„Analecta Cracoviensia" 53 (2021), s. 111–139
DOI: https://doi.org/10.15633/acr.5306

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A consequence of the notional existence of an effectively calculable yet non-recursive function[*]

Introduction

Alonzo Church (1905–1993) is the author of a certain hypothesis, first formulated in 1934 in connection with his research on lambda-calculus, and officially submitted to the AMS at its meeting on 22.03.1935 and first published (in terms of recursive functions) in an abstract in: *Bulletin of the American Mathematical Society*, vol. 41(1935) p. 332–333. The second version was published by Church in his article "An unsolvable problem of elementary number theory", which in his terms states as follows:

> We now define the notion, already discussed, of an effectively calculable function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ–definable function of positive integers).[1]

Church's student, S.C. Kleene reformulated it into a predicate form and coined it Church's Thesis (CT). Here is his formulation (Kleene 1952, 300):

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* Some parts of this article were written in collaboration with Jerzy Mycka, UMCS

[1] *The American Journal of Mathematics*, 58(1936), pp. 345–363; [Davis' anthology pp. 89–107].

Thesis I. Every effectively calculable function (effectively decidable predicate) is general recursive.

Church's thesis has kept many researchers up at night since then for some reasons. The main reason is that still for so many years, the logical value of the thesis has not been established. Secondly, it is otherwise known that a large part of computability theory is built using the thesis. We are not exaggerating when we say that it is one of the pillars of computability theory. Since its inception, many distinguished minds have worked on it, but the results are only partial. On this occasion a great deal of conceptual work has been done which has resulted, among other things, in the following findings.

Regarding the status of Church's thesis, the taxonomy of its formulations can be given:

- CT as an axiom or theorem (G. Kreisel, E. Mendelson. R. Gandy):
- CT as a definition (A. Church., K. Gödel);
- CT as a (empirical) working hypothesis (E. Post, J. Pepis);
- CT as a law of nature or a law of mind (E. Post);
- CT as an explication.[2]

Assume for the sake of further discussion that CT, as an expression of some language, has a structure that can be written in general:

(Schema of CT) (**I** = **S**):

where the variable **I** denotes an intuitive concept and the variable **S** a strict concept, while the equality sign should be understood as ambiguous, depending on the interpretation of CT. Following these observations, to the best of our knowledge, there are three main types of CT, which differ in the strength of the equality relation from the CT Schema. The strongest type of CT we have when '=' is understood as the sameness of two concepts taken by Church as definiendum and definiens (CT1).[3] Weaker types of thesis arise when '=' means the equivalence of both concepts (CT_2); and such a type when

[2] Cf. the paper "The Status of Church's Thesis" by Woleński and Murawski in: (Olszewski, Woleński, and Janusz 2006, 310–30).

[3] Church conceived CT, at least at one time, as a definition. The question of 'sameness' of concepts is a complicated matter and was considered in the paper (Olszewski 2009), sections 5.1.4 and 5.1.5. Yet the properties of the sameness of concepts are still unknown and we have to wait for scientists, probably cognitive scientists, to tell us what it means.

'=' means the identities of extensions of concepts (CT3). There are relations between them such that from CT1 follows CT2, and from CT2 follows CT3, i.e. CT3 is a necessary condition for the other types of the thesis.

(CT1) the identity of the concepts;
(CT2) the equivalence of the concepts;
(CT3) the sameness (identity) of extensions of the concepts.

Following G. Kreisel, we distinguish three versions of CT arising by replacing **I** in the general structure of CT by: the intuitive notion of an effectively computable function (E), the notion of a function computable by a physical process (P), and the notion of a mechanically computable function (M), and the right-hand side of the formal **S**, by the notion of recursiveness. Thus we have three versions:

(CT-H) human version ($E = R$);
(CT-P) physical version ($P = R$);
(CT-M) mechanical version ($M = R$).

On the other hand, CT variants are those formulations in which we replace the right-hand side of the identity by another term in any of the versions. For example, when we replace R by the notion of a function computable by a Turing machine (T), we get a human version variant in the form ($E = T$), and this is Turing's thesis, a variant of the first version, i.e. CT-H. This can be done for any formal computability model, for any version of CT, so there can be many such variants.[4]

Pepis on Church's Thesis

One of the proponents, of considering CT as an empirical working hypothesis, was the Polish logician and mathematician – Józef Pepis (1910–1941), a member of the Lviv-Warsaw School of Mathematics. In his doctoral thesis: *On the issue of decidability in terms of the narrower functional calculus*[5], he dealt with

[4] Note that CT is called by some, but quite often, the Church-Turing thesis. The above discussion clarifies how this issue is rearranged.

[5] Polish original: *O zagadnieniu rozstrzygalności w zakresie węższego rachunku funkcyjnego*; Archiwum Towarzystwa Naukowego we Lwowie: 7(8) (1937).

issues of computability and Church's thesis itself. In his dissertation, he wrote about CT:

> In regards to this hypothesis, the above mentioned authors (Church and Turing; added by A.O.) do not provide any convincing arguments in its support but they rely only on an *empirical fact* (emphasis by A.O.) that there are no known "calculable" functions except for those that are recursive. Due to such state of affairs, the question of the complete solubility of the problem of decidability for the system of the narrower functional calculus remains *open* (emphasis by A.O.).[6]

This sentence expressing doubt indicates that Pepis had a somewhat different position than Church on the understanding of CT, and even intended to construct an effectively computable, though non-recursive, function. His position was similar to Post's, which Church in turn criticized in his review of Post's article, submitted to the Journal of Symbolic Logic in October 1936. A letter is preserved which is Church's reply to Pepis' letter, which also concerned CT, and an excerpt of which we cite:

> I would say at the present time, however, that I have the impression that you do not fully appreciate the consequences which would follow from the construction of an effectively calculable non-recursive function.
> Therefore to discover a function which was effectively calculable but not general recursive would imply discovery of an utterly new principle of logic, not only never before formulated, but never before actually used in a mathematical proof – since all extant mathematics is formalisable within the system of Principia, or at least within one of its known extensions. Moreover this new principle of logic must be of so strange, and presumably complicated, a kind that its metamathematical expression as a rule of inference was not general recursive (for this reason, if such a proposal of a new principle of logic were ever actually made, I should be inclined to scrutinize the alleged effective applicability of the principle with considerable care). (Sieg 1997, 175–76)[7]

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[6] The reviewers of his doctoral dissertation were: Prof. Dr. St. Banach, and Prof. Dr. L. Chwistek. Prof. E. Żyliński, a mathematician and logician, was the supervisor of Pepis's doctoral dissertation. Cf. [Maligranda and Prytula, 2013, pp. 38–41].

[7] Church's letter to Pepis bears the date of June 8, 1937, and in it Church refers to the manuscript of Pepis' article sent to him: "Ein Verfahren der mathematischen Logik". As a humorous curiosity one can cite the fact that Church addresses Pepis per: "Dear Mgr. [Monsignore] Pepis". Presumably

In the second part of it, Church is talking about an entirely new logical principle, implied by the construction of effectively calculable and non-recursive function.[8] This new principle of logic, would have such features:

A. it would be unknown to previous mathematics, which is formalizable;[9]
B. it would be complicated and strange;
C. it would be metamathematically expressible;
D. it would be presented as a rule of inference.

The purpose of the remainder of this article (beginning with heading 1. Arithmetic), which is somewhat more technical in nature, will be to argue for the validity of Church's conjecture in the aforementioned case. In other words, it will be proposed a specific sense in which such a function is given and exists, and then in what sense it confirms the conjectures C., and D., while A., and B., to some limited extent.

A little methodology

It should be noted that the entire argument of this paper can be understood as a thought experiment, which is always subject to some kind of arbitrariness.

From the methodological point of view, the structure of the argument falls under a schema of *argumentum ad absurdum*. This argument, following N. Rescher[10] will be understand somewhat more loosely in the sense that the *ad absurdum* does not mean a mere contradiction, but means either *ad falsum* or *ad impossibile* or even *ad rudiculum*. Besides, to argue for CT, we do not presuppose its negation, but we assume an other sentence that implies the negation of CT. Let us emphasize that we want to argue for the truth of Church's thesis. So, to do this we assume a hypothesis *Hyp*:

- (*Hyp*) There exists effectively computable yet non-recursive function.;
- $Hyp \Rightarrow \neg CT$;
- $Hyp \Rightarrow F$; (will be showed in the paper)

Pepis signed his letter with: "Mgr." which in Polish is short for "Magister" which means "M.A." Church did not know this, so he deciphered the unfamiliar abbreviation as "Monsignore".

[8] In the letter there is also such a passage: "But it is proved in my paper in the *American Journal of Mathematics* that if the system of Principia Mathematica is omega-consistent, and if the numerical function f is not general recursive, then, whatever permissible choice is made of a formal definition of f within the system of Principia, there must exist a positive integer a such that for no positive integer b is the proposition $f(a) = b$ provable within the system of Principia."(Sieg 1997, 175–76).

[9] Formalizable in the frame of the Principia Mathematica system or its extension.

[10] (Rescher 2022).

- *F* satisfies the above features A.–D.;
- *F* is absurd (in Rescher's extended sense).

As a justification for this mode of argument, recall that a Jesuit priest, Giovanni Girolamo Saccheri (1667–1733), in his book *Euclides ab omni naevo vindicatus* (1733), pointed out conclusions resulting from, so called, the hypothesis of the acute angle. Such a hypothesis has, with respect to Euclid's fifth postulate, similar properties to our hypothesis *Hyp* with respect to CT, since, more precisely, it implies the negation of Euclid's fifth postulate and bears additionally absurd consequences. His hypothesis states that the summit angles a Saccheri's quadrilateral are acute, and this leads to the negation of Euclid's fifth postulate and to some counter-intuitive (absurd) conlusions that turned out to be certain principles of hyperbolic geometry. An important conclusion of recent article, with reference to the Italian Jesuit, is that it cannot be ruled out that an analogous situation could occur with our *Hyp*, as with Saccheri's hypothesis, because *Hyp* can stay in a similar relation to the current knowledge, as some theorems of hyperbolic geometry to the axioms of Euclidean geometry.

1. Arithmetic

Precise considerations of the foundations of mathematics are possible only after a clear understanding of the most basic concepts has been established.[11] In the context of the late nineteenth and early twentieth centuries research,[12] the formulation of a formal theory was achieved, designed for the aim of rigorously deriving possible theorems regarding the properties of natural numbers and of the relations and functions defined in their domain.

This theory, called (for historical reasons) Peano arithmetic, PA, can be characterized as follows. The language of the arithmetic is based on the language of first-order predicate logic with equality; however, it will be extended with some non-logical symbols:

[11] The reader well acquainted with Peano's arithmetic and the basics of the theory of computability can jump directly to the heading number 4 of the paper.

[12] In the history of arithmetic – a similar and imperfect approach was presented by Peano in his work "Arithmetices principia, Novo metodo exposita" (1889), while a fully mature and detailed version, almost in the modern version, was presented by D. Hilbert and P. Bernays in their work "Grundlagen der Mathematik" (1934).

- constant 0;
- unary function symbol *S*;
- binary function symbols +, •.

The understanding of these symbols will be consistent with their intuitive meaning assumed in arithmetic calculations. The first type of complex PA expression is a term, which is the representation of natural numbers given directly or as results of arithmetic operations. Hence a term is defined inductively as either an individual variable, a constant 0, or a simpler term preceded by the symbol *S* or two terms connected by the symbols + or •.

Now we can introduce the definition of formulas, i.e. the representation of statements about the properties of arithmetic expressions. The simplest formula is constructed as a combination of two terms by the equality sign; on the basis of such basic formulas complex formulas can be built by using logical functors (negation, conjunction, alternative, implication, equivalence) and quantifiers (limited and unlimited). A formula in which there are no variables outside of the range of the quantifier is called a sentence formula – its interpretation does not depend on the possible settings occurring within it variables.

In order to find theorems of PA theory, we need to define formulas that are the basis of reasoning (axioms), and methods of deriving conclusions from premises. In this way it will be possible, starting from the axioms, to construct a set of theorems by attaching further formulas to it.

The axioms of Peano arithmetic is a set of the following formulas describing the role of function symbols (variables x, y, z may be replaced by any term of the PA language):

- $S(x) = S(y) \rightarrow x = y$,
- $\neg(S(x) = 0)$,
- $x + 0 = x$,
- $x + S(y) = S(x + y)$,
- $x \cdot 0 = 0$,
- $x \cdot S(y) = x \cdot y + x$,

supplemented by an important axiomatic scheme of induction[13] (φ can be replaced by any formula of PA language with one free variable, i.e., not being within the scope of any quantifier):

- $\varphi(0) \wedge \forall x[\varphi(x) \rightarrow \varphi(S(x))] \rightarrow \forall x\varphi(x)$,

[13] An axiomatic scheme acts as a pattern for an infinite sequence of formulas, which is determined by replacing some component of the pattern by all possible expressions of a certain type.

and by equality axioms adapted to the original function symbols of PA arithmetic:

- $x = x$,
- $x = y \rightarrow y = x$,
- $x = y \wedge y = z \rightarrow x = z$,
- $x = y \rightarrow S(x) = S(y)$,
- $x = y \rightarrow z + x = z + y$,
- $x = y \rightarrow x + z = y + z$,
- $x = y \rightarrow x \cdot z = y \cdot z$,
- $x = y \rightarrow z \cdot x = z \cdot y$.

The above axioms define the basis of the properties of the functions set as the original components of the system: the successor, addition, and multiplication. These properties simultaneously guarantee a certain structure preserved by any model of PA theory.

What is still needed in order to develop a complete theory, is a logical apparatus allowing to derive theorems from the above set of axioms. The logical functors of negation, conjunction, alternative, implication, and equivalence satisfy the standard dependencies. The rules of inference may be defined in any way that ensures the possibility of proving all true formulas of the classical first order predicate calculus (for example, according to Hilbert's construction the rules may be restricted to modus ponens and the generalization principle, when extending the set of axioms by appropriate logical axioms).

In this way we obtain a complete designation of the theorems of Peano arithmetic – it is a set of formulas which can be derived using the appropriate (briefly characterized above) logical apparatus, beginning from the axioms of PA. The fact that a certain formula φ is a theorem of the theory PA will be written PA $\vdash \varphi$.

Within the consideration of formal theories, the property of consistency of a theory plays an important role. It can be expressed as the condition that for any sentence formula χ the theory cannot simultaneously prove χ and $\neg\chi$. A theory which does not satisfy this condition (a consistent theory) is not deductively interesting in the sense that its theorems will be all possible sentence formulas.

It now remains to be decided what kind of relations and functions can be described by the apparatus of Peano arithmetic. The technical means for expressing this relationship are the notions of representability and strong representability. A preliminary step is to assume that every natural number x

will be represented by an expression of the form $\underbrace{S\ldots S}_{x}0$ (x-th successor of 0), which we will denote by $\overline{x}$; of course, zero remains represented by the constant 0.

We say that the formula φ of PA with n free variables strongly represents the relation $R \subset \mathbb{N}^n$ if and only if for every $x_1,\ldots,x_n \in \mathbb{N}$ the following conditions hold:

- if $R(x_1, \ldots, x_n)$ then PA $\vdash \varphi(\overline{x}_1,\ldots, \overline{x}_n)$

and

- if $\neg R(x_1, \ldots, x_n)$ then PA $\vdash \neg\varphi(\overline{x}_1,\ldots, \overline{x}_n)$.

In turn, we say that the formula φ of $(n + 1)$ free variables represents the function $f : \mathbb{N}^n \rightarrow \mathbb{N}$ if and only if for each $x_1,\ldots, x_n \in \mathbb{N}$ it is true that PA $\vdash (\forall y)\left[\varphi(\overline{x}_1,\ldots, \overline{x}_n, y) \leftrightarrow y = \overline{f(x_1, \ldots, x_n)\}}\right]$.

The important relationship between the representation of functions and relations is described by a theorem that says that a relation $R \subset \mathbb{N}^n$ is strongly representable in PA if and only if the characteristic function of that relation is representable in PA.

A formal language can be associated with a structure **M**: a certain set M and a set of constants (belonging to the set M), functions (with arguments and results in M) and, if necessary, relations. Particular elements of the structure must be matched to the language: for every constant symbol there is a corresponding constant of the structure; for every function symbol, a function in the structure; for every relation symbol, a relation from the structure. On the basis of natural conditions (defined by Tarski[14]), the notion of the truth of a formula of a language in the structure can be introduced. The truth of a formula φ in the structure **M** is denoted by $\mathbf{M} \models \varphi$. A structure which makes every formula of a certain set of formulas Φ true is called the model of the set Φ. Kurt Gödel proved[15], that every formula that is true in every model of first-order logic has a proof. The aforementioned theorem can be expressed by appealing to somewhat broader notions. We say that a formula φ is a semantic consequence of a theory T (considered as a certain subset of formulas of a fixed language) if and only if φ is true in every model of the theory T, we denote such a relation by the symbol $\mathrm{T} \models \varphi$. Then the completeness theorem can

[14] Cf. any monograph on model theory.

[15] Gödel, strictly speaking, proved the theorem of the existence of a denumerable model, for any consistent set of sentence formulas of a first-order language. The completeness theorem for first-order logic easily follows from it (cf. Gödel 1930).

be stated as the statement that if there is T $\models \varphi$ then a proof can be found φ in the theory T: T $\vdash \varphi$.

Of course, we can take Peano arithmetic PA as the theory, in which case we obtain the following theorem as a conclusion.

Theorem 1. *Every formula which is true in any* PA *model also has a proof within the first-order predicate logic, when additional assumptions from the set of* PA *axioms are admitted.*

In what follows below, we will sometimes use certain models of the set of PA theorems. Among them the standard model **N** consisting of the set of natural numbers $\mathbb{N}$ and the usual successor, addition, and multiplication functions, plays a unique role.

2. Computability

As it turns out, to analyse the deductive capabilities of PA arithmetic, it is necessary to appeal to the formalized computability theory. We will therefore begin this section with a brief analysis introducing the essence of computability models.

A preliminary step in considering computable processes is establishing the conditions that allow to define certain problem-solving methods as algorithmic methods. As it turns out, in proposing various computation models based on the idea of processing in discrete steps, usually fairly uniform assumptions about the possibilities of constructing realistic computing devices were made.

Let us begin by presenting the analysis of the conditions of computability given by Alan Turing himself (cf. Turing 1937). The fundamental assumption is that conducted calculations must be based on the use of only a finite number of symbols, which are divided into groups of unambiguously distinguishable characters.

Furthermore, an analysis of various aspects of natural human computational activity leads to the following conditions:

- calculations are stored on a spatial medium divided into certain units – elementary cells;
- the number of symbols which can be entered into a single cell is limited by a fixed constant;
- the decision to modify the data is based on the observation of a finite and uniformly limited number of cells;

- a modification consists in changing the content of one of the cells;
- moving the attention to the next group of cells requires traveling only a finite distance from the current cell;
- in deciding whether to modify the data, some overall state of the structure may also be considered; the number of such different states for any particular algorithm must have an upper limit.

Without loss of generality, the constraints on the number of cells in the considered groups, the number of symbols in the cells, and the distance covered in changing the observed cells can be reduced to unity. This may cause some complication in the description of the computation, but it does not affect the solvability of problems.

The above conditions were clearly specified by the image of working on a paper tape with a pencil and an eraser. Alan Turing considered the conditions of efficient computation by describing a certain theoretical model of a mechanical calculator. This device can be described as follows: a machine consisting of an infinite tape, divided into identical cells, used to store input, output, and working information. All elements on the tape are inscriptions (strings of characters), with the rule of placing one character per cell in place. Without loss of generality, a certain alphabet is usually chosen as a character pool for inscriptions. In practice, the preference is often for a binary (zero-one) alphabet, which allows for a relatively convenient and concise representation of data. In addition, the design of the machine requires the definition of a finite set of states from which the element indicating the current situation (state) of the machine is derived. The transformation of the machine configuration is performed on the basis of a set of instructions which, on the basis of the state and symbol (contained in the currently observed cell), indicate the new state and the symbol with which the existing ones should be replaced, and decide the transfer of attention to one of the neighbouring cells.

Of course, this is not the only model that can be developed when considering the specifics of effective computation processes. However, having a description of such a model is an important basis for developing a general understanding of the essence of computable processes. From arithmetic's point of view, it would be advisable to propose a model of computability equal in its power to a Turing machine, but defined in terms of numerical functions. Since we do not admit information of infinite nature, the set of natural numbers becomes the numerical set sufficient for presenting the finite information resources.

In this context, we should note a special class of functions called the class of partial recursive functions PREC.

We say that a function with arguments and value in the set of natural numbers $\mathbb{N}$ is a partial recursive function (belongs to the PREC class) if and only if it:

- is one of the basic functions: $Z(x) = 0$; $S(x) = x + 1$; $I^i_n(x_1,\ldots, x_n) = x_i$, $n \in \mathbb{N}$, $1 \leq i \geq n$;
- is obtained as a composition of PREC class functions: if $f : \mathbb{N}^n \to \mathbb{N}$, $g_i : \mathbb{N}^k \to \mathbb{N}$, $i = 1,\ldots, n$ belong to PREC, then the newly defined function $h(x_1,\ldots x_k) = f(g_1(\mathrm{x}_1,\ldots x_k),\ldots g_n(x_1,\ldots x_k))$ also belongs to PREC;
- is obtained by a simple recursion operation applied to PREC functions: if $f : \mathbb{N}^{n+2} \to \mathbb{N}$, $g : \mathbb{N}^n \to \mathbb{N}$ belong to PREC, then the function $h : \mathbb{N}^{n+1} \to \mathbb{N}$, defined as follows:
  - $h(x_1,\ldots x_n, 0) = g(x_1,\ldots x_n)$
  - $h(x_1,\ldots x_n, S(x_{n+1})) = f(h(x_1,\ldots x_n, x_{n+1}), x_1,\ldots x_{n+1})$

also belongs to PREC;

- is obtained by applying minimisation operation to a PREC function: if $f : \mathbb{N}^{n+1} \to \mathbb{N}$ belongs to PREC, then $h : \mathbb{N}^n \to \mathbb{N}$ defined as follows:
  - $h(x_1,\ldots x_n) = y$ if and only if
    - $f(x_1,\ldots x_n, y) = 0$ and
    - for each $z < y$ the function $f(x_1,\ldots x_n, z)$ is defined with such a value that $f(x_1,\ldots x_n, z) \neq 0$

also belongs to PREC.

As can be seen, a function is partially recursive if it can be defined from simple basis functions, using operations that preserve the property of effective computability. In the definition three such operations are used:

- composition – after calculating the value of one function, its result becomes an argument for the calculation of another function; we are dealing here with the description of the sequentiality of calculations;
- simple recursion – we employ a function repeatedly, using as its argument in successive repetitions the previously determined value of the calculation of the same function (with a given initial argument); in this way we get a description of a repetition (a loop) of a certain calculation, where the number of repetitions is given as an initial parameter;
- minimization – we search the arguments of the function, until we find the one which will be its zero; we have here the process of searching for the smallest value satisfying a certain condition; the duration of this process is not explicitly given at the beginning of the search.

The last one of these operations poses a new problem for the class of partial recursive functions PREC. The function to which the minimization operation will be applied does not necessarily have a zero (even when it is a correctly defined computable function). Consequently, in some cases minimization will not be able to give a result, and the function defined by it will, for certain arguments, fall into infinite computation. Thus, we can divide the functions belonging to PREC into two groups:

- total functions – for each possible set of arguments they computationally determine the values (the designation of this class of total recursive functions is REC);
- partial functions – they are undefined (do not finish the computation correctly) for some data sets, but for those arguments that belong to their domain they still computationally determine the values.

Note that partial recursive functions for arguments outside the domain do not produce some special non-numerical value indicating indeterminacy. In such a case computation usually becomes an infinite process, an observer of which has no way of detecting whether the computation will eventually conclude or remain active forever.

As it turns out, functions that can be computed by Turing machines (in the chosen representation of numbers as inscriptions) are always partial recursive functions. Moreover, any partial recursive function can be implemented to be computed by a Turing machine. The same equivalence of computability applies to a much wider set of models such as Post machines, Kolmogorov machines, and Church calculus[16]. Therefore, the results regarding partial recursive functions gain special relevance as results that can be related to the intuitive notion of effective computability.

The notion of effective transformation (effective computation) was analysed by Hilbert, Gödel, Church, Post, and Kleene (Adams 2011). The work of Turing (Turing 1937), however, was an extremely important development, in which his analysis linking effectiveness to human computing activity proved crucial. In the context of the time, this step justified the correctness of the proposed computational models. Another aspect of his work is also worthy of note: by placing human activity in the foreground, Turing's analysis leads in a sense to the conclusion that the notion of effective computation is open to modification and evolution, relative to the dynamic nature of human potential i.e. human mind.

[16] For an overview of computational models cf. [Odifreddi (1989), (1989), pp. 31–86; Kolmogorov and Uspenskii (1958), (1958)].

3. Relation of computation theory and PA arithmetic

The main result on representability of relations and functions in the theory of Peano arithmetic (PA) is a theorem that relates representability to computability. We will call relations recursive if their characteristic functions belong to the set of recursive functions.

Theorem 3.1. *For each partial recursive function there corresponds a formula representing it in* PA *arithmetic.*

For each recursive (decidable) relation there corresponds a formula strongly representing it in PA *arithmetic.*

Using the apparatus of Peano arithmetic, it is possible to introduce new predicates and function symbols that will retain their recursive property (computability). An example of such a definition can be the "less than" predicate, defined as follows: $x < y = (\exists z \leq y)[\neg(z = 0) \wedge x + z = y]$.

We will now use the connection between recursive functions and Peano arithmetic to point out facts of great significance in mathematics. The first problem will be obtaining the possibility of a precise mathematical and computable description of the relations between formulas, their proofs, and the rules of inference used in the proofs. In order to achieve this, formulas must be expressed as numbers, in a process called arithmetization. The Gödel number of any formula (or term) φ will mean a natural number constructed according to appropriate rules, and will be denoted by $\ulcorner \varphi \urcorner$. The knowledge of Gödel numbers allows us to express all relations between formulas in terms of ordinary numerical relations. Moreover, thanks to the possibility of encoding finite sequences of numbers of any length belonging to $\mathbb{N}$ by single natural numbers, we can also effectively represent sequences of PA expressions (formulas or terms).

As can be easily seen, by applying the Gödel numbering method and relying on the possibility of defining various auxiliary recursive relations, two very important predicates can be defined:

- $\mathrm{Pr}(u, v)$ – signifies whether u is a code for a sequence of formulas (which can be reconstructed from the value of u), that form a proof in PA of the formula v;
- $\mathrm{Th}(v)$ – checks whether the formula signified by the number v is a PA theorem; Th can be defined with Pr as follows:

$$\mathrm{Th}(v) \equiv \exists u \, \mathrm{Pr}(u, v)$$

It should be noted that the relation, due to the use of the unbounded quantifier, is not a recursive relation. Our considerations will concern situations in which a theory will satisfy a rather basic condition of consistency. We can formulate it precisely in the following way: any theory T will be called consistent if and only if for any formula φ, $T \vdash \varphi$ and $T \vdash \neg\varphi$ do not occur simultaneously. Every consistent theory has a model.

Now we can state one of the most important results in the field of the foundations of mathematics: Gödel's theorem on the incompleteness of Peano arithmetic, documenting the lack of a complete match between the concepts of true formulas and formulas having a proof (theorems). As it turns out, within Peano arithmetic there is a formula that cannot be either proved or denied (that is, its negation cannot be proved).

Theorem 3.2 (Gödel's theorem (with Rosser's modification[17])). *If* PA *arithmetic is consistent, then there exists a sentence formula* χ *such that* $\mathrm{PA} \nvdash \chi$ *and* $\mathrm{PA} \nvdash \neg\chi$.

The above theorem reveals an extremely important fact. There are sentences describing arithmetical (that is, in a certain sense simple – based on addition and multiplication) properties of numbers, which cannot be proved, but also cannot be rejected by the proof of their negation. Since they are sentences with well-defined meaning, they have logical value (they are true or false in the standard model **N**). From this follows the conclusion that some true sentences (either χ or $\neg\chi$ in the theorem) do not have a proof. A particularly important example of this theorem is the consistency problem: whether it is possible within PA to construct a proof of the fact that PA is a consistent theory. Taking the formula $\varphi \equiv \neg\exists y\, \mathrm{Pr}(y, \ulcorner 0 = 1 \urcorner)$ as a representation of the consistency problem, it turns out that within PA such a formula φ has no proof (i.e., it cannot be shown that there is no proof of a false sentence stating the equality of 0 and 1)[18].

The formulation of the theorem suggests posing the question – can we somehow distinguish a class of such sentences of PA arithmetic which are true, and which with certainty have a proof. It turns out that such a result can be formulated in an elegant way (Boolos 1995) by referring to the notion of formulas

[17] Gödel proved the theorem under the assumption of ω-consistency, which is a concept stronger than mere consistency. Rosser weakened the assumption of the theorem to mere consistency.

[18] This is known as Gödel's second theorem.

of the $\sum_1^0$ type, that is, formulas which in PA have a proof of equivalence with formulas having at their beginning a group of existential quantifiers, followed by a subformula built only from restricted quantifiers, conjunction, alternative, implication, equivalence, and negation, combining equalities of PA terms.

Theorem 3.3. *If* χ *is a true sentence of* $\sum_1^0$ *type, then* PA *arithmetic guarantees a proof* χ*:* PA $\vdash \chi$.

Do we need Peano arithmetic's theory to obtain this extremely important result about the incompleteness of mathematical theories? The answer depends on the adopted point of view. If the aim is the simplest possible theory, the means of proof of which are not sufficient to obtain a complete set of theorems, then Robinson's arithmetic (called Q arithmetic) may be used. It differs from PA theory only in the absence of an axiomatic scheme of induction. This absence, however, causes the possibility of matching the Q theory with non-standard models, which do not guarantee the occurrence even of simple properties. For example, in Q neither $\forall x(0 + x = x)$ nor $\neg\forall x(0 + x = x)$ can be proved (cf. Smith 2007, 56).

For this reason, the Q theory does not appear to be particularly interesting, since it is shown to have too weak a system of axioms to prove the basic properties of arithmetic. Therefore, we should return to PA, which, although fully expresses the basic facts of arithmetic, at the same time illustrates the insurmountable difficulties of adequately complex deductive systems.

An important feature of a theory is the ability to internally determine the truth of formulas:

- is it possible, within a theory, to define such a relation with one variable, which will be true only for arguments which are the numbers of the formulas of this theory that are true in the standard model (under the standard understanding of the non-logical symbols of the language)?

It turns out that PA arithmetic does not allow us to define such a truth relation.

Theorem 3.4 (Tarski's theorem). *If Peano arithmetic is consistent, then there is no formula* $\Theta(x)$ *of Peano arithmetic, such that* ψ *is true (under the natural understanding of non-logical symbols) if and only if the relation* $\Theta(\ulcorner\psi\urcorner)$ *is true.*

The formula Θ represents the truth predicate, so it should become a true formula if and only if it speaks of a true argument. The argument representing a formula ψ is its Gödel number ⌈ψ⌉, which in PA arithmetic is written using the appropriate numeral $\overline{\ulcorner\psi\urcorner}$.

There remains one more area left, connecting the field of computability and the PA arithmetic theory. Since formulas can be represented by natural numbers, the question presents itself: can one algorithmically detect among formulas' numbers those, which are the numbers of theorems. Such a question leads to the following definitions. We call a formal theory decidable if the set of Gödel's numbers of its theorems is a recursive set. We call a formal theory undecidable if it is not decidable. It turns out that PA arithmetic unfortunately does not allow the algorithmic detection of theorems.

Theorem 3.5. *If* PA *arithmetic theory is consistent, then* PA *theory is also undecidable.*

Every algorithmic model (not only partial recursive functions, but also register machines, λ-calculus, Post machines, etc.), leads to the same metamathematical results. This situation is one of the motivations behind adopting Church's Thesis, a statement that restricts effective computability to the functions within the PREC set.[19] However, it also provides motivation for considering the problem: how would the provability possibilities of PA arithmetic (or other mathematical theories) change, if it became possible to introduce a function that is effectively computable but does not fall into the class of partial recursive functions?[20]

As can be seen from the above, computability depends on the essence of the notion of proof. Once it is accepted that a theory must guarantee the possibility of a formal proof, the notion of effective transformation determines permissible ways of constructing a set of axioms and the permissible form (structure?) of the rules of inference. The question of what an effective activity is, therefore, defines the structure of the development of mathematics, based on formalizable theories.

4. Computable and non-recursive extensions of arithmetic

Let us now imagine a situation in which we have a certain function f_Δ which does not belong to the PREC class, but which has the property of effective

[19] A discussion of some variants and consequences of Church's Thesis can be found in (Odifreddi 1989), (Olszewski 2009), (Copeland 2020)

[20] This question can also be taken as the primary question of the entire article.

and efficient computability, and which would be a numerical function defined in natural numbers. The existence of such a function is guaranteed by general considerations of the set theory concerning cardinal numbers of sets of mappings. Let us associate with this function a new function symbol Δ, corresponding to the unary function and having the property that for any argument (formally given by the numeral $\bar{x}$) we may determine the value of $\bar{y}$, such that in PA it always holds as an axiom that $\bar{y} = \Delta(\bar{x})$ if and only if $y = f_\Delta(x)$ (cf. Odifreddi 1989, 1:159–62).

We may now pose the question: what happens to the set of theorems of PA_Δ theory, that is, PA theory extended by the set of all possible axioms built for consecutive numbers $n \in \mathbb{N}$ according to the formula $\bar{y}_n = \Delta(\bar{n})$. That the set of theorems will expand, due to the new function symbol and new axioms is indubitable. The question that interests us, however, is whether it is possible within PA_Δ to obtain new theorems that are formulas without the symbol Δ, or whether any new theorems will be ones involving the symbol Δ. In technical terms, we ask whether PA_Δ is a conservative extension of PA? The answer to this question is (somewhat surprisingly) affirmative.

Theorem 4.1. *If* PA *is consistent, then* PA_Δ *is a conservative extension of* PA.

Proof. Let us consider a certain formula φ, which is a PA_Δ theorem: $\mathrm{PA}_\Delta \vdash \varphi$. Since PA is – by assumption – a consistent theory, it has a model. Let us consider any model **M** of all theorems of PA. We may extend it to a model **M**', in which we include a function f_Δ: $\mathbf{M}' = \mathbf{M} \cup \{f_\Delta\}$.

It is easy to see that if $\mathrm{PA}_\Delta \vdash \varphi$ then $\mathbf{M}' \models \varphi$. If the formula φ does not contain the symbol Δ, the actual model of this formula is **M**: $\mathbf{M} \models \varphi$. Since **M** was chosen to be any model of PA, it turns out that also $\mathrm{PA} \vdash \varphi$. □

We can also justify the above theorem by analyzing the structure of possible proofs within PA_Δ. Consider a formula φ having a proof in PA_Δ : $\mathrm{PA}_\Delta \vdash \varphi$. This proof is nothing else than a finite sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$, satisfying certain conditions (they are either axioms or they follow from previous elements of the sequence according to certain inference rules). In this sequence, appeals to Δ can only occur a finite number of times – so let us choose those formulas φ_i in which the symbol Δ occurs, and let us create their conjunction denoted as Φ_Δ. If we add Φ_Δ as assumptions to PA, then of course $\mathrm{PA} \cup \Phi_\Delta \vdash \varphi$. What is the status of the formulas from the set Φ_Δ? Their main determinant is the occurrence of the symbol Δ – it appears either in relation to specific numerical values (e.g. $\Delta(\bar{n}) = \bar{m}$), or in relation to general expressions based on finite number of values of the function Δ (e.g. $\forall x[(x < 5) \rightarrow \Delta(x) > 10]$), or in relation to expressions in which properties of Δ do not have actual significance (e.g. $\forall x[S(\Delta(x)) > \Delta(x))$. If we define a function h_Δ, which, on the initial segment of natural numbers used as arguments of Δ in Φ_Δ will return the same values as Δ (that is, for some k we have $\forall (x < k)\ [h_\Delta(x) = \Delta(x)]$), and outside of it will return zero, then we obtain a recursive function, i.e. one definable in PA. As can be easily seen, replacing in formulas of Φ_Δ the symbol Δ with such a function h_Δ does not in any way disturb the proof of φ, but at the same time will cause the formulas of the proof not to extend beyond the pure PA theory. Hence, we obtain the conclusion that also PA $\vdash \varphi$.[21]

The above theorem has important consequences.

Theorem 4.2. *If* PA *is consistent, then also* PA_Δ *theory is consistent.*

Proof. Let us assume that PA_Δ is contradictory – then every pair of formulas φ and $\neg\varphi$ has a proof in PA_Δ. However – due to the conservativeness of PA_Δ – formulas φ and $\neg\varphi$, which do not contain the Δ symbol have a proof in PA. Hence, PA would also be contradictory (contrary to the assumption). By virtue of *reductio ad absurdum* reasoning we therefore obtain the consistency of PA_Δ. □

Theorem 4.3. *If* PA *is consistent, then also the set (of indices) of theorems of* PA_Δ *theory is not a recursive set.*

Proof. If the set (of indices) of theorems of PA_Δ theory was to be recursive, then there would exist a characteristic function $\chi_{\mathrm{PA}_\Delta}$ belonging to the set of recursive functions, which would distinguish the theorems of PA_Δ. Since PA_Δ is a conservative extension of PA, then checking whether a formula is a theorem of PA_Δ and is written in the language of PA, would produce a recursive test for having a proof in PA. Thus PA would be decidable, which contradicts the results referenced earlier – hence, by virtue of *reductio ad absurdum*, we obtain the theorem. □

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[21] The accurate conduction of this reasoning would require explicit clarification of the rules of inference, and performing a reverse analysis showing that if the proved theorem does not contain Δ, then also in the formulas employed in the proof Δ does not have to occur.

Using the Δ symbol we can define a formula φ of PA_Δ, which corresponds to the function f_Δ: $\varphi(x, y) \equiv \Delta(x) = y$ (which signifies the weak representability of the function f_Δ in PA_Δ).

As the above results show, adding a nonrecursive function, although effectively computable in the intuitive sense will not add anything new to the deductive power of PA theory in terms of theorems concerning arithmetic. If the included function is not effectively computable, we still do not obtain a proper extension of PA.

It must be noted that modifying a theory by extending its syntactic apparatus (i.e., by adding new expressions that specify values of some special functions) is not sufficient. Such a 'black box' does not add anything substantially new to the deductive power of the theory. Only when this it is somehow integrated into the entire theory by a human (semantic) insight, translated as an appropriate expression, truly new consequences can be obtained.

5. Extensions of arithmetic with a given interpretation

We could, however, add a new function (or a relation) to PA_Δ, with a certain specific reference of its values to the system of formulas (that is, the indices of formulas) in PA_Δ. In other words, we would introduce an axiom that binds our new function symbol Δ with the formulas of PA_Δ. In this way we would obtain some kind of an internal (relative to PA_Δ) interpretation of the meaning of the function f_Δ. PA^+_Δ obtained in this way, may not be conservative with respect to PA.

Let us consider an example of the above statement. We begin by using a specific partial recursive function $U(x, y)$, which satisfies the following conditions:

- for each fixed $x \in \mathbb{N}$ a unary function $g(y) = U(x, y)$ is a partial recursive function;
- for each partial recursive function $f(y) \in$ PREC there exists its index $x \in \mathbb{N}$ such that for each argument y from the set of natural numbers either $U(x, y) = f(y)$, or the functions on either side of the equals sign are undefined.

Such a function U is called the universal function for the PREC set. An important property of the universal function U for PREC is that the function U itself is also a partial recursive function (it belongs to PREC), so it is represented in PA arithmetic.

Let us now define a relation $R(x, y)$ as follows:

$$R(x, y) \equiv \forall s \exists t U(x, s) = t \wedge \forall s \exists t U(y, s) = t \wedge \forall s U(x, s) = U(y, s).$$

The relation R states that numbers $x \in \mathbb{N}$ and $y \in \mathbb{N}$ are two indices of the same recursive function.

Theorem 5.1. *There exist such values x_0, y_0 for which there is in* PA *no proof of $R(x_0, y_0)$ and no proof of $\neg R(x_0, y_0)$.*

Proof. Let us negate the thesis and analyze consequences of the certainty that for every value either $R(x_0, y_0)$ or $\neg R(x_0, y_0)$ can be proved. For any partial recursive function $f: \mathbb{N} \to \mathbb{N}$ with the index x_f, then, one can calculate the index x_0 of the composition $Z(f(x))$, and then using the index y_0 of the function Z determine – through a computable process of generating all subsequent proofs in PA – whether $R(x_0, y_0)$ or $\neg R(x_0, y_0)$ holds. This would allow to recursively examine the totality of the function f. However, classical results of computability theory[22] indicate that there is no recursive function testing the totality of a function of PREC. Hence, we obtain the necessity of the existence of such x_0, y_0 for which PA $\nvdash R(x_0, y_0)$ and PA $\nvdash R(x_0, y_0)$. □

Let us now introduce a function symbol Δ' representing the function $f_{\Delta'}$ described as follows:

- $f_{\Delta'}(x) = y+1$ when the partial recursive function with index x is a total function and y is the smallest index of the function that is extensively identical to the function with index x;
- $f_{\Delta'}(x) = 0$ when a partial recursive function with index x is not total.

The function described above is not a partial recursive function (using it one can solve the unsolvable halting problem). However, it is conceivable that it is in some way intuitively computable, which would allow us to extent the axioms of PA with an infinite sequence of equations $\Delta'(\bar{n}) = \bar{m}$, one for each value of $n \in \mathbb{N}$. The assumption, which we adopted, that the above function is effectively computable in the intuitive sense, implies that the halting problem can be decidable when adopting the notion of computability as we have just extended it (and still remain undecidable with respect to the classical notion of computability).

Using the function symbol Δ' of this function, we can define a new relation in $\mathrm{PA}_{\Delta'}$ arithmetic, of the form: $R'(x, y) \equiv \Delta'(x) \neq 0 \wedge \Delta'(y) \neq 0 \wedge \Delta'(x) = \Delta'(y)$. This describes a relation the meaning of which is based on the meaning of the function f_{Δ}: x and y satisfy R' if and only if they are indices of the

[22] This is a special case of Rice's theorem; cf. (Rogers 1987, 36:34).

same (in the extensional sense) total recursive function. However, without further clarifications, the relation R' defines within the new theory only a certain relationship between numbers, which is associated with the symbol Δ', that has no specific meaning or use, other than to determine the value of $\Delta'(\bar{n}) = \bar{m}$. Let us consider the situation that arises when we add the axiom $R(x, y) \Leftrightarrow R'(x, y)$. Within the theory extended in this way, there appeared a tool which binds the symbol Δ' with concepts that can be analyzed within the theory itself. We thus have an internal interpretation of the meaning of Δ'. On the other hand, we can treat this axiom as a rule of the use of Δ' in proofs. It turns out that in this new theory we will obtain new theorems.

Theorem 5.2. *For any values of x_0, y_0, within the extended theory* $\mathrm{PA}_{\Delta'}$ *there exists either a proof of $R(x_0, y_0)$ or a proof of $\neg R(x_0, y_0)$.*

Proof. We need to calculate $\Delta'(x_0)$ and $\Delta'(y_0)$ for the given values of x_0, y_0; then the calculated values will be used in determining the logical value of $R'(x_0, y_0)$. In the final step it is sufficient to employ the axiom of equivalence of the relations R and R' to obtain the proof of $R(x_0, y_0)$ or of $\neg R(x_0, y_0)$. □

The above theorem shows that after choosing a suitable non-recursive function (effectively computable in the intuitive sense) and adding an axiom interpreting the meaning of this function, one can obtain an extension that is not conservative.

Let us try to use the reasoning given above to extend the deductive possibilities. For this purpose, we will use the result concerning formulas of the class $\sum_3^0$, that is, formulas equivalent to the formulas beginning with a sequence of an existential quantifier, a universal quantifier, and again an existential quantifier ($\exists\forall\exists$), followed by a subformula built only from restricted quantifiers, conjunction, alternative, implication, equivalence, and negation, combining equalities of PA terms.

Let us also introduce an extension (PA^+) of Peano arithmetic by adding to the set of axioms another axiom $\forall x P(x)$ for any predicate P such that for every number $n \in \mathbb{N}$ is $\mathrm{PA} \vdash P(n)$ by the usual means of proof.

Instead of extending the set of axioms, we can transfer a similar idea of extension to the realm of the rules of inference. Then we can supplement the arithmetic with an ω-rule:

- if $P(n)$ can be proved for any $n \in \mathbb{N}$ by classical methods of proof (without the ω-rule) then we can deduce (consider as proved) the sentence $\forall x P(x)$.

As it turns out, the approach of axiomatic extension and the addition of a new unconventional inference method are equivalent.

Theorem 5.3. *Sentence φ can be proven in arithmetic supplemented with the ω-rule if and only if $PA^+ \vdash \varphi$.*

The above modification of arithmetic has an interesting connection with deductive possibilities (completeness) for sentences of the $\sum_3^0$ class. As Boolos (1995, 191) states every true $\sum_3^0$ formula is provable when using the ω-rule. Hence we obtain the theorem.

Theorem 5.4. *If χ is a true sentence of type $\sum_3^0$, then* PA^+ *arithmetic guarantees a proof of χ :* $\mathrm{PA}^+ \vdash \chi$.

Let us now note that the problem of the ω-rule can be transformed into one connected with the halting problem and the set K_0 corresponding to it. In the case of any recursive function there arises the problem of checking which numbers belong to its domain – or, in a language closer to computer science, for which numerical arguments the function (or rather the algorithm implementing it) will complete its computation and produce a result. In more technical terms, it is the problem of checking whether the function with index k is defined for the argument n: $U(k, n)\downarrow$? This problem, called the halting problem, is associated with a numerical set K_0, which contains all and only these pairs, for which the halting problem has a positive solution: $K_0 = \{\langle x, y\rangle : | U(x, y)\downarrow\}$. The set K_0 is undecidable, that is, the characteristic function $\chi_{(K_0)}$ such that $\chi_{(K_0)}(\langle x, y\rangle) = 1$ if and only if $\langle x, y\rangle \in K_0$ and $\chi_{(K_0)}(\langle x, y\rangle) = 0$ if and only if $\langle x, y\rangle \notin K_0$ is not a recursive function. Let us analyze the inclusion of this function as effectively computable (though non-recursive) in the set of axioms of PA.

The method of constructing the new theory T makes it possible to generate successive theorems (and so also their codes). We can, therefore, introduce to our considerations a computable function $F_{\mathrm{PA}}(n)$, which generates the code of the n-th theorem of T. On its basis it is possible to define a partial recursive function $\chi_{\mathrm{PA}}(i) = \mu_y [F_{\mathrm{PA}}(y) = i]$, where $i = \ulcorner\psi\urcorner$ is the index of some sentence ψ. As can be easily seen, the function χ_{PA} is defined when its argument is an index of some theorem, otherwise it is undefined. Can it therefore be transformed in such a way that results in an always defined characteristic function of the set (of indices) of theorems? We can use for this purpose the set K_0, which checks whether partial recursive functions are defined or not for given arguments. This allows us to introduce a function:

$$\chi(\ulcorner P \urcorner, k) = \chi_{(K_0)}(\langle \ulcorner \chi_{PA} \urcorner, \ulcorner P(k) \urcorner \rangle),$$

which will equal one if and only if $P(k)$ is a theorem, and equal zero if and only if $P(k)$ is not a theorem. If we now add a function testing the relation P in a certain initial segment:

$$\chi'(\ulcorner P \urcorner, n) = \prod_{k=0}^{n} \chi_{(K_0)}(\langle \ulcorner \chi_{PA} \urcorner, \ulcorner P(k) \urcorner \rangle),$$

and look for the smallest value y, such that $P(y)$ has no proof:

$$\rho(\ulcorner P \urcorner) = \mu_y [\chi'(\ulcorner P \urcorner, n) = 0]$$

we will then obtain a recursive function ρ, such that $\rho(\ulcorner P \urcorner)$ is defined if and only if there exists some k such that T does not prove $P(k)$ and $\rho(\ulcorner P \urcorner)$ is undefined if and only if for every k the theory proves $P(k)$. The problem of proving $\forall x P(x)$ transforms into the question of checking whether a recursive function is defined. Therefore, once we have defined:

$$\rho_\forall(\ulcorner P \urcorner) = \chi_{(K_0)}(\langle \ulcorner \rho \urcorner, \ulcorner P \urcorner \rangle)$$

we obtain a test for $\forall x P(x)$ based on the properties of the particular $P(k)$: $\rho_\forall(\ulcorner P \urcorner) = 1$ if and only if for each k the arithmetic proves $P(k)$, which allows us to introduce an effective rule of inference that, based on the function $\rho_\forall(\ulcorner P \urcorner)$ decides whether $\forall x P(x)$.

In this way, the extension by a sequence of axioms describing the characteristic function K_0, extends the theory giving it possibilities corresponding to the ω-rule. This means that such a theory, together with the binding the function $\rho_\forall$ with universal quantifier becomes a Σ_3^0-complete theory.

An additional consequence is the decidability within the theory of all the theorems of that theory.

Theorem 5.5. *There exists a formula ρ of T, such that $\rho(\ulcorner \chi \urcorner)$ has a proof in this theory if and only if χ is a* PA *theorem; otherwise there exists within our theory a proof of $\neg\rho(\ulcorner \chi \urcorner)$.*

An illustration of the above results can be made by reference to Goldbach's hypothesis. Euler's formulation of this hypothesis, which reads: every even number greater than 2 is the sum of two prime numbers, can be represented in symbolic form as:

$$\forall n > 1 \exists p_1 \exists p_2 \, [2n = p_1 + p_2 \wedge Prime(p_1) \wedge Prime(p_2)].$$

Since Prime (p) is a recursive relation, the whole above sentence belongs to the $\prod_2^0$ class, which is contained within $\sum_3^0$. Hence, if Goldbach's hypothesis is true, we can be sure that there exists its proof within our theory.

The above considerations guarantee that the introduction of the characteristic function of the set K as an additional axiom allows us to simulate a finite ω-rule (i.e. ω-rule applied to a formula $\phi(x)$ such that all instances $\phi(\bar{k})$ can be proved without ω-rule). However, including a new function, allows us to define new functions which K is not able to check. This reasoning leads to a classic sequence, generated by the jump operation: $K = \phi'$, $K' = \phi''$, etc. A detailed analysis shows that a sufficiently deeply nested ω-rule (see Boolos 1995) allows to strengthen the deductive power of PA to true sentences of any chosen $\sum_n^0$ level. On the other hand, the truth-predicate for $\sum_n^0$ level also allow to strengthen the deductive power of PA to all true sentences of type $\sum_n^0$. Such predicate is also $\sum_n^0$ formula which allows us to reduce it to the corresponding jump ϕ^n.

In this way, we discover the adequacy of two methods for strengthening the proof power of the theory of arithmetic:

- by new axioms for some non-recursive functions (i.e., the characteristic function of ϕ^n);
- by new rules of inference (i.e., the nested finite ω-rule).

The ω-rule can be replaced by a function, the values of which will contain predictions of the course of a certain nonhomogeneous infinite process. In the ω-rule we have the possibility of using proofs of a predicate $P(i)$, which holds for each i, but each of these proofs might be significantly differ from the others, and there need not be a common scheme for carrying out the proof of $P(x)$. This might suggest that perhaps such a description of schemes in current mathematics is not sufficient, and one could achieve uniformity of proof by introducing new (effective but not recursive) means. Perhaps this is not the case, and there is no such possibility of conceptually improving the notion of effectiveness, but there do exist natural processes which, while acting according to unknown to us principles, but with known effects, capture the behaviour of heterogeneous processes and can extend (in a sense, experimentally) the power of deductive systems, as a kind of additional 'black boxes' with instructions, explaining how to utilize their results. Perhaps the human nature hides an ability to handle uniformly nonhomogeneous processes.

Concluding remarks

The results presented above show that by attaching a mechanism, allowing to efficiently compute the values of a certain non-recursive function to the set of PA axioms, we cannot obtain proofs of new theorems. Only with the introduction of the axiom linking a formula definable in PA with the values of a function, we are able to extend the deductive power of the theory. This may inspire proposing an analogy with Löb's theorem (cf. Boolos 1995, 54–59), which states that in PA arithmetic having a proof of the formula $\mathrm{Bew}(\ulcorner\rho\urcorner) \rightarrow \rho$ is equivalent to having a proof of ρ. (where Bew denotes an expression stating the existence of a proof of a formula with the given index). The Bew formula is recursively definable in PA, and therefore the analysis of provability can be associated with actual proofs, and, in this context, formulas' indices with the formulas themselves.

This leads us to note the need of adding axioms for the properties which do not have a recursive definition in PA. The role of such an axiom is to connect a non-recursive property P with a function that will verify the existence of this property:

$\chi_P(\ulcorner\rho\urcorner) \rightarrow \rho$ has the property P,

where the occurrence of a property can be somehow (though not recursively) defined in PA arithmetic. Such an approach would suggest that various extensions of PA would take place in a way similar to the implication of Löb's theorem, but due to the non-recursive character of the considered properties (which distinguishes them from Löb's theorem's Bew), these extensions require including the values of a function which may be effectively designated, but it has no recursive definition.

As can be seen from the above, what is of importance here is not only whether we have a certain computable and non-recursive function, but whether we are able to integrate (associate) it with classically computable functions. Without this integration we would only have a 'black box', producing values the meaning of which we cannot understand, and which we are unable to use in any proofs. In other words, we would have increased computational capabilities, but without an increase in deductive capabilities – we would not therefore extend our knowledge with new theorems.

We now present some conclusions obtained from the present research.

The falsity of Church's Thesis, as we understand it here, changes some results regarding the completeness of arithmetic. One needs to be able, however, to integrate the witness for the rejection of Church's Thesis (i.e., a certain effectively computable but non-recursive function) into the arithmetic. In other words, it is not sufficient to present a computable non-recursive function – it needs to be integrated through a suitable axiom (interpreting for the arithmetic what the function signifies, what its semantics are), in order to get an increase in the deductive possibilities. Without such a 'connection', there is no significant difference for a system, in our case PA, between a function that is non--recursive and effectively computable in the intuitive sense, and a function that is non-recursive and not effectively computable in the intuitive sense. The general properties of such a 'connection', a specific case of which was used in this paper, would be a matter for further studies.

In this context, the present paper reaffirms, by means of strict results, Church's intuition (cf. A.-D.) that an effectively computable non-recursive function could produce new means of proof.

It is also worth noting that in our understanding of CT, its formulation using the notion of an effectively computable – in the intuitive sense – function, presents, in a sense, a real characterization of the mind of human beings as *homo sapiens*. This notion can originate from the reflection of the mind on its own properties, that is, from self-awareness. Modifying the notion of an effectively computable function, as shown in the paper, results in changing mathematics, in the sense of changing the set of provable sentences. Thus CT, though not itself a mathematical sentence, has significant consequences for mathematics as such.

For it may turn out that in some today still unknown fully dimension or scale, the human mind can compute more than we currently think.

Abstract

A consequence of the notional existence of an effectively calculable yet non-recursive function[23]

The present paper is devoted to a discussion of the role of Church's thesis in setting limits to the cognitive possibilities of mathematics. The specific aim is to analyse the formalized theory of arithmetic as a fundamental mathematical structure related to the theory of computation. By introducing notional non-standard computational abilities into this theory, a non-trivial enlargement of the set of theorems is obtained. The paper also indicates the connection between the inclusion of new functions through the development of axioms and the potential modification of inference rules. In addition, the paper provides an explanation of the role of inclusion of a certain interpretation of the meaning of the axioms of the theory in that theory.

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[23] Some parts of this article were written in collaboration with Jerzy Mycka, UMCS

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