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On the stability of a certain Keynes-Metzler--Goodwin monetary growth model

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Abstract

The article has three aims. The first aim is to develop an improved version of the Keynes-Metzler-Goodwin (the KMG) monetary growth model originally presented and analysed in a series of publications by Carl Chiarella, Peter Flaschel and Willi Semler. The improvement of the model is obtained by modifying some of its equations in a way which ensures that they reflect real macroeconomic dependencies more properly. The equations that have been modified describe final demand expectations, determinants of production decisions, fixed capital accumulation, tax revenues, government budget deficit and money demand. The second aim is to transform the model into an intensive form described by seven non-linear differential equations and determine its unique steady state which shows proportions between variables on the balanced growth path. The third ultimate aim is to present a mathematical proof that the new improved version of the KMG model is locally asymptotically stable.

Keywords

- Keynesian macroeconomics
- disequilibrium macroeconomics
- monetary growth models
- nonlinear economic dynamics
- stability

JEL codes: C62, E12, E40

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Introduction

The mainstream economic theory has been increasingly questioned since the global financial crisis of 2007–2008. This raises interest in monetary models of economic growth which are related to Keynesian economics. Especially important in this regard are the works of three well known Keynesian economists: Carl Chiarella, Peter Flaschel and Willi Semmler who have been developing Keynesian economics over the last two decades. They have published jointly (sometimes with other co-authors) a series of books on Keynesian economics emphasing interrelations between real and financial spheres of the economy. Among others, one should mention the comprehensive monograph *The dynamics of Keynesian monetary growth* (Chiarella & Flaschel, 2000) and the fundamental trilogy *Reconstructing Keynesian macroeconomics* (Chiarella et al., 2012, 2013, 2014), in which they make an attempt at completely reinterpreting and reconstructing the whole Keynesian macroeconomics.[2] Other important works by these authors include Chiarella et al. (2000), Asada et al. (2003), Chiarella et al. (2005), and Charpe et al. (2011).

The most important model developed and analyzed (in various variants) by Chiarella, Flaschel and Semmler was named by them "the Keynes-Metzler–Goodwin model" (abbreviated as the KMG model) to emphasize its relationship to the concepts developed earlier by these economists. The KMG model is a disequilibrium monetary growth model which refers to ideas expressed in Keynes's *General theory* (1936) and in Goodwin's (1967) work on the interaction of growth and income distribution; these are the K and G components of the model. The KMG models take into account a gradual adjustment of inventories to its desired level. The dynamics of inventories is related also to the concept of expected sales, which is formulated in Metzler (1941) and constitutes therefore the M-component of the KMG approach. To the same extent, the KMG model refers also to the Keynes-Wicksell type models presented, among others, in Stein (1966), Rose (1966) and Fischer (1972). Besides, it is worth mentioning its similarity to the Keynesian model presented in Sargent (1987, Ch. 5).

All versions of the KMG model describe the functioning of the economy with the use of sixth or seventh dimensional systems of non-linear differential equations that reflect adaptive decision-making processes. A characteristic feature of any KMG model is that it can be transformed into so called intensive form model, in which all original variables are replaced with new variables describing proportions between them. The intensive form model enables in turn the derivation of the steady state of the economy, which describes proportions between variables of the original model maintained in the process of balanced growth at a certain constant rate.

[2] Another alternative and important current in economics referring to Keynes is post-Keynesian economics (Lavoie, 2014).

The main theoretical results which are obtained with the use of the intensive form of KMG models are stability theorems showing conditions under which the economy converges toward the steady state, which is equivalent to approaching the balanced growth path.

Since the intensive form model is a system of nonlinear differential equations to prove its (local) asymptotic stability, one has to show that all of the eigenvalues of the Jacobian matrix of this system are either negative numbers or complex numbers with negative real parts. Investigating the eigenvalues is a standard way of proving stability, which has been applied hundreds of times to different dynamic models. Since eigenvalues are also roots of the corresponding polynomial, their analysis is quite easy if one deals with a system of two differential equations; however, it becomes very difficult and sophisticated in the case of high dimensional systems such as the KMG model. In the latter case, probably the only way to show that the eigenvalues of a given Jacobian matrix guarantee stability is a subsequent zeroing of appropriate matrix parameters, which enables a multiple Laplace expansion of the determinant of the Jacobian matrix in order to obtain a sequence of matrices of an increasingly lower order whose eigenvalues can be more easily analysed. First, it is shown that the matrix of the lowest order has appropriate eigenvalues. Next, through a subsequent restoration of small positive values of previously zeroed parameters, it is demonstrated that also the original Jacobian matrix has eigenvalues which are either negative numbers or complex numbers with negative real parts.

The first rigorous proof of stability of the seven-dimensional KMG model based on the idea outlined above was presented twenty years ago in Chiarella et al. (2002). Other versions of the proof of stability (concerning modified KMG models) can be found in Asada et al. (2003) and Chiarella et al. (2006). In the latter publication the authors use the term *cascade of stable matrices approach* to name the general idea lying behind the proof.[3]

Work on the extension and modification of the KMG model is continuous. One example is Ogawa (2019a, 2019b, 2020), who extended the KMG model to a two-sector model. Another is Flaschel (2020), in which an extensive KMG model is used to analyze how taxes, transfer payments and government spending improve the social protection of the employee household sector. It is worth mentioning the most recent work by Chiarella et al. (2021), which is the culmination of the development of the Bielefeld school of macroeconomic thought. The book is authored by major representatives of this approach, namely Flaschel, Franke and the late Chiarella.

[3] Although the term *the cascade of stable matrices approach* comes from Chiarella et al. (2006), the idea of proving the stability hidden behind it was used independently of them by other authors in their proofs of stability of high dimensional dynamical systems completely different from the KMG model, e.g. Duménil and Lévy (1991), Kiedrowski (2018).

An interesting variant of the KMG model with private corporate debt, aimed at modeling the effects of fiscal and monetary policies, has been presented recently by Asada et al. (2018, 2019). The authors of these publications also deal with the question concerning the existence of limit cycles in the KMG model. Keynesian monetary growth models similar to the KMG model in which limit cycles or periodic orbits may exist have also been presented recently in Murakami (2016, 2018, 2020) and Araujo et al. (2020).

This article has three aims. The first aim is to demonstrate a new, improved KMG model which differs from the versions analyzed by the aforementioned authors through a number of modifications. The second aim is to transform the model into its intensive form and determine its unique steady state. The most important and challenging is the third aim, which is to prove that the new version of the KMG model is locally asymptotically stable, which means that the economy described by this model has an intrinsic ability to converge toward the balanced growth path.

The modifications are aimed at improving the model by eliminating from its equations some especially simplistic, questionable elements which seem to have been introduced not for their economic relevance but primarily for their simplicity, which essentially eases a mathematical analysis of the model. The first two modifications concern equations describing determinants of the growth rate of fixed capital $\dot{K}/K$ and the growth rate of expected demand $\dot{Y}^e/Y^e$. In all the KMG models presented by Chiarella et al. above equations contain parameter n added simply to other components of these equations. Depending on the particular version of the model, n expresses either the constant growth rate of labour supply or the growth rate of labuor productivity or the sum of these two. Adding n to these equations is very convenient since, in the steady state, the remining components of the equations become zero, which implies immediately that, in the steady state, both fixed capital and expected demand grow at rate n. In this article, the dynamics of fixed capital and expected demand are much more thoroughly elaborated. In the equation describing $\dot{K}/K$, the constant parameter n has been replaced by a variable expressing the growth rate of expected demand $\dot{Y}^e/Y^e$. At the same time, parameter n has been removed also from the equation for $\dot{Y}^e/Y^e$ and substituted with the growth rate of the real wage as one of the two factors influencing $\dot{Y}^e/Y^e$, which is economically much more justifiable. The growth rate of the real wage is an endogenous variable which depends on other model variables: primarily on the growth rate of the nominal wage and the inflation rate. Similar in character is the third modification concerning the equation showing three factors which determine output level Y. In this case, one doubtful factor (component) nN^d (N^d – desired level of inventories) has been replaced with $\dot{Y}^e$, i.e. the change in expected demand.

Other important modifications concern the assumption about taxes and real interest on bonds. In all versions of the KMG model presented by Chiarella et al., in order to ease the derivation of the intensive form of the model, it is assumed that lump-sum real taxes net of interest are collected in such a way that their ratio to the capital stock remains constant. As a consequence, the dependence of tax revenues on tax rates imposed on labuor or capital incomes is not visible. The dependency above is explicitly taken into account only in the present model, which allows for a more comprehensive analysis of fiscal policy regarding taxes and the government budget deficit. In particular and contrary to the previous models, the ratio of the government budget deficit to fixed capital ceases to be constant in time, despite the assumption about a constant ratio of government spending to capital.

The last modification introduced into the model concerns the money market and the interest rate. In the earlier versions of KMG models, a linear money demand function is considered, whose values depend on expected demand and the deviation of the actual interest rate from the steady state interest rate, which is known in advance.[4] As a consequence, money demand in the steady state is determined exclusively by the expected demand being independent of the value of the exogenously given steady state interest rate. This also raises some doubt. Therefore, in the present article, a nonlinear money demand function is considered which depends on expected demand and the actual, endogenously determined interest rate. The steady state interest rate is not known in advance.

The aforementioned improvements make the KMG model closer to reality. Despite the increased mathematical complexity of the modified model it is still possible to transform the model into its intensive form and determine its unique steady state which is the second aim of the article. Determination of the intensive form model and its steady state presented in the article opens the way to prove the stability of the model, i.e. to the realization of the third and the main aim of the article.

The proof of stability presented in the article exploits the aforementioned idea *cascade of stable matrices approach*, as named and originally applied by Chiarella et al. Despite this, due to at least two reasons, it is not just a re-narration of the proofs of stability of the earlier versions of the KMG model. The first reason is that the mathematical structure of the KMG model developed in the article is much more sophisticated than its earlier versions analyzed by Chiarella et al. This is especially visible in the intensive form of the model (described by seven non-linear differential equations with two additional conditions) for which the Jacobian matrix in the steady state had to be determined, and whose eigenvalues had to be examined. The second reason is that the proof of stability is obtained under a different set of assumptions about model parameters. Some of these assumptions concern tax rates, which are not present at all in the KMG models analysed by Chiarella et al.

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[4] Chiarella et. al. admit that the above assumption was introduced in order to ease the analysis of the model (Chiarella et al., 2000, p. 279).

To prove stability two fundamental difficulties had to be overcome. The first one was the derivation of the values of partial derivatives at the steady state of all functions defining the dynamics of the intensive form model in order to examine the Jacobian matrix. The second difficulty concerned the question as to which parameters in the Jacobian matrix should be zeroed and in which sequence, to show finally that all eigenvalues of the Jacobian matrix are either negative or complex numbers with negative real parts. There was no indication as to how to deal with this crucial question. The proper way of proceeding was found in a laborious heuristic process by undertaking a series of attempts that only finally led to success.

The article consists of three parts. Section 1 is devoted to the presentation of the new version of the KMG model. In Section 2, the model is transformed into its intensive form, and the steady state is determined. Finally, in Section 3, the proof of the local, asymptotic stability of the steady state is presented.

1. The model

This section provides an overview of the author's version of the KMG model, which is a modification of the KMG model considered in Chiarella and Flaschel (2000) as well as in Chiarella et al. (2013). The model is presented in the following seven sub-sections.

1.1. Consumption, wages and prices

Total final demand Y^d (in real terms) is the sum of private sector consumption C, private-sector gross investment I and government (public) sector demand G:

$$Y^d = C + I + G \tag{1}$$

Consumer demand is described by the following equation:

$$C = (1 - \tau_w)\omega L^d \tag{2}$$

where ω – real wage, L^d – labor demand (by assumption equal to employment), τ_w – tax rate on income from work.

Despite the simplicity of Equation (2), the consumption dynamics is the result of a complex processes on the one hand being formed by production dynamics, which determines employment, and on the other hand being determined by real wage $\omega = \frac{w}{p}$, (the ratio of nominal wage to price level the ratio of nominal wage w to price level p).

The growth rate of real wage $\hat{\omega} = \dot{\omega} / \omega$ equals the growth rate of nominal wage $\hat{w}$ minus inflation rate $\pi = \hat{p}$ (the growth rate of the price level):

$$\hat{\omega} = \hat{w} - \hat{p} \tag{3}$$

Wage and price dynamics are determined by two separate equations of two Phillips curves.

The rate of growth in the nominal wage is calculated according to the following equation of the wage Phillips curve:

$$\hat{w} = \beta_w \left(V - \overline{V} \right) + \kappa_w \hat{p} + \left(1 - \kappa_w \right) \pi^e + n \tag{4}$$

where $\beta_w > 0$ is a parameter of sensitivity of the nominal wage growth rate to the deviation of the employment rate $0 < V < 1$ (employment to labor supply ratio) from natural employment rate $\overline{V}$. It is also influenced by the labor productivity growth rate ($n > 0$) and the linear combination of the current inflation $\pi = \hat{p}$ and the expected inflation π^e $(0 < \kappa_w < 1)$.

Inflation $\pi = \hat{p}$, in turn, is described by the equation of the price Phillips curve

$$\hat{p} = \beta_p \left(u - \overline{u} \right) + \kappa_p \left(\hat{w} - n \right) + \left(1 - \kappa_p \right) \pi^e \tag{5}$$

according to which inflation depends on the deviation of the capacity utilization rate $0 \leq u \leq 1$ from its normal level $0 < \overline{u} < 1$ ($\beta_p > 0$ is the response parameter) and on the linear combination of the surplus of nominal wage growth rate $\hat{w}$ over labour productivity growth rate and the expected inflation $(0 < \kappa_p < 1)$.[5] Coefficient $\beta_p > 0$ is the reaction parameter of the price level to deviation $u - \overline{u}$.

The change in expected inflation $\dot{\pi}^e$ depends on the difference between the linear combination of current inflation $\hat{p}$ and its normal value $\overline{\pi}$ (equal to inflation at steady state $\overline{\pi} = \mu - n$) and expected inflation π^e:

$$\dot{\pi}^e = \beta_{\pi^e} \left(\alpha \hat{p} + (1 - \alpha) \overline{\pi} - \pi^e \right) \tag{6}$$

where $\beta_{\pi^e} > 0$ is the adjustment parameter.[6]

[5] The capacity utilization rate $0 < u < 1$ and the employment rate $0 < V < 1$ are model variables defined in section 1.5 (Equations (22) and (25)).

[6] The basic idea of Equation (6) is borrowed from Groth (1988, p. 254). Here, the revisions of π^e are a combination of two rules with weighting factor α, where the adjustments take place at a general speed of adjustment β_{π^e}. The polar case $\alpha = 1$ represents adaptive expectations. The other extreme case $\alpha = 0$ is a regressive mechanism.

1.2. Capital dynamics and investment demand

Private sector investment demand I equals the sum of net investments, increasing the fixed capital stock K of firms, and restitution investments incurred to replace depreciated capital. Net investments are described by the derivative of capital with respect to time $\dot{K}$, while restitution investments are given as δK, where $0 < \delta < 1$ is the capital depreciation index. Thus:

$$I = \dot{K} + \delta K \tag{7}$$

The increase in fixed capital $\dot{K}$ is described by the following behavioural equation:

$$\dot{K} = i_1\left(\rho^e - (r - \pi^e)\right)K + i_2\left(u - \overline{u}\right) + \hat{Y}^e K \tag{8}$$

where: ρ^e – the expected rate of return on fixed capital, r – the nominal interest rate on government bonds $r - \pi^e$ – the expected real interest rate, $\hat{Y}^e = \dot{Y}^e / Y^e$ – the expected growth rate of final demand, $i_1 > 0$ and $i_2 > 0$ – reaction parameters.

The expected rate of return on capital is defined as the ratio of expected profit to capital, where the expected profit is the difference between the expected amount of final demand Y^e and the costs of labour and capital depreciation:

$$\rho^e = \frac{Y^e - \omega L^d - \delta K}{K} \tag{9}$$

The growth rate of expected final demand $\hat{Y}^e = \dot{Y}^e / Y^e$ is assumed to be shaped according to equation:

$$\hat{Y}^e = \hat{\omega} + \beta_{y^e} \frac{Y^d - Y^e}{Y^e} \tag{10}$$

where $\hat{\omega} = \dot{\omega} / \omega$ is the real wage growth rate, (resulting from (3)–(5)), $\frac{Y^d - Y^e}{Y^e}$ is the relative error in final demand expectations. Symbol $\beta_{y^e} > 0$ denotes a reaction parameter.

The view that an increase in the real wage increases demand expectations is quite obvious and expressed by many economists (e.g. Napoletano et al., 2012). Introducing such an assumption to the KMG model is a novelty proposed by the author.[7]

[7] In all earlier versions of the KMG model it is assumed that $\hat{Y}^e = n + \beta_{y^e} \frac{Y^d - Y^e}{Y^e}$, where n is an exogenously given constant labour growth rate or the sum of labour growth rate and the growth rate of labour productivity (e.g. Chiarella et al., 2013, p. 247). Such an assumption is quite doubtful from the economic point of view.

According to (Equation 8), the first two factors contributing to high fixed capital dynamics are:

- an excess of the real expected rate of return on fixed capital ρ^e over the expected real interest rate on bonds $r - \pi^e$,
- an excess of the capacity utilization rate over its natural level ($u > \overline{u}$).

The third term $\hat{Y}^e K$ in equation (8) is another novelty introduced by the author.[8] According to it, the growth rate of fixed capital $\dot{K}/K$ depends also on the expected growth rate of final demand $\dot{Y}^e / Y^e$. Such an assumption is consistent with the Keynesian theory, which emphasizes the key role of final demand in the economy. "The feature that is uniquely Keynesian in growth models, and is found in all such models, however, is the role of aggregate demand as a determinant of growth" (Dutt, 2012, p. 42).

It is worth emphasizing that component $\hat{Y}^e K$ in Equation (8) introduces a new large loop into the model since, on the one hand, investments $I = \dot{K} + \delta K$ are a component of final demand $Y^d = C + I + G$ and, on the other, they are dependent on final demand. The last dependence is realized directly through Equations (8) and (10), and indirectly by the fact that, according to (19)–(20), final demand influences production decisions, which in turn, through (22), (24)–(25) and (3)–(5), have an effect on growth in the real wage and expected demand, and ultimately on capital growth.

1.3. Government budget deficit, issuance of bonds and money creation

Following Sargent (1987), Asada (2011), Asada et al. (2012) and Chiarella et al. (2000, 2005), for the sake of simplicity, the government sector's demand is assumed to be proportional to fixed capital, i.e.:

$$G = gK \tag{11}$$

where g is a constant ratio.

Government expenditures comprise also interest payments on bonds, paid to the private sector. Government expenditures are covered mainly by taxes. The government's total tax income (in real terms) is the following sum:

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[8] In other KMG models, fixed capital dynamics is described by a simpler equation of the form: $\dot{K} = i_1\left(\rho^e - \left(r - \pi^e\right)\right)K + i_2\left(u - \overline{u}\right) + nK$ (e.g. Chiarella et al., 2013, p. 246). Since in the steady state $\rho^e = (r - \pi^e)$ and $u = \overline{u}$, this implies immediately that, in the steady state, the growth rate of fixed capital $\dot{K}/K$ equals labour growth rate n. Such an equation makes it easier not only to derive the steady state of the model but also to prove its stability.

$$T = \tau_w \omega L^d + \tau_c \left(\rho K + \frac{rB}{p} \right) \tag{12}$$

where $\tau_w \omega L^d (0 < \tau_w < 1)$ are taxes on wages ωL^d and $\tau_c \left(\rho K + \frac{rB}{p} \right)$ are taxes on capital gains imposed on profits from fixed capital ρK and profits (in real terms) from government bonds $\frac{rB}{p}$ (both capital gains are taxed by the same capital tax rate $0 < \tau_c < 1$).[9]

Similarly, as in other KMG models (Asada, 2012; Asada et al., 2011; Chiarella et al., 2000) it is also assumed that the government budget deficit is financed either by the government selling new bonds to the public sector or through open market operations by the central bank, which buys short-term bonds from asset holders when issuing new money. The open market operations by the central bank are a unique channel through which money enters the economy. Hence the government budget deficit equation has the form:

$$\dot{B} + \dot{M} = pG + rB - pT \tag{13}$$

where B denotes government fixed-price bonds in the hands of the public, $\dot{B}$ describes changes in bonds and $\dot{M}$ reflects changes in the amount of money in the hands of the private sector.

The central bank's monetary policy rule is to keep a constant growth rate of money supply $\mu > 0$, so:

$$\hat{M} = \frac{\dot{M}}{M} = \mu \tag{14}$$

In view of (14), changes in bonds supplied by the government $\dot{B}$ which appear in Equation (13) are given residually.

Since the money market cannot be in disequilibrium, the constant growth rate of money supply necessitates an identical growth in money demand, so that in every moment of time:

$$M = M^d \tag{15}$$

where M^d is money demand.

[9] Assumptions about taxes in the presented KMG model differ essentially from those made in its earlier original versions. Usually, in the KMG models, real taxes, net of interest, remain in a constant proportion to the capital stock, e.g. $\frac{T - rB/p}{K} = t^n = \text{const}$. Such an assumption, together with $G = gK$, imply that the ratio of government budget deficit $G + rB - T$ to fixed capital K remains constant in time. Moreover, it eliminates both tax rates and bonds from the stage which makes easier derivation of intensive form model and calculation of its steady state (Chiarella et al., 2013, p. 247). A similar simplifying assumption can be found also in Sargent (1987, p. 16) and Rødseth (2000, p. 122).

Unlike in the earlier versions of the KMG model a non-linear money demand function is considered:

$$M^d = h\frac{pY^e}{r} \tag{16}$$

in which pY^e is the nominal value of expected demand in the goods market, r is the interest rate on bonds and $h > 0$ is a reaction parameter.[10]

In view of (14)–(16), to enable a constant growth in money supply, the interest rate on bonds in every moment of time must satisfy the following equation:

$$r = h\frac{pY^e}{M} \tag{17}$$

where $\hat{M} = \mu$.

1.4. Determinants of production decisions

To counteract the difficulty with maintaining the continuity of sales caused by too low levels of stocks, or the reduction in revenues resulting from too high levels of stocks, producers strive to maintain a desired ratio of stocks to expected demand. Hence the desired level of stocks N^d satisfies equation:

$$N^d = \beta_{N^d}Y^e \tag{18}$$

where β_{N^d} is the desired ratio of inventory to the expected demand.

Change in actual inventories $\dot{N}$ equals the difference between output Y and demand (equivalent to sales) Y^d:

$$\dot{N} = Y - Y^d \tag{19}$$

The decision about output level is based on three factors: currently expected demand Y^e, change in expected demand $\dot{Y}^e$ (resulting from Equation (10)) and the deviation of actual inventories from their desired level $N^d - N$. These assumptions are reflected in the following behavioural equation:

[10] Money demand function in the earlier KMG models is usually a linear function of the form $M^d = h_1pY^e + h_2pK(r_o - r)$, where r_o is the steady state interest rate and r is the actual interest rate (Chiarella et al., 2013, p. 247). Such a formula assumes that the steady state interest rate is known in advance. In the present KMG model money demand depends not on deviations $r_o - r$ but on the actual interest rate r. Consequently, the steady state interest rate (denoted in the article by $\overline{r}$) is an endogenous variable whose value depends on the model parameters.

$$Y = Y^e + \beta_Z \dot{Y}^e + \beta_n (N^d - N) \tag{20}$$

where $\beta_n > 0$, $\beta_Z > 0$ are reaction parameters.[11]

1.5. Constraints of output by capital and labor resources

Production requires inputs of fixed capital K and labor L. The dynamics of K are described by Equation (8). Labour supply is constant in time and equal to $\overline{L}$. Capital and labour are complementary, so both production factors are necessary in specific amounts to generate a given volume of production (no substitution possible). As a consequence, the production technology is described by two coefficients: the potential capital efficiency coefficient y^p and the labour productivity coefficient x. Coefficient y^p is ratio $y^p = \frac{Y^p}{K}$, where Y^p denotes potential output defined as the maximum production that can be obtained with the use of fixed capital K (and sufficient labour supply).

It is assumed that decisions on production made in line with Equation (20) are always feasible with respect to capital, which means that in every moment of time:

$$Y \leq Y^p = y^p K \tag{21}$$

The utilization degree of the existing fixed capital is measured by the capacity utilization rate $0 \leq u \leq 1$ representing the ratio of output Y (determined by (20)) to potential output $Y^p = y^p K$:

$$u = \frac{Y}{Y^p} \tag{22}$$

Deviations of the capacity utilization rate from its natural level $0 < \overline{u} < 1$ influence price dynamics, as shown in Equation (5). (In particular, according to (5), if u exceeds $\overline{u}$ firms have limited possibilities of increasing output in a short time and then are more likely to raise prices.) By assumption, labour demand L^d never exceeds labour supply $\overline{L}$, so labour demand is identical

[11] The equivalent of Equation (20) in the earlier versions of the KMG models does not contain component $\beta_Z \dot{Y}^e$ and takes a simpler form: $Y = Y^e + nN^d + \beta_n(N_d - N)$ (e.g. Chiarella et al., 2013, p. 247). The above equation is interpreted in such a way that output Y is designed to meet expected demand Y^e and adjust inventories to the desired level given by $Z = nN^d + \beta_n(N_d - N)$. Factor nN^d however has no clear economic meaning and is probably added to ensure balanced growth in the steady-state.

to employment. Hence, labour productivity coefficient x, defined as output per worker, is expressed as ratio $x = \frac{Y}{L^d}$, and labuor demand equals $L^d = \frac{1}{x}Y$.

Labour productivity x grows exogenously at a constant rate n:

$$n = \frac{\dot{x}}{x} \tag{23}$$

Thanks to this, despite constant labour supply $\overline{L}$, the constraint:

$$L^d = \frac{1}{x}Y \leq \overline{L} \tag{24}$$

is satisfied at any moment of time, even when output grows infinitely (later a balanced growth path, at which all quantitative variables, such as output, fixed capital, consumption and investments, grow at rate n will be considered).

A counterpart to capacity utilization rate in the case of labour market is the employment rate defined as a ratio of employment to labour supply:

$$V = \frac{L^d}{\overline{L}} \tag{25}$$

Deviations of V from its natural level $0 < \overline{V} < 1$ influence wage dynamics according to Equation (4).

1.6. Investments and savings

So far, only sources of the financing of private consumption C and government spending G have been presented. As far as investments $I = \dot{K} + \delta K$ are concerned the focus has been on the behavioural Equation (8):

$$\dot{K} = i_1\left(\rho^e - (r - \pi^e)\right)K + i_2\left(u - \overline{u}\right) + \hat{Y}^e K$$

which presents factors determining the decision on net investments $\dot{K}$ without indicating sources of their financing. To fill this gap the standard identity $\dot{K} = S$ will be derived below which implies that ex post net investments $\dot{K}$ equal total saving S composed of private savings S_p and government savings S_g:

$$S = S_p + S_g \tag{26}$$

Private savings S_p equal taxed capital income (on fixed capital and bonds) and are expressed by the Equation below:

$$S_p = (1-\tau_c)\left(\rho K + \frac{rB}{p}\right) \tag{27}$$

Equation (27) results from the simplifying assumption (reflected by Equation (2)) that, rather than saving, workers spend their whole taxed labour income entirely on consumption while capital owners do not consume at all, devoting their whole taxed capital income (from fixed capital and bonds) to savings (an assumption often found in post-Keynesian literature).

Government savings S_g (if negative—government deficit) are the difference between tax revenues and the sum of government spending on goods and services G and the real value of paid interest on bonds B/p:

$$S_g = T - \left(G + \frac{rB}{p}\right) \tag{28}$$

Equations (27)–(28) imply that:

$$S = \left(1-\tau_c\right)\left(\rho K + \frac{rB}{p}\right) + T - \left(G + \frac{rB}{p}\right) \tag{29}$$

According to (9) the realized profit on fixed capital equals $\rho K = Y^d - \omega L^d - \delta K$. Taxes in view of (12) satisfy Equation $T = \tau_w \omega L^d + \tau_c\left(\rho K + \frac{rB}{p}\right)$. Substituting both equations into (29) results in:

$$S = Y^d - (1 - \tau_w)\omega L^d - \delta K - G \tag{30}$$

In view of $C = (1 - \tau_w)\omega L^d$ and $I = \dot{K} + \delta K$ (Equations (2) and (7)) the final demand $Y^d = C + I + G$ equals:

$$Y^d = (1 - \tau_w)\omega L^d + \dot{K} + \delta K + G \tag{31}$$

Equations (30)–(31) imply directly that:

$$\dot{K} = S \tag{32}$$

which means that ex post net investments $\dot{K}$ are financed by total savings S.

2. The steady state stability of the intensive form KMG

In this section, the KMG model is reduced to an intensive form and its steady state is derived. Next, a theorem about local asymptotic stability is formulated.

2.1. Derivation of the intensive form KMG model

As in the case of many other models of economic growth, an interesting question is whether the economy described by this version of KMG model can evolve sustainably along the balanced growth path. Balanced growth is defined as a growth in which all quantitative variables of the model grow at the same growth rate. This implies that, in the process of balanced growth, proportions between model variables remain constant. For this reason, to verify if balanced growth is a possible outcome in the presented model, one needs to find such proportions between variables of the model that allow for such growth. To do this, the original model must be transformed into the intensive form model, whose variables describe proportions between variables of the original model.

The variables of the intensive form KMG model are as follows:

real labour income ωL^d per unit of output: $U = \omega L^d/Y$,
effective labour supply xL per unit of capital: $l = xL/K$,
real money supply M/p per unit of capital: $m = M/pK$,
expected final demand per unit of capital: $y^e = Y^e/K$,
inventory stock per unit of capital: $v = N/K$,
real value of bonds B/p per unit of capital: $b = B/pK$,
expected inflation: π^e,
final demand per unit of capital: $y^d = Y^d/K$,
output per unit of capital: $y = Y/K$.

To derive the intensive form KMG model the start is from the real labour income ωL^d per unit of output: $U = \omega L^d/Y$. Ratio Y/L^d is labour productivity, denoted by x. Hence $U = \omega/x$, which implies that the growth rate of U equals:

$$\hat{U} = \hat{\omega} - \hat{x} \tag{33}$$

The formula for the real wage $\omega = w/p$ implies that $\hat{\omega} = \hat{w} - \hat{p}$. According to (23), labour productivity grows exponentially at a constant rate n, so $\hat{x} = x$. In view of these:

$$\hat{U} = \hat{w} - \hat{p} - n \tag{34}$$

According to (4) and (5), the growth rate of nominal wage $\hat{w}$ and the inflation rate $\hat{p}$ are given by equations:

$$\hat{w} = \beta_w (V - \overline{V}) + \kappa_w \hat{p} + (1-\kappa_w)\pi^e + n \tag{35}$$

$$\hat{p} = \beta_p \left(u - \overline{u}\right) + \kappa_p \left(\hat{w} - n\right) + \left(1 - \kappa_p\right)\pi^e \tag{36}$$

Solving the above system of equations for and yields:

$$\hat{w} = \kappa\left(\beta_w \left(V - \overline{V}\right) + \kappa_w \beta_p \left(u - \overline{u}\right)\right) + \pi^e + n \tag{37}$$

$$\hat{p} = \kappa\left(\kappa_p \beta_w \left(V - \overline{V}\right) + \beta_p \left(u - \overline{u}\right)\right) + \pi^e \tag{38}$$

where $\kappa = \dfrac{1}{1 - \kappa_w \kappa_p}$, $\kappa_w \kappa_p \neq 1$.

Substituting (37)–(38) into (34) after simplifications results in the first equation of the intensive form model:

$$\hat{U} = \kappa\left(\beta_w \left(1 - \kappa_p\right)\left(\frac{y}{l} - \overline{V}\right) - \beta_p \left(1 - \kappa_w\right)\left(\frac{y}{y^p} - \overline{u}\right)\right) \tag{39}$$

where $V = \dfrac{y}{l}$, $u = \dfrac{y}{y^p}$.

The formula for effective labor supply per unit of capital, $l = xL/K$, implies that:

$$\hat{l} = \hat{x} + \hat{L} - \hat{K} \tag{40}$$

According to (8), the growth rate of fixed capital is given by:

$$\hat{K} = i_1\left(\rho^e - \left(r - \pi^e\right)\right) + i_2\left(u - \overline{u}\right) + \hat{Y}^e \tag{41}$$

where:

$$\rho^e = \frac{Y^e - \omega L^d - \delta K}{K} = \frac{Y^e}{K} - \frac{\omega L^d}{K} - \frac{\delta K}{K} = y^e - Uy - \delta \tag{42}$$

$$\hat{Y}^e = \hat{\omega} + \beta_{y^e} \frac{Y^d - Y^e}{Y^e} = \hat{\omega} + \beta_{y^e}\left(\frac{y^d}{y^e} - 1\right) \tag{43}$$

$$r = h\frac{pY^e}{M} = h\frac{py^e}{m} \tag{44}$$

By assumption labor supply is constant, so $\hat{L}=0$. Moreover $\hat{x}=n$. Hence in view of (41)–(44) the second equation of the intensive form model is obtained:

$$\hat{l}=n-\left(\hat{U}+n+\beta_{y^e}\left(\frac{y^d}{y^e}-1\right)\right)-i_1\left(y^e-Uy-\delta-\left(\frac{hy^e}{m}-\pi^e\right)\right)-i_2\left(\frac{y}{y^p}-\overline{u}\right) \quad (45)$$

where $u = y/y^p$ and $\hat{U}+n=\hat{\omega}$ (see Equation (34)).

Now proceed to variable $m = M/pK$ which is the real money supply M/p per unit of fixed capital. The growth rate of m equals:

$$\hat{m}=\hat{M}-\hat{p}-\hat{K} \quad (46)$$

By assumption, the nominal money supply grows at a constant rate μ, so $\hat{M}=\mu$. The inflation rate $\hat{p}$ is described by (38). The growth rate of capital $\hat{K}$, according to (40) and Equations $\hat{x}=n$, $\hat{L}=0$ equals $\hat{K}=n-\hat{l}$. Taking all of these into account the third equation is obtained:

$$\hat{m}=\mu-\pi^e-n-\kappa\left(\kappa_p\beta_w\left(\frac{y}{l}-\overline{V}\right)+\beta_p\left(\frac{y}{y^p}-\overline{u}\right)\right)+\hat{l} \quad (47)$$

The fourth equation of the intensive form model describes changes in inflation expectations. It is the same as in Section 1.1, i.e.:

$$\dot{\pi}^e=\beta_{\pi^e}\left(\alpha\hat{p}+(1-\alpha)\overline{\pi}-\pi^e\right) \quad (48)$$

where $\hat{p}$ satisfies (38).

The formula for expected demand per unit of capital implies that:

$$\hat{y}^e=\hat{Y}^e-\hat{K} \quad (49)$$

Hence, by substituting (43) and $\hat{K}=n-\hat{l}$ to (48) the fifth equation is obtained:

$$\dot{y}^e=y^e\left(\hat{U}+n+\beta_{y^e}\left(\frac{y^d}{y^e}-1\right)-n+\hat{l}\right) \quad (50)$$

where $\hat{l}$ satisfies (45).

The sixth equation describes the dynamics of inventory stocks per unit of capital $v=\frac{N}{K}$. To derive its intensive form start is from Equation:

$$\dot{v}=\frac{\dot{N}K-N\dot{K}}{K^2}=\frac{\dot{N}}{K}-\frac{N}{K}\frac{\dot{K}}{K}=\frac{\dot{N}}{K}-\frac{N}{K}\frac{\dot{K}}{K}=\frac{\dot{N}}{K}-v\hat{K} \quad (51)$$

According to (19), $\dot{N} = Y - Y^d$. Hence, by using $\hat{K} = n - \hat{l}$, $y = Y/K$, and $y^d = Y^d/K$, the sixth Equation is obtained:

$$\dot{v} = y - y^d - v(n - \hat{l}) \tag{52}$$

The seventh equation of the intensive form model reflects changes in the real value of bonds per unit of capital $= \frac{B}{pK}$. The definition of b implies that:

$$\dot{b} = \frac{\dot{B}}{Kp} - \frac{B}{Kp}\left(\hat{K} + \hat{p}\right) = \frac{\dot{B}}{Kp} - b\left(\hat{K} + \hat{p}\right) \tag{53}$$

According to (13) and (11), $\dot{B} = pG + rB - pT - \dot{M}$ and $G = gK$. Hence:

$$\dot{b} = g + rb - \frac{T}{K} - \frac{\dot{M}}{Kp} - b\left(\hat{K} + \hat{p}\right) \tag{54}$$

In the next step Equations (12), (5) and $\dot{M} = \mu M$ and $\hat{K} = n - \hat{l}$ to (54) are substituted which gives the final form of the seventh equation of the intensive form model:

$$\begin{aligned} \dot{b} = {} & g + rb - \tau_c\left(y^d - Uy - \delta + rb\right) - \tau_w Uy - \mu m - \\ & - b\left(n - \hat{l} + \kappa\left(\beta_p\left(\frac{y}{y^p} - u\right) + \kappa_p \beta_w\left(\frac{y}{l} - \overline{V}\right)\right) + \pi^e\right) \end{aligned} \tag{55}$$

where $m=M/pK$.

By collecting together the seven equations derived above the following system of seven nonlinear differential equations is obtained which describe the dynamics of proportions between variables of the original KMG model from Section 1.

$$\dot{U} = U\kappa\left(\beta_w\left(1 - \kappa_p\right)\left(\frac{y}{l} - \overline{V}\right) - \beta_p\left(1 - \kappa_w\right)\left(\frac{y}{y^p} - \overline{u}\right)\right) \tag{56}$$

$$\begin{aligned} \dot{l} = {} & l\left(n - \left(\hat{U} + n + \beta_{y^e}\left(\frac{y^d}{y^e} - 1\right)\right) - \right. \\ & \left. - i_1\left(y^e - Uy - \delta - \left(\frac{hy^e}{m} - \pi^e\right)\right) - i_2\left(\frac{y}{y^p} - \overline{u}\right)\right) \end{aligned} \tag{57}$$

$$\dot{m} = m\left(\mu - \pi^e - n - \kappa\left(\kappa_p \beta_w\left(\frac{y}{l} - \overline{V}\right) + \beta_p\left(\frac{y}{y^p} - \overline{u}\right)\right) + \hat{l}\right) \tag{58}$$

$$\dot{\pi}^e = \beta_{\pi^e}\left(\alpha\left(\kappa\left(\kappa_p\beta_w\left(\frac{y}{l}-\bar{V}\right)+\beta_p\left(\frac{y}{y^p}-\bar{u}\right)\right)+\pi^e\right)+(1-\alpha)\bar{\pi}-\pi^e\right) \tag{59}$$

$$\dot{y}^e = y^e\left(\hat{U}+\beta_{y^e}\left(\frac{y^d}{y^e}-1\right)+\hat{l}\right) \tag{60}$$

$$\dot{v} = y - y^d - v(n-\hat{l}) \tag{61}$$

$$\dot{b} = g + rb - \tau_c\left(y^e - Uy - \delta + rb\right) - \tau_w Uy - \mu m -$$
$$- b\left(n-\hat{l}+\kappa\left(\beta_p\left(\frac{y}{y^p}-u\right)+\kappa_p\beta_w\left(\frac{y}{l}-\bar{V}\right)\right)+\pi^e\right) \tag{62}$$

It should be emphasized that besides variables $U, l, m, \pi^e, y^e, v, b$ there are also two additional variables which appear in Equations (56)–(62). These are the final demand per unit of capital $y^d = \frac{Y^d}{K}$ and the production per unit of capital $y = \frac{Y}{K}$. To identify interrelations between the above variables it should be noted that, according to (1) and (20), variables Y^d and Y appear in the following equations of the original model from Section 1:

$$Y^d = C + I + G \tag{63}$$

$$Y = Y^e + \beta_Z \dot{Y}^e + \beta_n(N^d - N) \tag{64}$$

By dividing both equations by the result is:

$$y^d = \frac{C}{K}+\frac{I}{K}+\frac{G}{K} \tag{65}$$

$$y = y^e + \beta_Z\frac{\dot{Y}^e}{K}+\beta_n\left(\frac{N^d}{K}-\frac{N}{K}\right) \tag{66}$$

where $y^e = \frac{Y^e}{K}$ and $v = N/K$.

Substituting Equations (2), (7) and (11) into (65) and Equations (43), (18) into (66) results in the following system of two linear equations (variables Y^d and Y appear on both sides of these equations):

$$y^d = (1-\tau_w) yU + i_1\left(y^e - yU - \delta - \left(\frac{hy^e}{m} - \pi^e\right)\right) + i_2(u - \overline{u}) + \delta + g +$$
$$+\kappa\left(\beta_w(1-\kappa_p)\left(\frac{y}{l} - \overline{V}\right) - \beta_p(1-\kappa_w)\left(\frac{y}{y^p} - \overline{u}\right)\right) + n + \beta_{y^e}\left(\frac{y^d}{y^e} - 1\right) \quad (67)$$

$$y = y^e + \beta_n \beta_{N^d} y^e - \beta_n v +$$
$$+ y^e \beta_Z\left(\kappa\left(\beta_w(1-\kappa_p)\left(\frac{y}{l} - \overline{V}\right) - \beta_p(1-\kappa_w)\left(\frac{y}{y^p} - \overline{u}\right)\right) + n + \beta_{y^e}\left(\frac{y^d}{y^e} - 1\right)\right) \quad (68)$$

Since variables y^d and y appear also in differential Equations (56)–(62), Equations (67)–(68) must be taken into account in any analysis of the above system of differential equations. This means that the complete KMG model in intensive form is composed not only of Equations (56)–(62) but also of Equations (67)–(68).

2.2. The steady state

The economy described by the KMG model presented in section 1 remains in the steady state if the proportions between its variables allow for a balanced growth of the economy at a constant growth rate equal to the growth rate of labour productivity n.

Formally, the steady state is described by a vector $\left(\overline{U}, \overline{l}, \overline{m}, \overline{\pi}^e, \overline{y}^e, \overline{v}, \overline{b}, \overline{y}, \overline{y}^d\right)$ for which the right hand sides of all Equations (56)–(62) are equal to zero and additionally Equations (67)–(68) are satisfied.

By solving Equations (56)–(62) with left hand sides zeroed out and considering (67)–(68) it is easy to obtain analytically that the formulas describing the steady state values of the intensive form KMG model are as follows:

$$\left.\begin{array}{l}
\overline{U} = \dfrac{\gamma\overline{u}y^p - (n+\delta+g)}{(1-\tau_w)\overline{u}y^p}, \ \overline{l} = \dfrac{\overline{u}y^p}{\overline{V}}, \ \overline{m} = \gamma\dfrac{h\overline{u}y^p}{\overline{r}} \\
\overline{\pi} = \mu - n, \ \ \overline{y}^d = \overline{y}^e = \gamma\overline{u}y^p, \ \ \overline{v} = \dfrac{1-\gamma}{n}\overline{u}y^p \\
\overline{b} = \dfrac{g - \tau_c\left(\gamma\overline{u}y^p - \dfrac{\overline{c}}{(1-\tau_w)} - \delta\right) - \tau_w\dfrac{\overline{c}}{(1-\tau_w)} - \mu\overline{m}}{n - \gamma\overline{u}y^p + \dfrac{\overline{c}}{(1-\tau_w)} + \delta + \tau_c\left(\gamma\overline{u}y^p - \dfrac{\overline{c}}{(1-\tau_w)} - \delta + \overline{\pi}\right)}
\end{array}\right\} \quad (69)$$

where:

$$\gamma = \frac{n+\beta_n}{n+\beta_Z n^2+\beta_n\beta_{N^d}n+\beta_n}, \bar{r} = \frac{g+\mu-\tau_w\left(\mu-n-\delta+\gamma\bar{u}y^p\right)}{\left(1-\tau_w\right)},$$

$$\bar{c} = \gamma\bar{u}y^p - \left(n+\delta+g\right)^{12}$$

To show that proportions described by the above formulas indeed allow for balanced growth of the economy at the rate n the first focus is on the effective labor supply per unit of capital $l = xL/K$. In the steady state, the growth rate of l is zero, so $\hat{l} = \hat{x} + \hat{L} - \hat{K} = 0$. By assumption, labour supply L is constant while labour productivity x grows at rate n. Hence, in the steady state, the growth rate of fixed capital equals the growth rate of labour productivity, $\hat{K} = n$. According: to (21) and (22), $Y \leq Y^p = y^p K$ and $u=Y/Y^p$. In view of this, $\bar{u}y^p = \bar{Y}/\bar{K}$. Since $\bar{u}y^p$ is constant in time, output $\bar{Y}$ in the steady state must also grow at rate n. When examining other formulas, one can easily notice that in the steady state also other quantitative variables, such as investments, consumption and government purchases, grow at rate n. Additionally, taking into account that in the steady state the expected inflation equals the actual inflation, it is easy to find that in the steady state also the growth rate of the real wage $\hat{\omega} = \hat{w} - \hat{p}$ is equal to n. What seems to be interesting is that, according to equation $\bar{c} = \gamma\bar{u}y^p - \left(n+\delta+g\right)$, consumption per unit of capital (and thus also per unit of output) decreases in the steady state when the growth rate of labour productivity n (equal to the balanced growth rate) increases. Moreover, an increase in n also lowers the ratio $\bar{U}$ of labor income to output. Finally, it is worth stressing that the only variable which depends in the steady state on the growth of money supply μ is inflation rate $\bar{\pi} = \mu - n$. Thus money neutrality is confirmed.

2.3. Stability of the steady state

The main topic of the paper is the local asymptotic stability of the discussed KMG model starting from the formal definition of stability of the steady state of the KMG model in intensive form from the previous subsection.

Definition. The steady state $(\bar{U}, \bar{l}, \bar{m}, \bar{\pi}^e, \bar{y}^e, \bar{v}, \bar{b})$ of model (56)–(62), (67)–(68) is locally asymptotically stable if there is such neighborhood of this steady

[12] Due to the limited number of words imposed by the publisher, the article omits the derivation of Equation (69). However, this derivation as well as all other derivations and other details may be made available to readers by the author on request at his e-mail address.

state that every solution of the model starting from this neighborhood, converges toward it when time tends to infinity, i.e.:

$$(U_t, l_t, m_t, \pi_t^e, y_t^e, v_t, b_t) \underset{t\to\infty}{\rightarrow} (\overline{U}, \overline{l}, \overline{m}, \overline{\pi}^e, \overline{y}^e, \overline{v}, \overline{b})$$

To prove the stability defined above the following assumptions will be used.

Assumption 1. Parameters $\beta_z, \beta_n, \alpha, \beta_p, \beta_w, \beta_{y^e} > 0$ occurring in equations:

- $Y = Y^e + \beta_z \dot{Y}^e + \beta_n (N^d - N)$
- $\dot{\pi}^e = \beta_{\pi^e}\left(\alpha\hat{p} + (1-\alpha)(\mu - n) - \pi^e\right)$
- $\hat{p} = \beta_p(u - \overline{u}) + \kappa_p(\hat{w} - n) + (1-\kappa_p)\pi^e$
- $\hat{w} = \beta_w(V - \overline{V}) + \kappa_w\hat{p} + (1-\kappa_w)\pi^e + n$
- $\hat{Y}^e = \hat{\omega} + \beta_{y^e}\dfrac{Y^d - Y^e}{Y^e}$

are sufficiently small.

According to Assumption 1 output is determined mainly on the basis of demand expectations Y^e, whose growth rate $\hat{Y}^e$ depends mainly on the growth rate of the real wage $\hat{\omega} = \hat{w} - \hat{p}$.

Changes in inflation expectations $\dot{\pi}^e$ are stabilized strongly by a constant ($\mu - n$) factor reflecting inflation in the steady state.

Moreover, deviations of the capacity utilization rate from its normal level $u - \overline{u}$ weakly influence the inflation rate $\hat{p}$, and deviations of the employment rate from its natural level $V - \overline{V}$ weakly influence the growth rate of nominal wage $\hat{w}$.

Assumption 2. The growth rate of the nominal money supply μ cannot exceed the sum of labour productivity growth rate n and the capital depreciation rate δ:

Assumption 3. Parameter occurring in the investment equation:

$$I = i_1\left(\rho^e - (r - \pi^e)\right)K + i_2(u - \overline{u})K + \hat{Y}^e K + \delta K$$

is sufficiently large.

According to Assumption 3, investment demand is assumed to be highly sensitive to the difference between the expected profit from fixed capital and the expected real interest rate.

Assumption 4. Parameter κ_p, $(0 < \kappa_p < 1)$ occurring in the price dynamics equation:

$$\hat{p} = \beta_p(u - \overline{u}) + \kappa_p(\hat{w} - n) + (1-\kappa_p)\pi^e$$

is sufficiently close to 1.

According to Assumption 4, the inflation rate is more sensitive to the difference between nominal wage growth rate $\hat{w}$ and the rate of growth in labour productivity $n = \frac{\dot{x}}{x}$ than to the expected inflation rate π^e.

Assumption 5. The growth rate of money supply satisfies the inequality:

$$(1-\tau_c)\overline{r} < \mu$$

Assumption 6. The nominal interest rate in the steady state is positive, e.g.:

$$\overline{r} = \frac{g + \mu - \tau_w(\mu - n - \delta + \gamma\overline{u}y^p)}{(1-\tau_w)} > 0$$

It is worth noting that Assumption 5 is satisfied for a sufficiently high capital tax rate τ_c, while Assumption 6 is met for a sufficiently low labour income tax rate τ_w.

Assumptions 1–6 allow for the proof of the main result of the paper, which is the following stability theorem.

Theorem 1. *If Assumptions* 1–6 *are satisfied, then the steady state of model* (56)–(62), (67)–(68) *is locally asymptotically stable.*

3. The proof of stability of the KMG model

3.1. General remarks about the proof

To prove Theorem 1, it must be shown that all of the eigenvalues (characteristic roots) of the 7 × 7 Jacobian matrix J_7 of model (56)–(62), (67)–(68) in the steady state $\overline{x} = \left(\overline{U}, \overline{l}, \overline{m}, \overline{\pi}^e, \overline{y}^e, \overline{v}, \overline{b}\right)$ are either negative numbers or complex numbers with negative real parts (see Gandolfo, 1996, pp. 360–362).[13]

As already mentioned in the introduction, the examination of the eigenvalues of the Jacobian matrix J_7 is based on the idea of *the cascade of stable matrices approach* applied originally by Chiarella, Franke, Flaschel and Semmler in the stability proof of their version of the KMG model (Chiarella

[13] Examination of the eigenvalues of Jacobian matrix is a standard way of proving local asymptotic stability. In most cases it is applied however to two dimensional dynamical systems (e.g. Filipowicz et al., 2016). The difficulties in examining eigenvalues grow very rapidly with the dimension of the system becoming an extremely complex matter in a case of high dimensional systems like the KMG model.

et al., 2006). The implementation of this idea in the stability proof presented below is realized in four steps.[14]

In the first step, the Jacobian matrix J_7 is determined. In the second step, the characteristic polynomial of Jacobian matrix J_7 (considered to be determinant $W_7(\lambda) = \det(J_7 - \lambda I)$) is reduced to the second degree polynomial W_2^{000} by resetting the values of some parameters to zero. This is realized in three stages. The first vector of parameters $\beta = (\beta_n, \beta_z, \alpha)$ is reset to zero. The next parameter β_w and finally parameter β_p are also reset to zero. The third step consists in showing that both roots of polynomial W_2^{000} have negative real parts. In the last (fourth) step, by gradually restoring the positive values of the parameters previously reset to zero, it is demonstrated that all eigenvalues of Jacobian matrix J_7 have roots with negative real parts.

To examine the Jacobian matrix subsequent functions which appear on the right hand sides of Equations (56)–(62), (67)–(68) are denoted by:

$$F_i = F_i(X) \qquad (i = 1, 2, \ldots, 7)$$

where:

$$X = (x_1 = U, x_2 = l, x_3 = m, x_4 = \pi^e, x_5 = y^e, x_6 = v, x_7 = b)$$

Consequently, the Jacobian matrix J_7 in the steady state can be expressed in the following way:

$$J_7 = \begin{pmatrix} F_{1U} & F_{1l} & F_{1m} & F_{1\pi^e} & F_{1y^e} & F_{1v} & F_{1b} \\ F_{2U} & F_{2l} & F_{2m} & F_{2\pi^e} & F_{2y^e} & F_{2v} & F_{2b} \\ F_{3U} & F_{3l} & F_{3m} & F_{3\pi^e} & F_{3y^e} & F_{3v} & F_{3b} \\ F_{4U} & F_{4l} & F_{4m} & F_{4\pi^e} & F_{4y^e} & F_{4v} & F_{4b} \\ F_{5U} & F_{4l} & F_{5m} & F_{5\pi^e} & F_{5y^e} & F_{5v} & F_{5b} \\ F_{6U} & F_{6l} & F_{6m} & F_{6\pi^e} & F_{6y^e} & F_{6v} & F_{6b} \\ F_{7U} & F_{7l} & F_{7m} & F_{7\pi^e} & F_{7y^e} & F_{7v} & F_{7b} \end{pmatrix} \tag{70}$$

where elements of matrix J_7 are derivatives of functions F_i with respect to model variables calculated in the steady state $\overline{x} = \left(\overline{U}, \overline{l}, \overline{m}, \overline{\pi}^e, \overline{y}^e, \overline{v}, \overline{b}\right)$, which means that:

$$F_{ij} = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\overline{x}}, \quad (i, j = 1, 2, \ldots, 7)$$

[14] It is worth emphasizing that due to many modifications introduced into the KMG model the proof of its stability presented in the article although based on the idea of the *cascade of stable matrices approach* differs essentially in many details from the proofs of stability of other versions of KMG models.

The eigenvalues $\lambda_1, \ldots, \lambda_7$ of matrix J_7 are the roots of the following characteristic equation of this matrix:

$$W_7(\lambda) = \det(J_7 - \lambda I) = 0 \tag{71}$$

As stated at the beginning of this section at the first stage the vector of three reaction parameters is zeroed: $\beta = (\beta_n, \beta_z, \alpha) = 0$. This reduces derivatives $F_{ij} = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\bar{x}}$ (e.g. elements of Jacobian matrix J_7) to derivatives:

$$F_{ij}^0 = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\bar{x}} (\beta = 0), (i, j = 1, \ldots, 7)$$

Similarly symbol $J_7(\beta = 0)$ or simply J_7 is used to denote the 7 × 7 matrix of such derivatives.

Let $F_i(X, 0)$, $(i = 1, 2, \ldots, 7)$, be functions obtained from $F_i(X)$ by setting $\beta = (\beta_n, \beta_z, \alpha) = 0$.

On the basis of (56)–(62), it can be easily shown that:

$$F_{ij}^0 = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\bar{x}} (\beta = 0) = \left.\frac{\partial F_i(X, 0)}{\partial x_j}\right|_{x=\bar{x}}, \qquad (i, j = 1, \ldots, 7) \tag{72}$$

Equation (72) simplifies the derivation of Jacobian matrix J_7 because, instead of determining derivatives $F_{ij} = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\bar{x}}$ and then reducing them to $F_{ij}^0 = \left.\frac{\partial F_i(X)}{\partial x_j}\right|_{x=\bar{x}} (\beta = 0)$, one can obtain F_{ij}^0 more easily by nullifying the first parameters $\beta = (\beta_n, \beta_z, \alpha)$ in functions $F_i(X)$ and then determining derivatives $\left.\frac{\partial F_i(X,0)}{\partial x_j}\right|_{x=\bar{x}}$.

In particular, the derivation of Jacobian matrix J_7^0 in the method described above reveals that some of its elements F_{ij}^0 are zeros:

$$J_7^0 = \begin{pmatrix} 0 & F_{1l}^0 & 0 & 0 & F_{1y^e}^0 & 0 & 0 \\ F_{2u}^0 & F_{2l}^0 & F_{2m}^0 & F_{2\pi^e}^0 & F_{2y^e}^0 & 0 & 0 \\ F_{3u}^0 & F_{3l}^0 & F_{3m}^0 & F_{3\pi^e}^0 & F_{3y^e}^0 & 0 & 0 \\ 0 & 0 & 0 & F_{4\pi^e}^0 & F_{4y^e}^0 & 0 & 0 \\ F_{5u}^0 & 0 & F_{5m}^0 & F_{5\pi^e}^0 & F_{5y^e}^0 & 0 & 0 \\ F_{6u}^0 & F_{6l}^0 & F_{6m}^0 & F_{6\pi^e}^0 & F_{6y^e}^0 & F_{6v}^0 & 0 \\ F_{7u}^0 & F_{7l}^0 & F_{7m}^0 & F_{7\pi^e}^0 & F_{7y^e}^0 & F_{7v}^0 & F_{7b}^0 \end{pmatrix} \tag{73}$$

Because of the limited length of this paper, it is not possible to present all the formulas describing elements of J_7^0. At the next stages of the proof only some of them will be presented where this is especially needed.

3.2. Reducing polynomials degrees

Lemma 1. *Matrix* J_7^0 *has three negative eigenvalues* $\lambda_1^0 = -\beta_{\pi^e} < 0, \lambda_2^0 = -n < 0$ *and* $\lambda_3^0 = (1-\tau_c)\overline{r} - \mu < 0$.

Proof. In view of (73), polynomial $W_7^0(\lambda) = \det(J_7^0 - \lambda I)$ can be expanded to the following form:

$$W_7^0(\lambda) = (F_{7b}^0 - \lambda)(F_{4\pi^e}^0 - \lambda)(F_{6v}^0 - \lambda) \cdot W_4^0(\lambda) \tag{74}$$

where:

$$W_4^0(\lambda) = \det(J_4^0 - \lambda I) = \det \begin{pmatrix} -\lambda & F_{1l}^0 & 0 & F_{1y^e}^0 \\ F_{2u}^0 & F_{2l}^0 - \lambda & F_{2m}^0 & F_{2y^e}^0 \\ F_{3u}^0 & F_{3l}^0 & F_{3m}^0 - \lambda & F_{3y^e}^0 \\ F_{5u}^0 & 0 & F_{5m}^0 & F_{5y^e}^0 - \lambda \end{pmatrix} \tag{75}$$

The first step in obtaining formula for $F_{7b}^0 = \left.\dfrac{\partial F_7^0}{\partial b}\right|_{x=\overline{x}}$ is to note that in view of (62) function $F_7 = F_7(U, l, m, \pi^e, y^e, v, b)$ has the following form:

$$F_7 = g + \frac{hy^e}{m}b - \tau_c\left(y^e - Uy - \delta + \frac{hy^e}{m}b\right) - \tau_w Uy - \mu m -$$
$$- b\left(n - \hat{l} + \kappa\left(\beta_p\left(\frac{y}{y^p} - u\right) + \kappa_p\beta_w\left(\frac{y}{l} - \overline{V}\right)\right) + \pi^e\right)$$

In general, according to (72), F_{ij}^0 is a derivative of function F_i^0 obtained from function F_i by nullifying the vector of parameters $\beta = (\beta_n, \beta_z, \alpha) = 0$. Function F_7, however, does not contain the above parameters, so $F_7^0 = F_7$. Hence:

$$\frac{\partial F_7}{\partial b}=\frac{\partial F_7^0}{\partial b}=\frac{\partial}{\partial b}\left(g+\frac{hy^e}{m}b-\tau_c\left(y^e-Uy-\delta+\frac{hy^e}{m}b\right)-\tau_w Uy-\mu m-\right.$$

$$\left.-b\left(n-\hat{l}+\kappa\left(\beta_p\left(\frac{y}{y^p}-u\right)+\kappa_p\beta_w\left(\frac{y}{l}-\bar{V}\right)\right)+\pi^e\right)\right)=$$

$$=\frac{hy^e}{m}-\tau_c\left(-U\frac{\partial y}{\partial b}+\frac{hy^e}{m}\right)-$$

$$-\tau_w U\frac{\partial y}{\partial b}-\left(n-\hat{l}+\kappa\left(\beta_p\left(\frac{y}{y^p}-u\right)+\kappa_p\beta_w\left(\frac{y}{l}-\bar{V}\right)\right)+\pi^e\right)-$$

$$-b\left(\kappa\left(\beta_p\frac{\frac{\partial y}{\partial b}}{y^p}+\kappa_p\beta_w\frac{\frac{\partial y}{\partial b}}{l}\right)\right)$$

It follows form Equations (68) and (20) that y does not depend on b, hence:

$$\frac{\partial y}{\partial b}=0$$

This in turn implies that:

$$\frac{\partial F_7}{\partial b}=\frac{\partial F_7^0}{\partial b}=\frac{hy^e}{m}-\tau_c\frac{hy^e}{m}-\left(n-\hat{l}+\kappa\left(\beta_p\left(\frac{y}{y^p}-u\right)+\kappa_p\beta_w\left(\frac{y}{l}-\bar{V}\right)\right)+\pi^e\right) \quad (76)$$

Substituting values of the variables U, l, m, π, y, v in the steady state into above Equation yields the formula for the derivative $\frac{\partial F_7^0}{\partial b}$ in the steady state:

$$F_{7b}^0=\left.\frac{\partial F_7^0}{\partial b}\right|_{x=\bar{x}}=\bar{r}-\tau_c\bar{r}-\mu=(1-\tau_c)\bar{r}-\mu$$

which in view of Assumption 5 implies that $F_{7\mathrm{b}}^0<0$.

Similarly it can be derived that: $F_{6v}^0=-n$ and $F_{4\pi^e}^0=-\beta_{\pi^e}$. Hence, in view of (74) the polynomial $W_7^0(\lambda)$ has three negative roots $\lambda_1^0=-\beta_{\pi^e}<0$, $\lambda_2^0=-n<0$ and $\lambda_3^0=(1-\tau_c)\bar{r}-\mu<0$, which are the eigenvalues of matrix J_7^0.

■

The remaining roots $\lambda_4^0, \lambda_5^0, \lambda_6^0, \lambda_7^0$ of polynomial $W_7^0(\lambda)$ are also roots of the fourth degree polynomial $W_4^0(\lambda)$ defined by (75).

Let be J_4^{00} the matrix obtained from J_4^0 under assumption that not only $\beta = (\beta_n, \beta_z, \alpha) = 0$ but also $\beta_w = 0$. The characteristic polynomial of matrix J_4^{00} is given by:

$$W_4^{00}(\lambda) = \det(J_4^{00} - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & 0 & F_{1y^e}^{00} \\ F_{2u}^{00} & -\lambda & F_{2m}^{00} & F_{2y^e}^{00} \\ F_{3u}^{00} & 0 & F_{3m}^{00} - \lambda & F_{3y^e}^{00} \\ F_{5u}^{00} & 0 & F_{5m}^{00} & F_{5y^e}^{00} - \lambda \end{pmatrix} \tag{77}$$

Expanding $W_4^{00}(\lambda)$ yields:

$$W_4^{00}(\lambda) = \lambda \cdot W_3^{00}(\lambda) = \lambda \cdot \det(J_3^{00} - \lambda I) = \lambda \cdot \det\begin{pmatrix} -\lambda & 0 & F_{1y^e}^{00} \\ F_{3u}^{00} & F_{3m}^{00} - \lambda & F_{3y^e}^{00} \\ F_{5u}^{00} & F_{5m}^{00} & F_{5y^e}^{00} - \lambda \end{pmatrix} \tag{78}$$

which implies that one of the roots $\lambda_4^{00}, \lambda_5^{00}, \lambda_6^{00}, \lambda_7^{00}$ of polynomial $W_4^{00}(\lambda)$ is zero. Suppose that:

$$\lambda_4^{00} = 0 \tag{79}$$

Finally add to Assumptions $\beta = (\beta_n, \beta_z, \alpha) = 0$, $\beta_w = 0$ that also $\beta_p = 0$. Matrix J_3^{00} reduces then to J_3^{000}. The polynomial:

$$W_3^{000}(\lambda) = \det(J_3^{000} - \lambda I) = \det\begin{pmatrix} -\lambda & 0 & 0 \\ F_{3u}^{000} & F_{3m}^{000} - \lambda & F_{3y^e}^{000} \\ F_{5u}^{000} & F_{5m}^{000} & F_{5y^e}^{000} - \lambda \end{pmatrix} \tag{80}$$

after expansion takes the form:

$$W_3^{000}(\lambda) = \lambda \cdot W_2^{000}(\lambda) = \lambda \cdot \det(J_2^{000} - \lambda I) = \lambda \cdot \det\begin{pmatrix} F_{3m}^{000} - \lambda & F_{3y^e}^{000} \\ F_{5m}^{000} & F_{5y^e}^{000} - \lambda \end{pmatrix} \tag{81}$$

This implies that one of the roots $\lambda_5^{000}, \lambda_6^{000}, \lambda_7^{000}$ of $W_3^{000}(\lambda)$ is zero. Suppose that:

$$\lambda_5^{000} = 0 \tag{82}$$

Lemma 2. *Both eigenvalues* λ_6^{000} *and* λ_7^{000} *of matrix* $J_2^{000} = \begin{pmatrix} F_{3m}^{000} & F_{3y^e}^{000} \\ F_{5m}^{000} & F_{5y^e}^{000} \end{pmatrix}$ *are either negative or complex numbers with negative real parts.*

Proof. The characteristic equation $\det(J_2^{000} - \lambda I) = 0$ of matrix J_2^{000} can be expressed as:

$$\lambda^2 + a_1\lambda + a_2 = 0 \tag{83}$$

where:

$$a_1 = -\mathrm{tr}\, J_2^{000} = -(F_{3m}^{000} + F_{5y^e}^{000}) \tag{84}$$

$$a_2 = \det J_2^{000} = F_{3m}^{000} F_{5y^e}^{000} - F_{5m}^{000} F_{3y^e}^{000} \tag{85}$$

It can be verified that elements of matrix J_2^{000} are described by the following formulas:

$$F_{3m}^{000} = -i_1\,\overline{r}\left(\frac{\overline{u}y^p}{\overline{u}y^p - \beta_{y^e}}\right) \tag{86}$$

$$F_{3y^e}^{000} = -\frac{h\overline{u}y^p}{\overline{r}}\left(\frac{\kappa\beta_w}{\overline{u}y^p}\overline{V} + \frac{\kappa_w\kappa\beta_p}{y^p} + \beta_{y^e}\left(\frac{y_{y^e}^d - 1}{\overline{u}y^p}\right) + i_1\left(1 - \overline{U} - \frac{\overline{r}}{\overline{u}y^p}\right) + \frac{i_2}{y^p}\right) \tag{87}$$

$$F_{5y^e}^{000} = \beta_{y^e} - i_1(\delta - \mu + n) - \overline{u}i_2 \tag{88}$$

$$F_{5m}^{000} = -\frac{i_1}{h}\overline{r}^2 \tag{89}$$

Assumptions 2, 3 and 6 imply that $F_{3m}^{000} < 0$ and $F_{5y^e}^{000} < 0$. Hence:

$$a_1 = -\mathrm{tr}\, J_2^{000} > 0 \tag{90}$$

Inserting (86)–(89) into (85) after simplifications implies that:

$$a_2 = i_1\overline{r}\beta_{y^e}\left(\frac{-\beta_{y^e} - \overline{u}y^p + (n + \delta + g) + 2i_1(\delta - \mu + n) - \beta_{y^e}\overline{u}y^p}{\overline{u}y^p - \beta_{y^e}}\right) \tag{91}$$

Assumption 2 implies that $\delta - \mu + n > 0$. Hence, $i_1 > 0$ if is sufficiently large (Assumption 3) and parameter $\beta_{y^e} > 0$ is sufficiently small (Assumption 1), then:

$$a_2 = \det J_2^{000} > 0 \tag{92}$$

In view of Routh-Hurwitz theorem (Gandolfo, 2005, pp. 221–222), conditions (90) and (92) imply that both eigenvalues λ_6^{000} and λ_7^{000} of matrix J_2^{000} have negative real parts.

3.3. Restoring positive values of reaction parameters

In the further part of the proof of Theorem 1 the positive values of temporarily nullified parameters $\beta=(\beta_n, \beta_z, \alpha)$, β_w, β_p will be gradually restored. First by restoring positive value of parameter β_p and exploiting Lemma 2 the following Lemma 3 on the eigenvalues of matrix J_3^{00} (which appears in Equation (78)) will be proved.

Lemma 3. *Suppose that* $\beta=(\beta_n, \beta_z, \alpha) = 0$ and $\beta_w = 0$. *Then, for sufficiently small values of parameters* $\beta_p > 0$ *and* $\beta_{y^e} > 0$, *all* λ_5^{00}, λ_6^{00}, λ_7^{00} *of matrix* J_3^{00} *are either negative or complex numbers with negative real parts.*

Proof. The starting point are matrices:

$$J_3^{00} = \begin{pmatrix} 0 & 0 & F_{1y^e}^{00} \\ F_{3u}^{00} & F_{3m}^{00} & F_{3y^e}^{00} \\ F_{5u}^{00} & F_{5m}^{00} & F_{5y^e}^{00} \end{pmatrix} \quad \text{and} \quad J_3^{000} = \begin{pmatrix} 0 & 0 & 0 \\ F_{3u}^{000} & F_{3m}^{000} & F_{3y^e}^{000} \\ F_{5u}^{000} & F_{5m}^{000} & F_{5y^e}^{000} \end{pmatrix}$$

Matrix J_3^{000} is obtained from J_3^{00} by nullifying parameter $\beta_p > 0$ in all elements of J_3^{00}.

Moreover: $$J_3^{00} \to J_3^{000} \quad \text{when} \quad \beta_p \to 0 \tag{93}$$

According to (80) and (81):

$$W_3^{000}(\lambda) = \det(J_3^{000} - \lambda I) = \lambda \cdot \det(J_2^{000} - \lambda I) \tag{94}$$

which implies that the two eigenvalues of matrix J_2^{000} are also eigenvalues of matrix J_3^{000}. Hence in view of Lemma 2 and (94) it is concluded that matrix J_3^{000} has one zero eigenvalue $\lambda_5^{000} = 0$ and two eigenvalues λ_6^{000} and λ_7^{000} which are either negative or complex numbers with negative real parts. In both cases:

$$\lambda_6^{000} \cdot \lambda_7^{000} > 0 \tag{95}$$

Due to the continuity of matrix J_3^{00} with respect to $\beta_p \geq 0$ (condition (93)) and the continuity of the eigenvalues of matrix J_3^{00} with respect to its elements for a sufficiently small value of $\beta_p > 0$, the two eigenvalues λ_6^{00}, λ_7^{00} of matrix J_3^{00} (corresponding to λ_6^{000} and λ_7^{000}) are also either negative or complex numbers with negative real parts, satisfying inequality

$$\lambda_6^{00} \cdot \lambda_7^{00} > 0 \tag{96}$$

To complete the proof the third eigenvalue λ_5^{00} corresponding to $\lambda_5^{000} = 0$ must be examined. For this purpose the determinant will be considered:

$$\det J_3^{00} = \det\begin{pmatrix} 0 & 0 & F_{1y^e}^{00} \\ F_{3u}^{00} & F_{3m}^{00} & F_{3y^e}^{00} \\ F_{5u}^{00} & F_{5m}^{00} & F_{5y^e}^{00} \end{pmatrix} = F_{1y^e}^{00} \cdot \det\begin{pmatrix} F_{3u}^{00} & F_{3m}^{00} \\ F_{5u}^{00} & F_{5m}^{00} \end{pmatrix} \tag{97}$$

The determinant of any matrix equals the product of its eigenvalues. Hence:

$$\det J_3^{00} = \lambda_5^{00} \cdot \lambda_6^{00} \cdot \lambda_7^{00} \tag{98}$$

The elements of matrix are described by the following formulas:

$$F_{1y^e}^{00} = -\bar{U}\kappa\left(\frac{\beta_p\left(1-\kappa_w\right)}{y^p}\right) \tag{99}$$

$$F_{3u}^{00} = -\frac{h}{\bar{r}}(\bar{u}y^p)^2\left(\frac{\beta_{y^e} - \tau_w\beta_{y^e} - i_1\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right) \tag{100}$$

$$F_{3m}^{00} = -i_1\left(\bar{u}y^p - \frac{\bar{u}y^p - (n+\delta+g)}{(1-\tau_w)} - \delta + \mu - n\right)\left(\frac{\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right) \tag{101}$$

$$F_{5u}^{00} = (\bar{u}y^p)^2 i_1 \tag{102}$$

$$F_{5m}^{00} = -\frac{i_1}{h}\left(\bar{u}y^p - \frac{\bar{u}y^p - (n+\delta+g)}{(1-\tau_w)} - \delta + \mu - n\right)^2 \tag{103}$$

Inserting (99)–(103) into (97) yields after simplifications:

$$\det J_3^{00} = -i_1\bar{r}\bar{U}\kappa\left(\frac{\beta_p(1-\kappa_w)}{y^p}\right)(\bar{u}y^p)^2\left(\left(\frac{(1-\tau_w)\beta_{y^e}}{\bar{u}y^p - \beta_{y^e}}\right)\right) \tag{104}$$

In view of inequalities $0 < \kappa_w < 1$ and $0 < \tau_w < 1$, it follows from (60) that for a sufficiently small $\beta_{y^e} > 0$:

$$\det J_3^{00} < 0$$

Hence, in view of (96) and (98) it is obvious that that for a sufficiently small $\beta_p > 0$ and $\beta y^e > 0$, the third eigenvalue of matrix J_3^{00} is a negative number $\lambda_5^{000} < 0$.
■

Below, by restoring the positive value of parameter β_w and using Lemma 3 it will be proven Lemma 4 on the eigenvalues of the following matrix:

$$J_4^0 = \begin{pmatrix} 0 & F_{1l}^0 & 0 & F_{1y^e}^0 \\ F_{2u}^0 & F_{2l}^0 & F_{2m}^0 & F_{2y^e}^0 \\ F_{3u}^0 & F_{3l}^0 & F_{3m}^0 & F_{3y^e}^0 \\ F_{5u}^0 & 0 & F_{5m}^0 & F_{5y^e}^0 \end{pmatrix} \tag{105}$$

appearing in (73)–(74), in which $\beta = (\beta_n, \beta_z, \alpha) = 0$ while $\beta_p > 0$ and $\beta_w > 0$.

Lemma 4. Suppose that $\beta = (\beta_n, \beta_z, \alpha) = 0$ and $\beta_p > 0$, $\beta_w > 0$. Then, under Assumptions 1–6 for sufficiently small values of parameters $\beta_p > 0$ and $\beta_w > 0$, all eigenvalues λ_4^0, λ_5^0, λ_6^0, λ_7^0 *of matrix* J_4^0 *are either negative or complex numbers with negative real parts.*

Proof. The proof of Lemma 4 is similar to that of Lemma 3. The start will be by considering the following matrix:

$$J_4^{00} = \begin{pmatrix} 0 & 0 & 0 & F_{1y^e}^{00} \\ F_{2u}^{00} & 0 & F_{2m}^{00} & F_{2y^e}^{00} \\ F_{3u}^{00} & 0 & F_{3m}^{00} & F_{3y^e}^{00} \\ F_{5u}^{00} & 0 & F_{5m}^{00} & F_{5y^e}^{00} \end{pmatrix}$$

which appears in (46). Matrix J_4^{00} is obtained from J_4^0 by nullifying parameter β_w in all elements of J_4^{00}. Beside this:

$$J_4^0 \to J_4^{00} \text{ when } \beta_w \to 0 \tag{106}$$

According to (75):

$$W_4^{00}(\lambda) = \det(J_4^{00} - \lambda I) = \lambda \cdot \det(J_3^{00} - \lambda I) = \lambda \cdot \det \begin{pmatrix} -\lambda & 0 & F_{1y^e}^{00} \\ F_{3u}^{00} & F_{3m}^{00} - \lambda & F_{3y^e}^{00} \\ F_{5u}^{00} & F_{5m}^{00} & F_{5y^e}^{00} - \lambda \end{pmatrix} \tag{107}$$

It follows from Equation (107) that matrix J_4^{00} has three eigenvalues λ_5^{00}, λ_6^{00}, λ_7^{00} which are identical to the eigenvalues of matrix J_3^{00}. As shown in the proof of Lemma 3, the two eigenvalues λ_6^{00} and λ_7^{00} are either negative or complex numbers with negative real parts. The third eigenvalue λ_5^{00} of J_4^{00} is a negati-ve number:

$$\lambda_5^{00} < 0 \tag{108}$$

Equation (108) also implies that the fourth eigenvalue of equals zero:

$$\lambda_4^{00} = 0 \tag{109}$$

Hence, in view of (106) and the continuity of the eigenvalues of matrix J_4^0 with respect to its elements, for a sufficiently small $\beta_w > 0$ the three eigenvalues λ_5^0, λ_6^0, λ_7^0 of matrix (corresponding to λ_5^{00}, λ_6^{00}, λ_7^{00}) are also either negative or complex numbers with negative real parts. What remains to be investigated is the fourth eigenvalue λ_4^0 of J_4^0 corresponding to $\lambda_4^{00} = 0$ of J_3^{00}.

For this purpose the sign of the determinant will be determined:

$$\det J_4^0 = \lambda_4^0 \cdot \lambda_5^0 \cdot \lambda_6^0 \cdot \lambda_3^0 \tag{110}$$

and use the inequalities:

$$\lambda_5^{00} < 0, \quad \lambda_6^{00} \cdot \lambda_7^{00} > 0 \tag{111}$$

(the second inequality is identical to (96)).

It follows from (105) that:

$$\det J_4^0 = -F_{1l}^0 \cdot A_1 - F_{1y^e}^0 \cdot A_2 \tag{112}$$

where $F_{1l}^0 = -\bar{U}\kappa\beta_w(1-\kappa_p)\dfrac{\bar{V}^2}{\bar{u}y^p} < 0, \quad F_{1y^e}^0 = \bar{U}\kappa\left(\dfrac{\beta_w(1-\kappa_p)}{\bar{u}y^p}\bar{V} - \dfrac{\beta_p(1-\kappa_w)}{y^p}\right)$

$$A_1 = \det\begin{pmatrix} F_{2u}^0 & F_{2m}^0 & F_{2y^e}^0 \\ F_{3u}^0 & F_{3m}^0 & F_{3y^e}^0 \\ F_{5u}^0 & F_{5m}^0 & F_{5y^e}^0 \end{pmatrix}, \quad A_2 = \det\begin{pmatrix} F_{2u}^0 & F_{2l}^0 & F_{2m}^0 \\ F_{3u}^0 & F_{3l}^0 & F_{3m}^0 \\ F_{5u}^0 & 0 & F_{5m}^0 \end{pmatrix}$$

Elements of determinants and have the form:

$$F_{2l}^0 = \kappa\beta_w(1-\kappa_p)\bar{V}\left(\frac{\bar{u}y^p - 2\beta_{y^e}}{\bar{u}y^p - \beta_{y^e}}\right) \tag{113}$$

$$F_{1y^e}^0 = \bar{U}\kappa\left(\frac{\beta_w(1-\kappa_p)}{\bar{u}y^p}\bar{V} - \frac{\beta_p(1-\kappa_w)}{y^p}\right) \tag{114}$$

$$F_{2m}^0 = -\frac{i_1}{\bar{V}h}\left(\frac{\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right)\left(\bar{u}y^p - \frac{(\bar{u}y^p - (n+\delta+g)}{(1-\tau_w)} - \delta + \mu - n\right)^2 \tag{115}$$

$$F_{2u}^0 = -\frac{(\bar{u}y^p)^2}{\bar{V}}\left(\frac{(1-\tau_w)\beta_{y^e} - i_1\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right) \tag{116}$$

$$F_{2m}^0 = -i_1\left(\bar{u}y^p - \frac{(\bar{u}y^p - (n+\delta+g)}{(1-\tau_w)} - \delta + \mu - n\right)\left(\frac{\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right) \tag{117}$$

$$F_{3u}^0 = -\frac{h}{\bar{r}}(\bar{u}y^p)^2\left(\frac{\beta_{y^e} - \tau_w\beta_{y^e} - i_1\bar{u}y^p}{\bar{u}y^p - \beta_{y^e}}\right) \tag{118}$$

$$F_{3l}^0 = \frac{h}{\bar{r}}\kappa\beta_w\bar{V}^2\left(\frac{\bar{u}y^p - 2\beta_{y^e} + \kappa_p\beta_{y^e}}{\bar{u}y^p - \beta_{y^e}}\right) \tag{119}$$

$$F_{5m}^0 = -\frac{i_1}{h}\left(\bar{u}y^p - \frac{(\bar{u}y^p - (n+\delta+g)}{(1-\tau_w)} - \delta + \mu - n\right)^2 \tag{120}$$

$$F_{5u}^0 = (\bar{u}y^p)^2 i_1 \tag{121}$$

$$F_{1l}^0 = -\bar{U}\kappa\beta_w(1-\kappa_p)\frac{\bar{V}^2}{\bar{u}y^p} \tag{122}$$

It follows from the above formulas that the first component of determinant $\det J_4^0$, is equal to $(-F_{1l} \cdot A_1)$ and converges to zero when $0 < \kappa_p < 1$ converges to 1 (Assumption 4).

After being expanded and simplified, determinant takes the form:

$$A_2 = F_{5u}(-F_{3l}F_{2m}) + F_{5m}(F_{2u}F_{3l}) \tag{123}$$

Inserting (113)–(122) into (123) yields, after simplification:

$$A_2 = (\bar{u}y^p)^2 i_1\bar{r}\kappa\beta_w\bar{V}\left(\frac{\bar{u}y^p - 2\beta_{y^e} + \kappa_p\beta_{y^e}}{\bar{u}y^p - \beta_{y^e}}\right)\left(\frac{(1-\tau_w)\beta_{y^e}}{\bar{u}y^p - \beta_{y^e}}\right) \tag{124}$$

It follows from (124) that $\beta_{y^e} > 0$ if is sufficiently small (Assumption 1), then:

$$A_2 > 0$$

At the same time, when $0 < \kappa_p < 1$ converges to 1 (Assumption 4), then:

$$F_{1y^e}^0 = \bar{U}\kappa\left(\frac{\beta_w(1-\kappa_p)}{\bar{u}y^p}\bar{V} - \frac{\beta_p(1-\kappa_w)}{y^p}\right) < 0$$

Consequently, if $0 < \kappa_p < 1$ is sufficiently close to 1 (Assumption 4) and parameters $\beta_p > 0, \beta_w > 0, \beta_{y^e} > 0$ and are sufficiently small (Assumption 1), then:

$$\det J_4^0 = \lambda_4^0 \cdot \lambda_5^0 \cdot \lambda_6^0 \cdot \lambda_7^0 > 0 \tag{125}$$

Since $\lambda_5^{00} < 0$, the corresponding eigenvalue λ_5^0 of J_4^0 for a sufficiently small $\beta_w > 0$ can be either a negative number or a complex number with a negative real part.

If $\lambda_5^0 < 0$, then, in view of the continuity of the eigenvalues of matrix J_4^0 with respect to its elements, taking into account (106) and (111), for a sufficiently small $\beta_w > 0$:

$$\lambda_5^0 \cdot \lambda_6^0 \cdot \lambda_7^0 < 0 \tag{126}$$

is obtained, which in view of (125) implies directly that $\lambda_4^0 < 0$.

The second possibility should be considered that when eigenvalue λ_5^0 is a complex number with a negative real part. Then, since complex eigenvalues (roots of a polynomial) always appear as conjugate numbers, the fourth eigenvalue λ_4^0 must be the conjugate of λ_5^0 having the same negative real part as λ_5^0[15].

It has been demonstrated earlier that λ_5^0, λ_6^0 and λ_7^0 are either negative or complex numbers with negative real parts. Thus, the two remarks above on λ_4^0 complete the proof of Lemma 4.

■

To complete the whole proof of stability Theorem 1 it should be noted that in view of lemmas 1 and 4 all eigenvalues $\lambda_1^0, \lambda_2^0, \lambda_3^0, \lambda_4^0, \lambda_5^0, \lambda_6^0, \lambda_7^0$ of matrix J_7^0 defined by (73) are either negative or complex numbers with negative real parts, since:

$$W_7^0(\lambda) = \det(J_7^0 - \lambda I) = \left((1-\tau_c)\overline{r} - \mu - \lambda\right)(-n-\lambda)(-\beta_{\pi^e} - \lambda) \cdot W_4^0(\lambda) \tag{127}$$

Matrix J_7^0 is obtained from Jacobian matrix J_7 by nullifying the vector of parameters $\beta = (\beta_n, \beta_z, \alpha) > 0$. It can also be verified that:

$$J_7^0 \to J_7 \text{ when } \beta \to 0$$

In view of this and the continuity of the eigenvalues of matrix with respect to its elements, it may be concluded that all eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7$ of Jacobian matrix J_7 are either negative or complex numbers with negative real parts. This proves theorem 1 stating that the steady state of model (56)–(62), (67)–(68) is locally asymptotically stable.

■

[15] The product of conjugate complex numbers $\lambda_4^0 = a + ib$ and $\lambda_5^0 = (a - ib)$ different from zero is always positive since $(a + ib)(a - ib) = a^2 + b^2 > 0$. Hence, in view of (65), there is $\lambda_4^0 \cdot \lambda_5^0 > 0$ and $\lambda_6^0 \cdot \lambda_7^0 > 0$, which implies that $\det J_4^0 = \lambda_4^0 \cdot \lambda_5^0 \cdot \lambda_6^0 \cdot \lambda_7^0 > 0$.

Conclusions

In their (sometimes co-authored) books, Flaschel, Franke and Chiarella have contributed significantly to the development of Keynesian monetary macroeconomics. One of their main achievements is the development of the KMG model and a stability analysis of its various variants. Proving the stability of complex, high dimensional systems like the KMG model is always a complex, difficult task. For this reason, to make the stability analysis easier, they simplified some equations of the model at the expense of its adequacy to reality.

In the present article, an attempt has been made to improve the way the KMG model describes the functioning of the economy by modifying some of its equations. The modifications introduced have resulted in the appearance of new loops in the model, thereby increasing its complexity. This is particularly evident in the intensive form model (56)–(62) with additional Equations (67)–(68). What is worth emphasizing is that all of these modifications have been introduced in a way which retains the possibility of transforming the KMG model into its intensive form and deriving its steady state, which is necessary for proving the stability of the model.

The modifications of the KMG model have influenced all the results presented in the article. Firstly, they are reflected in the formulas describing values of the variables in the steady state presented in section 2.2. In particular, this is visible how tax rates (which are not present in other versions of the KMG model) influence the values of some variables in the steady state, such as the interest rate, the real labour income per unit of output, the ratio of the real value of bonds to fixed capital. (On the other hand, an interesting finding is that, in the steady state, the ratio of consumption to fixed capital does not depend on tax rates.) Secondly, the Jacobian matrix of the intensive form of the new KMG model differs from the Jacobian matrices of KMG models analysed by Chiarella et al. This meant that the proof of stability presented in section 3, although based on the general idea of the cascade of stable matrices approach, differs essentially from the proofs of stability of other versions of KMG models. Thirdly, due to the equation's modifications, the set of Assumptions 1–6 exploited in the proof also differs from that used by Chiarella et al. In particular, Assumptions 5 and 6, which feature tax rates on labour and capital incomes are completely new, as these tax rates are not considered at all in other versions of the KMG model.

According to the proven stability theorem, when Assumptions 1–6 are satisfied, the economy described by the modified KMG model approaches the balanced growth path when time goes to infinity. Since the proven stability theorem concerns only local stability, the convergence to the balanced growth path is guaranteed only if the initial structure of the economy does not differ too much from that on the balanced growth path, described by the steady state of the intensive form model. How much the initial structure of the economy may depart from the steady state without losing the stability of the model may be verified only through computer simulations. Also, the speed at which the economy converges toward the balanced growth path can only be tested by computer experiments.

Despite these limitations, mathematical proofs of the stability of economic systems like that presented in the article are important for the development of economic growth theory since they reveal an intrinsic ability of the analyzed economy to achieve a structure which allows for balanced growth. Lack of stability is a serious deficiency of the economy because it is equivalent to the existence of a self-deepening disequilibrium mechanism leading to economic collapse.

References

Araujo, R. A., Flaschel, P., & Moreira, H. N. (2020). Limit cycles in a model of supply-side liquidity/profit—rate in the presence of a Phillips curve. *Economia*, *21*(2), 145–159.

Asada, T. (2012). Modeling financial instability. *Intervention: European Journal of Economics and Economic Policies*, *9*, 215–232.

Asada, T., Chiarella, C., Flaschel, P., & Franke, F. (2003). *Open economy macrodynamics*. Springer.

Asada, T., Chiarella, C., Flaschel, P., Mouakil, T., Proaño, C., & Semmler, W. (2011). Stock flow interactions, disequilibrium macroeconomics and the role of economic policy. *Journal of Economic Surveys*, *25*(3), 569–599.

Asada, T., Demetrian, M., & Zimka, R. (2018). On dynamics in a Keynesian model of monetary and fiscal policy with debt effect. *Communications in Non-linear Science and Numerical Simulations*, *58*, 131–146.

Asada, T., Demetrian, M., & Zimka, R. (2019). On dynamics in a Keynesian model of monetary and fiscal stabilization policy mix with twin debt accumulation. *Metroeconomica*, *70*(3), 365–383.

Charpe, M., Chiarella, C., Flaschel, P., & Semmler, W. (2011). *Financial assets, debt and liquidity crises: A Keynesian approach*. Cambridge University Press.

Chiarella, C., & Flaschel, P. (2000). *The dynamics of Keynesian monetary growth*. Cambridge University Press.

Chiarella, C., Flaschel, P., Franke, R. (2005). *Foundations for a disequilibrium theory of the business cycle*. Cambridge University Press.

Chiarella, C., Flaschel, P., Franke, R., Araujo, R., Charpe, M., Proaño, Ch., & Szczutkowski, A. (2021). *Unbalanced growth from a balanced perspective*. Edward Elgar.

Chiarella, C., Flaschel, P., Franke, R., & Semmler, W. (2002). *Stability analysis of a high-dimensional macrodynamic model of real-financial interaction: A cascade of matrices approach*. Working Paper, 123. School of Finance and Economics, UTS Business School, University of Technology, Sydney.

Chiarella, C., Flaschel, P., Franke, R., & Semmler, W. (2006). A high-dimensional model of real-financial market interaction: The cascade of stable matrices approach. In C. Chiarella, R. Franke, P. Flaschel & W. Semmler (Eds.), *Quantitative and empirical analysis of nonlinear dynamic macromodels, contributions to economic analysis* (vol. 277, s. 359–384). Elsevier.

Chiarella, C., Flaschel, P., Groh, G., & Semmler, W. (2000). *Disequilibrium, growth and labor market dynamics*. Springer.

Chiarella, C., Flaschel, P., & Semmler, W. (2012). (2013). (2014). *Reconstructing Keynesian macroeconomics*, vol. 1–3: *Integrated approaches*. Routledge.

Duménil, G., & Lévy, D. (1991). Micro adjustment toward long-term equilibrium. *Journal of Economic Theory*, *52*(2), 369–395.

Dutt, A. (2012). Distributional dynamics in post Keynesian growth models. *Journal of Post Keynesian Economics*, *34*(3), 431–451.

Filipowicz, K., Misiak, T., & Tokarski, T. (2016). Bipolar growth model with investment flows. *Economics and Business Review*, *2*(3), 32–56 https://doi.org/10.18559/ebr.2016.3.4

Fischer, S. (1972). Keynes-Wicksell and neoclassical models of money and growth. *American Economic Review*, *62*, 880–890.

Flaschel, P. (2020). *A baseline model of 'Social Protection' in open economies of the KMG variety*. Working Paper: Bielefeld University.

Gandolfo, G. (2005). *Economic dynamics*. Springer Verlag.

Goodwin, R. M. (1967). A growth cycle. In C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth* (pp. 54–58). Cambridge University Press.

Groth, C. (1988). IS-LM dynamics and the hypothesis of adaptive-forward-looking expectations. In P. Flaschel & M. Krüger (Eds.), Recent approaches to economic dynamics (s. 251–266). Verlag Peter Lang.

Keynes, J .M. (1936). *The general theory of employment, interest and money*. Macmillan.

Kiedrowski, R. (2018). Profit rates equalization and balanced growth in a multi-sector model of classical competition. *Journal of Mathematical Economics*, *77*, 39–53.

Lavoie, M. (2014). *Post-Keynesian economics: New foundations*. Edward Elgar.

Metzler, L. A. (1941). The nature and stability of inventory cycles. *Review of Economic Statistics*, *23*, 113–129.

Murakami, H. (2016, December). Alternative monetary policies and economic stability in a medium-term Keynesian model. *Evolutionary and Institutional Economics Review*, *13*(2), 323–362.

Murakami, H. (2018). Existence and uniqueness of growth cycles in post Keynesian systems. *Economic Modelling*, *75*, 293–304.

Murakami, H. (2020). Monetary policy in the unique growth cycle of post Keynesian systems. *Structural Change and Economic Dynamics*, *52*(C), 39–49.

Napoletano, M., Dosi, G., Fagiolo, G., & Roventini, A. (2012). Wage formation, investment behavior and growth regimes: An agent-based analysis. *Revue de L'OFCE*, *5*(124), 235–261. https://www.cairn.info/revue-de-l-ofce-2012-5-page-235.htm

Ogawa, S. (2019a). Dynamic analysis of a disequilibrium macroeconomic model with dual labor markets. *Metroeconomica*, *70*(3), 525–550.

Ogawa, S. (2019b). *Effective demand and quantity constrained growth: A simple two--sector disequilibrium approach*. MPRA Paper 93336. University Library of Munich.
Ogawa, S. (2020). *Monetary growth with disequilibrium: A non-Walrasian baseline model*. MPRA Paper 101236. University Library of Munich.
Rødseth, A. (2000). *Open economy macroeconomics*. Cambridge University Press.
Rose, H. (1966). Unemployment in a theory of growth. *International Economic Review*, *7*, 260–282.
Sargent, T. (1987). *Macroeconomic theory*. Academic Press.
Stein, J. (1966). Money and capacity growth. *Journal of Political Economy*, *74*, 451–465.