k\left(\theta_{q(\theta)}, x_{p(x)}\right)\right|$$

$$+ \left|\Omega^{(l)}\left(b_n^{-1}(t - T_k)\right) \frac{\delta_m}{\bar{\mathrm{G}}(Y_m)} \Omega^{(l)}\left(b_n^{-1}\left(v_{p_t} - T_k\right)\right)\right|$$

$$\leq \sup_{t \in S_{\mathbb{R}}} C \frac{\left|t - v_{p_t}\right|}{b_n^{l+1}} \left(\frac{1}{n\mathbb{E}\left(\Gamma_1\left(\theta_{q(\theta)}, x_{p(x)}\right)\right)}\right.$$

$$\left.\sum_{k=1}^{n} \Gamma_k\left(\theta_{q(\theta)}, x_{p(x)}\right) \frac{1}{\bar{G}(Y_k)}\right) \leq \frac{Cl_n}{b_n^{l+1}\phi(a_n)} = \mathcal{O}\left(\frac{l_n}{b_n^{l+1}\phi(a_n)}\right).$$

Now, the fact that $\lim_{n\to\infty} n^{\nu} b_n = \infty$, choosing $l_n = n^{-(3\nu+1)/2}$, and using the second part of (A6), implies that

$$\frac{l_n}{b_n^{l+1}\phi(a_n)} = o\left(\sqrt{\frac{\log n}{nb_n^{2l-1}\phi(a_n)}}\right),$$

as $n \to \infty$, therefore, it follows

$$F_5 \leq F_3 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right).$$

Concerning F_4 let us consider $\varepsilon = \varepsilon_0 \sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}$. Since

$$\mathbb{P}\left(F_4 > \varepsilon_0 \sqrt{\frac{\log d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right) \leq \mathbb{P}\left(\max_{q \in \left\{1,\cdots,d_n^{\Theta_{\mathcal{H}}}\right\}} \max_{p \in \left\{1,\cdots,d_n^{S_{\mathcal{H}}}\right\}} \max_{j \in \{1,\cdots,\tau_n\}} \left|\Sigma_k^{(l)} - \mathbb{E}\Sigma_k^{(l)}\right| > \varepsilon\right)$$

$$\leq \tau_n d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\, \mathbb{P}\left(\left|\Sigma_k^{(l)} - \mathbb{E}\Sigma_k^{(l)}\right| > \varepsilon\right).$$

Applying Bernstain's exponential inequality, under (H5) and (H7), to get $\forall q \leq d_n^{\Theta_{\mathcal{H}}}$, $\forall p \leq d_n^{S_{\mathcal{H}}}$ and $\forall p_t \leq \tau_n$,

$$\mathbb{P}(|\Sigma_k - \mathbb{E}\Sigma_k| > \varepsilon) \leq 2\left(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{-C\varepsilon_0^2}.$$

Choosing $\tau_n \leq Cn^{(3\nu+1)/2}$, one obtains

$$\mathbb{P}(F_4 > \varepsilon) \leq C'^{\left(d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}\right)^{1-C\varepsilon_0^2}}.$$

Putting $C\varepsilon_0^2 = \beta$ and using (A6)

$$F_4 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right). \tag{4.1}$$

- Concerning F_1 and F_2

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{t\in S_{\mathbb{R}}} \left|\widetilde{\Psi}_N^{(l)}(\theta,t,x) - \widetilde{\Psi}_N^{(l)}\left(\theta,t,x_{p(x)}\right)\right| &\leq \left|\Gamma_k(\theta,x) - \Gamma_k\left(\theta,x_{p(x)}\right)\right| \\
&+ \frac{1}{nb_n^l \mathbb{E}\left(\Gamma_1(\theta,x)\right)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{t\in S_{\mathbb{R}}} \sum_{k=1}^n \left|\frac{\delta_k}{\overline{G}(Y_k)}\right| \left|\Omega_k^{(l)}(t)\right| \\
&\leq \frac{1}{nb_n^l\phi(a_n)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sum_{k=1}^n \left|\Delta_k(\theta,x) - \Delta_k\left(\theta,x_{p(x)}\right)\right| \\
&\leq \frac{1}{b_n^l\phi(a_n)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \frac{1}{n}\sum_{k=1}^n 1_{B_\theta(x,h)\cup B_\theta(x_{p(x)},h)}(X_k) \\
&\leq \frac{C}{b_n^l} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \frac{1}{n}\sum_{k=1}^n \Upsilon_k(\theta,x).
\end{aligned}$$

Therefore, similar to the arguments for (4.1)

$$F_1 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right).$$

$$\begin{aligned}
\sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{t\in S_{\mathbb{R}}} \left|\widetilde{\Psi}_N^{(l)}(\theta,t,x) - \widetilde{\Psi}_N^{(l)}\left(\theta_{q(\theta)},t,x_{p(x)}\right)\right| &\leq \left|\Gamma_k\left(\theta,x_{p(x)}\right) - \Gamma_k\left(\theta_{q(\theta)},x_{p(x)}\right)\right| \\
&+ \frac{b_n^{-l}}{n\mathbb{E}\left(\Gamma_1(\theta,x)\right)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \sup_{t\in S_{\mathbb{R}}} \sum_{k=1}^n \left|\frac{\delta_k}{\bar{G}(Y_k)}\right| \left|\Omega_k^{(l)}(t)\right| \\
&\leq \frac{Cb_n^{-l}}{\phi(a_n)} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \frac{1}{n}\sum_{k=1}^n \left|\Delta_k\left(\theta,x_{p(x)}\right) - \Delta_k\left(\theta_{q(\theta)},x_{p(x)}\right)\right| \\
&\leq \frac{C}{b_n^l} \sup_{\theta\in\Theta_{\mathcal{H}}} \sup_{x\in S_{\mathcal{H}}} \frac{1}{n}\sum_{k=1}^n \Xi_k(\theta,x).
\end{aligned}$$

Similar to the deduction of (4.1), it yields

$$F_2 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right).$$

On the other hand, since $F_7 \leq F_1$ and $F_6 \leq F_2$, it also leads to

$$F_6 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right) \text{and } F_7 = \mathcal{O}_{a.co.}\left(\sqrt{\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{nb_n^{2l-1}\phi(a_n)}}\right).$$

Then the proof of Lemma 4.3 can be completed.

Corollary 4.1. Under the hypotheses of Theorem 4.1

$$\sup_{x\in S_{\mathcal{H}}} \left|\hat{\vartheta}_\theta(\gamma,x) - \vartheta_\theta(\gamma,x)\right| \underset{n\to\infty}{\longrightarrow} 0, a.co.,$$

and

$$\sup_{\theta\in\Theta_{\mathcal{H}}}\sup_{x\in S_{\mathcal{H}}} \left|\hat{\vartheta}_\theta(\gamma,x) - \vartheta_\theta(\gamma,x)\right| = \mathcal{O}\left(\left(a_n^{\alpha_1} + b_n^{\alpha_2}\right)^{\frac{1}{l}}\right) + \mathcal{O}_{a.co.}\left(\left(\frac{log\, d_n^{S_{\mathcal{H}}} d_n^{\Theta_{\mathcal{H}}}}{n\phi_{\theta,x}(a_n)}\right)^{\frac{1}{2l}}\right).$$

5. Simulation (see Akkal, Kadiri, and Rabhi, 2021; Kadiri, Rabhi, and Bouchentouf, 2018)

In this section the authors consider simulated data studies to assess the finite sample performance of the proposed estimator and compare it to its competitor. To study the behaviour of the estimator, this part considers a comparison of the CFSIM (2.3) model (functional single index model with censored data) with that of CNPFDA (5.1) (censored non-parametric functional data analysis), for more details refer to (Chaouch, Bouchentouf, Traore, and Rabhi, 2020; Kadiri, Rabhi, Khardani, and Akkal, 2021).

$$\widehat{\Psi}_n(\theta,t,x) = \frac{\sum_{k=1}^{n} \frac{\delta_{\mathrm{k}}}{\bar{G}_n(T_{\mathrm{k}})}\, \Gamma\left(a_n^{-1} d(x,X_k)\right)\Omega\left(b_n^{-1}(t-T_k)\right)}{\sum_{k=1}^{n} \Gamma\left(a_n^{-1} d(x,X_k)\right)}. \tag{5.1}$$

Consider the following regression model where the covariate is a curve and the response is a scalar:

$$T_{\mathrm{k}} = R(X_{\mathrm{k}}) + \epsilon_k, k = 1,\cdots,n,$$

where ϵ_i is the error supposed to be generated by anautoregressive model defined by

$$\epsilon_k = \frac{1}{2}(\epsilon_{k-1} + \eta_k), k = 1, \cdots, n,$$

with $(\eta_k)_k$ a sequence of i.i.d. random variables normally distributed with a variance equal to 0.1. Functional covariate X is supposed to be a diffusion process defined on $[0, 2\pi\ /\ 3]$ and generated by the following equation:

$$X(t) = 2 - \cos\left(W\left(t - \frac{2\pi}{3}\right)\right), t \in \left]0, \frac{2\pi}{3}\right[,$$

where W is an α process generated by $W_k = \frac{2}{9} + \epsilon, k = 1, \cdots, 215$ and ϵ are i.i.d. $\mathcal{N}(0,1)$ (standard normal distribution) independently from W_k (the standard normal W_0 is independently generated). The authors carried out the simulation with a 215-sample of curve X represented by the following graphs (see Figure 1). Here a nonlinear regression function is considered such that:

$$R(X) = \frac{1}{4} exp\left\{2 - {}^{1}\Big/\left(\int_0^1 X'(t)dt\right)^2\right\}.$$

On the other hand, n i.i.d. random variables $(C_k)_k$ aresimulated through the exponential distribution $\mathcal{E}(1.5)$.

Given $X = x, T \hookrightarrow \mathcal{N}(R(X), 0.2)$ and thus, the conditional median, the conditional mode and theconditional mean functions coincide and are equal to $R(x)$ for any fixed x.

The computation of the estimator is based on the observed data $\left(Y_k, \delta_{k,} X_k\right)_{k=1,\cdots,n}$, where $Y_k = \min\{T_\mathrm{k}, C_\mathrm{k}\}$; $\delta_k = 1_{\{T_k \leq C_k\}}$ and single index θ which is unknown and has to be estimated.

In practice, this parameter can be selected by a cross-validation approach (see AitSaïdi, Ferraty, Kassa, and Vieu, 2008).

In this section one can select the real-valued function $\theta(t)$ among the eigenfunctions of the covariance operator $\mathbb{E}[(X' - \mathbb{E}X') < X', . >_{\mathcal{H}}]$ where $X(t)$ is a diffusion processes defined on arealinterval $[a, b]$ and $X'(t)$ its first derivative (see Attaoui and Ling, 2016). Thus for a chosen training sample $\mathcal{L}$, by applying the principal component analysis (PCA) method, the computation of the eigen vectors of the covariance operator estimated by its empirical covariance operator $\frac{1}{|\mathcal{L}|}\sum_{i\in\mathcal{L}}(X_i' - \mathbb{E}X')^t(X_i' - \mathbb{E}X')$ is the best approximation of functional parameter θ. Now, denote with θ^* the first eigen functions corresponding to the first higher eigenvalue to replace θ during the simulation step.

In practice, some tuning parameters have to be fixed: kernel $\Gamma(.)$ is chosen to be the quadratic function defined as $\Gamma(u) = \frac{3}{2}(1 - u^2)1_{[0,1]}$ and cumulative df $\Omega(u) = \int_{-\infty}^{u} \frac{3}{4}(1 - z^2)1_{[-1,1]}(z)dz$. Taking into account the smoothness of curves $X_k(t)$ (Figure 1), the distance in $\mathcal{H}$ is selected as:

$$d(\chi_1, \chi_2) = \left(\int_0^1 (\chi_1'(t) - \chi_2'(t))^2 dt \right)^{1/2},$$

as semi-metric.

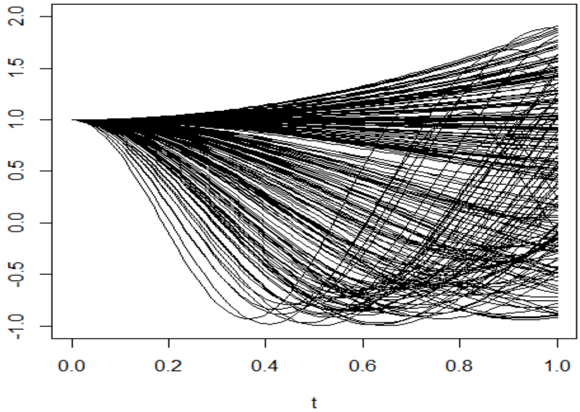

Fig. 1. A sample of curves $\{X_k(t), t \in [0,1]\}_{k=1,\cdots,215}$

Source: (Akkal, Rabhi, and Keddani, 2021).

In the following graphs, the covariance operator for $\mathcal{L} = \{1, \cdots, 215\}$ gives the discretisation of the first eigenfunctions θ(presented by a continuous curve), and all the eigenfunctions $\theta_k(t)$ (Figures 2 and 3).

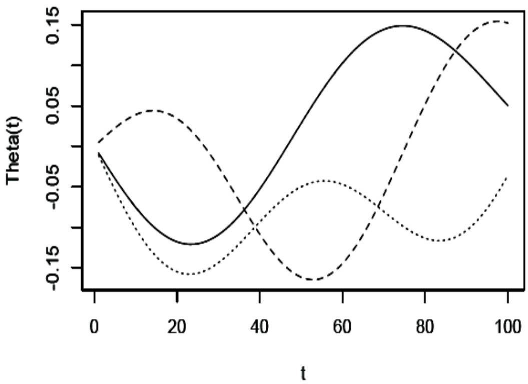

Fig. 2. The first three eigenfunctions (respectively, continuous, dashed and dotted lines) representing $\theta_k(t)$, $k = 1; 2; 3$

Source: (Chaouch, Bouchentouf, Traore, and Rabhi, 2020).

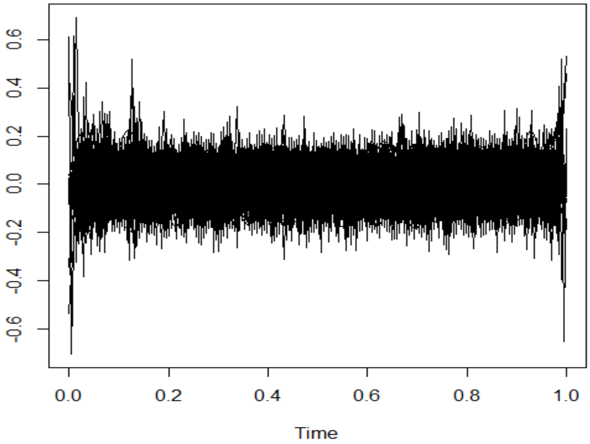

Fig. 3. Curves $\theta_k(t)$, $k = 1, \cdots, 215$

Source: own calculations.

Next, to simplify the implementation of the applied methodology, the study takes the bandwidths $b_n \sim a_n = h$, where $h$is chosen by the cross-validation method on the k-nearest neighbours (see Ferraty and Vieu, 2006, p. 102), and denote by θ^* the first eigenfunction corresponding to the first higher eigenvalue of the empirical covariance operator

$$\frac{1}{|\mathcal{L}|} \sum_{k \in \mathcal{L}} (X'_k - \mathbb{E}X')^t (X'_k - \mathbb{E}X'),$$

and follow these steps:

Step 1: Compute the inner product: $< \theta^*, X_1 >$,...,$< \theta^*, X_{215} >$, generate independently variables $\varepsilon_1, \dots, \varepsilon_{215}$, then simulate the response variables $Y_k = r(< \theta^*, X_k >) + \varepsilon_k$, where $r(< \theta^*, X_k >) = \exp(10(< \theta^*, X_k > -0.05))$ and generate independently variables $\varepsilon_1, \dots, \varepsilon_{215}$.

Step 2: For each k in the test sample $\mathcal{J} = 161, \cdots, 215$, one computes

$\hat{Y}_k = \hat{\vartheta}_{\theta^*}(\gamma, X_k)$ and $\hat{Y}_k = \hat{\vartheta}(\gamma, X_k)$, where $\vartheta_\theta(\gamma, x) = \inf\{y \in R, \Psi^\chi(y) \geq \gamma\}$ and

$$\widehat{\Psi}^\chi(y) = \frac{\sum_{k=1}^n \Gamma\big(h^{-1} d(x, X_k)\big) \Omega\big(h^{-1}(y - Y_k)\big)}{\sum_{k=1}^n \Gamma\big(h^{-1} d(x, X_k)\big)}, \forall y \in \mathbb{R}.$$

Step 3: Finally, the authors present the results by plotting the predicted values versus the true values and compute the mean squared error (MSE):

$$MSE = \frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} (Y_j - \hat{Y}_j)^2.$$

Then using the learning sample to compute the estimator of $\hat{Y}_k = \hat{\vartheta}_{\theta^*}(\gamma, x)$ and $\hat{Y}_{nk} = \hat{\vartheta}(\gamma, x)$ for $j = \{161, \cdots, 215\}$. Lastly, the results are shown by plotting the true values versus the predicted values for the MSE under censored data for both estimators with different censored rate (CR) (2.3) and (5.1) which are defined as

$$CFSIM.MSE = \frac{1}{55}\sum_{k=161}^{215}(Y_j - \hat{Y}_k)^2, CNPFDA.MSE = \frac{1}{55}\sum_{k=161}^{215}(Y_j - \hat{Y}_{nk})^2.$$

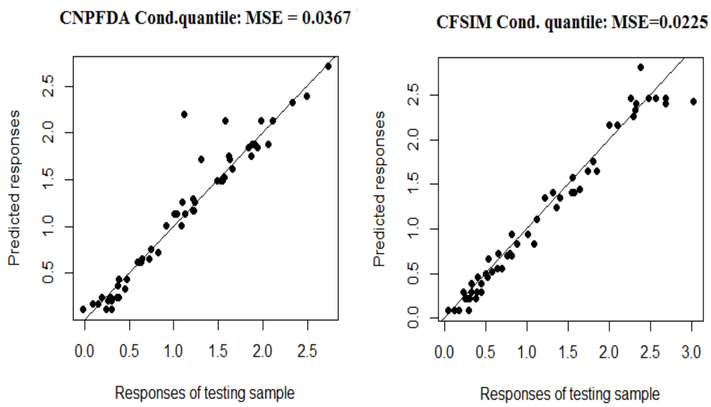

Fig. 4. Comparison between CNPFDA and CFSIM with $CR{\sim}3\%$

Source: own calculations.

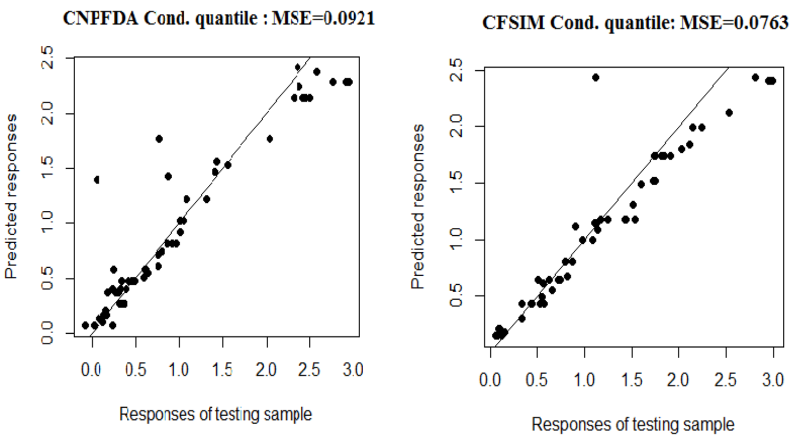

Fig. 5. Comparison between CNPFDA and CFSIM with $CR{\sim}18\%$

Source: own calculations.

One can see that the sum of mean square error (MSE) of the method (Censored--Single-Index-Method) is less than that of the Censored Non-Parametric Functional Data Analysis (CNPFDA). This is confirmed by the following graphs comparing the conditional quantile by censored single index methods (CFSIM) against the conditional quantile by censored non-parametric functional data analysis (CNPFDA). Hence the estimator is acceptable.

6. Conclusion

This paper focused on the non-parametric estimation of a conditional quantile for independent data under random censorship. Both the almost complete convergence (with rates), and the resulting estimator were shown to be asymptotically normally distributed under some regularity conditions. Naturally, the plug-in rules were used to obtain an estimator of the asymptotic variance term. The authors point out that here it is possible to prove that the variance estimator is almost completely consistent, using analogous ideas.

The proofs are based on a combination of the existing techniques. The author's prime aim was to improve the performance of this model for the conditional quantile with the censored response variable. The simulations experiments in this paper show that this methodology can be easily implemented and works very well for both simulated and real data. It is well known that the kernel choice does not affect substantially the quality of the estimator. By contrast, the bandwidth choice is very crucial in non-parametric estimation. In addition, in order to explore the effectiveness of this method in real situations, the authors applied the CNPFDA estimator to data constituting hourly electricity demand for the Rocky Mountain region of the United States, as well as spectrometric data.

This paper examines conditional distribution based on the single-index model in the censorship model when the sample is considered as an independent and identically distributed (i.i.d.) random variables. The asymptotic properties such as point-wise almost complete consistency, and the uniform almost complete convergence of the kernel estimator with rates, are presented under some mild conditions. In this case, the asymptotic properties of the estimation of the conditional hazard function and the asymptotic normality of the conditional quantile in the single functional index model are being investigated in other works by these authors.

References

Aït-Saidi, A., Ferraty, F., and Kassa, R. (2005). Single functional index model for a time series. *Revue Roumaine de Mathématique Pures et Appliquées*, (50), 321-330.

Aït-Saidi, A., Ferraty, F., Kassa, R., and Vieu, P. (2008). Cross-validated estimation in the single functional index model. *Statistics*, (42), 475-494.

Akkal, F., Rabhi, A. and Keddani, L. (2021). Some asymptotic properties of conditional density function for functional data under random censorship. *Applications and Applied Mathematics: An International Journal (AAM*), *16*(1), 12-42.

Akkal, F., Kadiri, N., and Rabhi, A. (2021). Asymptotic normality of conditional density and conditional mode in the functional single index model. *Econometrics. Ekonometria. Advances in Applied Data Analysis*, *25*(1), 1-24.

Ataoui, S., and Ling, N. (2016). Asymptotic results of a nonparametric conditional cumulative distribution estimator in the single functional index modeling for time series data with applications. *Metrika: International Journal for Theoretical and Applied Statistics*, *79*(5), 485-511.

Bouchentouf, A. A., Djebbouri, T., Rabhi, A., and Sabri, K. (2014). Strong uniform consistency rates of some characteristics of the conditional distribution estimator in the functional single index model. *Appl. Math.*, *41*(4), 301-322.

Bouraine M., AïtSaidi, A., Ferraty, F., and Vieu Ph. (2010). Choix optimal de l'indice multifunctionnel: Méthode de validation croisée. *Rev. Roumaine de Math. Pures et Appliquées*., *55*(5), 355-367.

Cai, Z. (2002). Regression quantiles for time series. *Econometric Theory*, (18), 169-192.

Chaouch, M., Bouchentouf, A., Traore, A., and Rabhi, A. (2020). Single functional index quantile regression under generaldependencestructure.*J. Nonparametr. Stat*., *32*(3), 725-755.

Deheuvels, P., and Einmahl, J. H. J. (2000). Functional limit laws for the increments of Kaplan-Meier product-limit processes and applications. *Ann. Probab.*, *28*(3), 1301-1335.

Ferraty, F., Laksaci, A., Tadj, A., and Vieu, P. (2010). Rate of uniform consistency for nonparametric estimates with functional variables. Statist. *Inference Stoch. Process.*, (140), 335-352.

Ferraty, F., Park, J., and Vieu, Ph. (2011). Estimation of a functional single index model. In F. Ferraty (Ed.), *Recent advances in functional data analysis and related topics. Contribution to statistics*. Physica.

Ferraty, F., Peuch, A., and Vieu, P. (2003). Modèle à indice functionnel simple, C.R. *Mathématiques, Paris*, (336), 1025-1028.

Ferraty, F., and Vieu, P. (2002). The functional nonparametric model and application to spectrometric data. *Computat. Statist. Data Anal*., *17*(4), 545-564.

Ferraty, F., and Vieu, P. (2003). Functional nonparametric statistics: A double infinite dimensional framework. In M. Akritas and D. Politis (Ed.), *Recent advances and trends in nonparametric statistics*. Elsevier.

Ferraty, F., and Vieu, P. (2006). *Nonparametricfunctional data analysis: Theory and practice*. New York: Springer.

Gannoun, A., Saracco, J., and Yu, K. (2003). Nonparametric prediction by conditional median and quantiles. *J. Statist. Plann. Inference*, (117), 207-223.

Gefeller, O., and Michels, P. (1992). A review on smoothing methods for the estimation of the hazard rate based on kernel functions. In Y. Dodge and J. Whittaker (Eds.), *Computational statistics* (pp. 459-464). Physica-Verlag.

Härdle, W., Hall, P., and Ichumira, H. (1993). Optimal smoothing in single-index models, *Ann.Statist*., (21), 157-178.

Hassani, S., Sarda, P., and Vieu, P. (1986). Approche non-paramétrique en théorie de la fiabilité, revue bibliographique. *Rev. Statist. Appl.*, *35*(4), 27-41.

Izenman, A. (1991). Developments in nonparametric density estimation. *J. Amer. Statist. Assoc*., (86), 205-224.

Kadiri, N., Rabhi, A., and Bouchentouf, A. (2018). Strong uniform consistency rates of conditional quantile estimation in the single functional index model under random censorship. *Dependence Modeling*, *6*(1), 197-227.

Kadiri, N., Rabhi, A., Khardani, S., and Akkal, F. (2021). CLT for single functional index quantile regression under dependence structure. *Acta Universitatis Sapientiae, Mathematica*, *13*(1), 45-77.

Keilegomvan, I., and Veraverbeke, N. (2001). Hazard rate estimation in nonparametric regression with censored data. *Ann. Inst. Statist. Math.*, (53), 730-745.

Khardani, S., Lemdani, M., and Ould-Saïd, E. (2010). Some asymptotic properties for a smooth kernel estimator of the conditional mode under random censorship. *Journal of Korean Statistical Society*, (39), 455-469.

Khardani, S., Lemdani, M., and Ould Saïd, E. (2011). Uniform rate of strong consistency for a smooth kernel estimator of the conditional mode under random censorship. *J. Statist.Plann. and Inf.*, (141), 3426-3436.

Khardani, S., Lemdani, M., and Ould Saïd, E. (2012). On the strong uniform consistency of the mode estimator for censored time series. *Metrika*, (75), 229-241.

Lecoutre, J-P., and Ould-Saïd, E. (1995). Hazard rate estimation for strong mixing and censored Processes. *J. Nonparametr. Stat.*, (5), 83-89.

Lemdani, M., Ould-Saïd, E., and Poulin, N. (2009). Asymptotic properties of a conditional quantileestimator with randomly truncated data. *J. Multivariate Anal*., (100), 546-559.

Liang, H., and de Uña-Álvarez, J. (2010). Asymptotic normality for estimator of conditional modeunder left-truncated and dependent observations. *Metrika*, (72), 1-19.

Ould Saïd, E., and Djabrane, Y. (2011). Asymptotic normality of a kernel conditional quantile estimator under strong mixing hypothesis and left-truncation. *Comm. Statist. Theory Methods*, (40), 2605--2627.

Ould Saïd, E., and Tatachak, A. (2011). A nonparametric conditional mode estimate under RLT model and strong mixing condition. *Int. J. Stat. Econ*., (6), 76-92.

Padgett, W.-J. (1988). Nonparametric estimation of density and hazard rate functions when samples are censored. *Handbook of Statist*., (7), 313-331.

Pascu, M., and Vaduva, I. (2003). Nonparametric estimation of the hazard rate: a survey. *Rev. Roumaine Math. Pures Appl.*, (48), 173-191.

Ramsay, J., and Silverman, B. (2005). *Functional Data Analysis, 2nd Ed*. New York: Springer Series in Statistics Springer.

Roussas, G. (1969). Nonparametric estimation of the transition distribution function of a Markov Process. *Ann. Statist.*, (40), 1386-1400.

Samanta, M. (1989). Nonparametric estimation of conditional quantiles. *Statist. Probab. Lett.*, (7), 407-412.

Singpurwallam, N., and Wong, M.Y. (1983). Estimation of the failure rate: a survey of nonparametric models. Part I: Non-Bayesian methods. *Comm. Statist. Theory Methods*., (12), 559-588.

Tanner, M., and Wong, W. -H. (1983). The estimation of the hazard function from randomly censored data by the kernel methods. *Ann. Statist.*, (11), 989-993.

Wang, H. and Zhao, Y. (1999). A kernel estimator for conditional *t*-quantiles for mixing samples and its strong uniform convergence (in Chinese). *Math. Appl. (Wuhan)*, (12), 123-127.

Xia, X., and Härdle, W. (2006). Semi-parametric estimation of partially linear single-index models. *Journal of Multivariate Analysis*, (97), 1162-1184.

Zhou, Y., and Liang, H. (2003). Asymptotic properties for *L*1 norm kernel estimator of conditional median under dependence. *J. Nonparametr. Stat.*, (15), 205-219.

REGRESJA KWANTYLOWA POJEDYNCZEGO WSKAŹNIKA FUNKCJONALNEGO DLA NIEZALEŻNYCH DANYCH FUNKCJONALNYCH Z CENZUROWANIEM PRAWOSTRONNYM

Streszczenie: Głównym celem artykułu jest prezentacja nieparametrycznej estymacji kwantyli rozkładu warunkowego na podstawie modelu jednoindeksowego w modelu cenzury, gdy próba jest traktowana jako niezależne zmienne losowe o identycznym rozkładzie. Przede wszystkim wprowadzono estymator jądrowy dla funkcji skumulowanego rozkładu warunkowego (cond-cdf). Następnie podano oszacowanie kwantyli przez odwrócenie oszacowanego cond-cdf. Właściwości asymptotyczne są określane, gdy obserwacje są połączone ze strukturą jednoindeksową. Na koniec przeprowadzono badanie symulacyjne, aby ocenić skuteczność tego oszacowania.

Słowa kluczowe: dane cenzurowane, estymator jądrowy, normalność, estymacja nieparametryczna, prawdopodobieństwo *small ball*.