STATISTICS IN TRANSITION new series, September 2022
Vol. 23, No. 3, pp. 65–78, DOI 10.2478/stattrans-2022-0030
Received – 21.12.2021; accepted – 26.05.2022

Interval shrinkage estimation of the parameter of exponential distribution in the presence of outliers under loss functions

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ABSTRACT

In this paper, we studied estimators based on an interval shrinkage with equal weights point shrinkage estimators for all individual target points $\bar{\theta} \in (\theta_0, \theta_1)$ for exponentially distributed observations in the presence of outliers drawn from a uniform distribution. Estimators obtained from both shrinkage and interval shrinkage were compared, showing that the estimators obtained via the interval shrinkage method perform better. Symmetric and asymmetric loss functions were also used to calculate the estimators. Finally, a numerical study and illustrative examples were provided to describe the results.

Key words: interval information, mean square error, shrinkage estimator, exponential distribution, uniform distribution, outliers, Linex loss function.

1. Introduction

We are interested in working on an exponential distribution due to its various applications in life testing in case we encounter some outliers. Suppose $(X_1, X_2, ..., X_n)$is a random sample of size n whose k out of n observations seem to be outliers and taken from a uniform distribution. Studying previous works shows that Epstein and Sobel (1954) obtained the minimum variance unbiased estimator (MVUE) for scale parameter and location parameter of exponential distribution. Bhattacharia and Srivastava (1974) work on a shrinkage estimator for scale parameter. Stein (1956) proposes non-sample information in shrinkage estimation. The shrinkage estimation contents are an innovative combination of classical estimators of parameter and a guess value for it, which is called a shrinkage target. Based on Hawkins (1980) an outlier is an observation that deviates so much from other observations and it might have been generated by a different procedure. Dixit and Nasiri (2001) estimate parameters of exponential distribution in the presence of outliers generated from uniform distribution. Nasiri and Jabbari (2009) discuss estimation of parameters of the generalized exponential distribution in the existence of outliers. Finally, Golosnoy and Liesenfeld (2011) obtain an interval for shrinkage estimators. Based of this review, we show that the interval shrinkage estimator does better than another estimator. This paper is in some way related to the investigation by Nasiri and Ebrahimi (2019), whenever now we consider the outliers generated from uniform distribution.

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The LINEX loss function was introduced by Varian (1975), and several others including Zellner (1986), Basu and Ebrahimi Rojo (1987) and Soliman (2000), who have used this loss function in different estimation and prediction problems. The LINEX loss function is given by:

$$L(\Delta) = e^{a\Delta} - a\Delta - 1, \;\; a \neq 0,$$

With $\Delta = \frac{\hat{\theta}}{\theta}$, where $\hat{\theta}$ is an estimate of θ and a represents the shape parameter of the loss function. The behaviour of the LINEX loss function changes with the choice of a. Particularly, if a is close to zero (see Pandey (1997)), this loss function is almost equivalent to the Squared Error Loss Function(SELF) and therefore almost symmetric.

In shrinkage estimation when θ_g, a guess value of θ is available, the shrinkage estimator and its properties following Thompson (1968) is defined as

$$\hat{\theta}_{sh} = \theta_g + \omega(\hat{\theta} - \theta_g), \quad 0 \leq w < 1 \tag{1}$$

To find ω we have to consider MSE of estimator as:

$$MSE(\hat{\theta}_{sh}) = E[\hat{\theta}_{sh} - \theta]^2$$

In equation (1), to obtain MSE($\hat{\theta}_{sh}$), we consider $\hat{\theta}_{sh}$ as the shrinkage estimator, that is $\hat{\theta}_{sh} = \theta_g + \omega(\hat{\theta} - \theta)$, where $0 \leq \omega < 1$ and θ_g is our guess from parameter space (see Thompson 1968). Hence, MSE($\hat{\theta}_{sh}$) $= E(\hat{\theta}_{sh} - \theta)^2 = E(\theta_g + \omega(\hat{\theta} - \theta_g) - \theta)^2$, so

$$\begin{aligned} MSE(\hat{\theta}) &= E[\theta_g + \omega(\hat{\theta} - \theta_g) - \theta]^2 \quad (2) \\ &= E[\omega(\hat{\theta} - \theta) + (\omega - 1)(\theta - \theta_g)]^2 \\ &= \omega^2 MSE(\hat{\theta}, \theta) + (\omega - 1)^2 * (\theta - \theta_g)^2 + 2\omega(\omega - 1)(\theta - \theta_g)E(\hat{\theta} - \theta) \\ &= \frac{\omega^2\theta^2}{n} + (\omega - 1)^2(\theta - \theta_g)^2, \quad (3) \end{aligned}$$

Now, we have to minimize the MSE,

$$\frac{dMSE(\hat{\theta}_{sh})}{d\omega} = \frac{2\omega\theta^2}{n} + 2(\theta_g - \theta)^2(\omega - 1) = 0, \tag{4}$$

$$\omega^* = \frac{(\theta_g - \theta)^2}{\frac{\theta^2}{n} + (\theta_g - \theta)^2}, \tag{5}$$

So the shrinkage estimator is given by

$$\hat{\theta}_{sh} = \theta_g + [\frac{(\theta_g - \theta)^2}{\frac{\theta^2}{n} + (\theta_g - \theta)^2}](\hat{\theta} - \theta_g) \tag{6}$$

and

$$\begin{aligned} MSE(\hat{\theta}_{sh}) &= [\hat{\theta}_{sh} - \theta]^2 \\ &= E[\theta_g + B_1(\hat{\theta} - \theta_g) - \theta]^2 \\ &= {B_1}^2 MSE(\hat{\theta}) + (1 - B_1)^2 (\theta_g - \theta)^2, \end{aligned}$$

where

$$B_1 = \frac{(\theta_g - \theta)^2}{\frac{\theta^2}{n} + (\theta_g - \theta)^2}. \tag{7}$$

In Section 2, we have obtained the joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of k outliers. In Section 3, 4 and 5 we deal with the shrinkage estimator with the presence of outliers, a feasible interval shrinkage estimator and an interval shrinkage estimator under LINEX loss function. In Section 6, we compare the MSE and LINEX risk of the interval shrinkage estimators.

2. Joint distribution of $(X_1, X_2, ..., X_n)$ with presence of outliers

Let $X_1, X_2, ..., X_n$ be n non-negative continuous random variables such that for a given combination $(i_1, i_2, ..., i_{n-k})$ of the integers $(1, 2, ..., n)$, the following conditions hold:

a) The random variables $X_{i_1}, X_{i_2}, ..., X_{i_{n-k}}$ are independent, each having the probability density function $f(x)$.

b) The remaining random variables are also independent, each having the probability density function $g(x)$.

c) The two sets of the random variables are also independent.

d) Further, it is assumed that the combinations $(i_1, i_2, ..., i_{n-k})$ of the integers $(1, 2, 3, ..., n)$ are chosen at random with equal probability $[c(n,k)]^{-1}$ for each combination, where

$$c(n,k) = \frac{n!}{k!(n-k)!}$$

The joint density of $X_1, X_2, ..., X_n$ is given as (See Dixit and Nasiri (2001))

$$f(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i) \sum_{(i_1, i_2, ..., i_{n-k})} \prod_{j=1}^{k} [c(n,k)]^{-1} \frac{g(x_{i_j})}{f(x_{i_j})}$$

Dixit and Nasiri (2001) consider estimation of parameters of an exponential distribution in the presence of outliers generated from a uniform distribution. So, if we have random variables $(X_1, X_2, ..., X_n)$ such that k of them are a distribution with pdf $f_1(x;\theta)$

$$f_1(x;\theta) = \frac{1}{\theta}, 0 < x < \theta, \tag{8}$$

and the remaining $(n-k)$ random variables are distributed with pdf $f_2(x;\theta)$ function

$$f_2(x;\theta)=\frac{1}{\theta}e^{-\frac{x}{\theta}},x>0,\theta>0, \tag{9}$$

then the joint distribution of $(X_1,X_2,...,X_n)$ is

$$f(x_1,x_2,...,x_n;\theta)=\left[\frac{k!(n-k)!}{n!}\right]^{-1}\prod_{i=1}^{n}f_2(x_i,\theta)\sum^{*}\prod_{j=1}^{k}\frac{f_1(x_{A_j};\theta)}{f_2(x_{A_j};\theta)}, \tag{10}$$

where

$$\sum^{*}=\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}$$

.

For $f_1(x;\theta)$ and $f_2(x;\theta)$, $f(x_1,x_2,...,x_n;\theta)$ is

$$\begin{aligned} f(x_1,x_2,...,x_n;\theta) &= \frac{k!(n-k)!}{n!}\ \frac{e^{\frac{-\sum x_i}{\theta}}}{\theta^{n-k}}\sum^{*}\prod_{j=1}^{k}\frac{\frac{1}{\theta}I_{(0,\theta)}(x_{A_j})}{e^{\frac{-x_{A_j}}{\theta}}} \\ &= \frac{k!(n-k)!}{n!\ \theta^{n}}\ e^{\frac{-\sum x_i}{\theta}}\sum^{*}\prod_{j=1}^{k}\frac{I_{(0,\theta)}(x_{A_j})}{e^{\frac{-x_{A_j}}{\theta}}} \\ &= \frac{k!(n-k)!}{n!\ \theta^{n}}\ e^{\frac{-\sum x_i}{\theta}}\sum^{*}\prod_{j=1}^{k}e^{\frac{x_{A_j}}{\theta}}I_{(0,\theta)}(x_{A_j}), \end{aligned}$$

For $k=1$; $f(x_1,x_2,...,x_n;\theta)=\frac{1}{n\theta^n}e^{\frac{-\sum x_i}{\theta}}\sum_{A_1=1}^{n}e^{\frac{x_{A_1}}{\theta}}I(\theta-x_{A_1})$.
For $k=2$; $f(x_1,x_2,...,x_n;\theta)=\frac{2}{n(n-1)\theta^n}e^{\frac{-\sum x_i}{\theta}}\sum_{A_1=1}^{n-1}\sum_{A_2=A_1+1}^{n}e^{\frac{x_{A_1}+x_{A_2}}{\theta}}\ I(x_{A_1}-\theta)\ I(x_{A_2}-\theta)$.

Dixit (1987), based on the joint distribution

$$f(x_1,x_2,...,x_n)=\prod_{j=1}^{n}\frac{g(x_{ij})}{f(x_{ij})}[C(n,k)]^{-1}$$

show that the marginal distribution of X_i is given by

$$h(x_i)=\frac{k}{n}g(x_i)+\frac{n-k}{n}f(x_i)$$

Hence,

$$f(x;\theta) = \frac{k}{n} f_1(x;\theta) + \frac{n-k}{n} f_2(x;\theta)$$
$$= \frac{k}{n}\frac{1}{\theta} I_{(0,\theta)}(x) + \frac{n-k}{n\theta}\ e^{\frac{-x}{\theta}} I_{(0,\infty)}(x), \tag{11}$$

So we have

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = E(X) = \frac{k}{n}\int_0^{\theta} \frac{1}{\theta} x\, dx + \frac{n-k}{n}\int_0^{\infty} \frac{1}{\theta} x e^{-\frac{x}{\theta}} dx = \frac{(2n-k)\theta}{2n}$$
$$V(\bar{X}) = \Big(1 - \frac{2k}{3n} + \frac{k^2}{4n^2}\Big)\frac{\theta^2}{n}, \tag{12}$$

It is easy to show that

$$\hat{\theta} = \frac{2n}{2n-k}\bar{X}. \tag{13}$$

which is unbiased with expectation and variance as:

$$E(\hat{\theta}) = \theta \qquad and \qquad V(\hat{\theta}) = A^2 C \frac{\theta^2}{n}, \tag{14}$$

where $A = \frac{2n}{2n-k}$ and $C = \Big(1 - \frac{2k}{3n} + \frac{k^2}{4n^2}\Big)$.

Note: The sample size n and the number of outliers k are given parameters. But in the actual application, k is unknown and should be estimated. One of the methods is that k can be selected by evaluating the likelihood for different values of k choosing the one that maximizes the likelihood.

3. Feasible interval shrinkage estimator

In 2011, Golosnoy and Liesenfeld (2011) show the shrinkage estimator towards the interval $\theta \in [\theta_0, \theta_1] \subset R$ for unbiased conventional sample estimator of $\hat{\theta}$ with $E(\hat{\theta}) = \theta$ is given by

$$\widetilde{\theta}_{sh} = \hat{\theta} + \sqrt{V(\hat{\theta})}\frac{\theta - \hat{\theta}}{\theta_1 - \theta_0}\Big[arctan\Big(\frac{\theta_1 - \theta}{\sqrt{V(\hat{\theta})}}\Big) - arctan\Big(\frac{\theta_0 - \theta}{\sqrt{V(\hat{\theta})}}\Big)\Big]$$
$$+ \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} \ln\Big(\frac{V(\hat{\theta}) + (\theta_1 - \theta)^2}{V(\hat{\theta}) + (\theta_0 - \theta)^2}\Big), \tag{15}$$

and

$$E(\widetilde{\theta}_{sh}) = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} \ln\Big[\frac{V(\hat{\theta}) + (\theta_1 - \hat{\theta})^2}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\Big], \tag{16}$$

for $E(\hat{\theta}) = \theta$, we have

$$\widetilde{\theta}_{sh} = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln\Big[\frac{V(\hat{\theta}) + (\theta_1 - \hat{\theta})^2}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\Big]. \tag{17}$$

For different values of lower and upper bound of the interval, when θ_1 is far from θ_0 or $V(\hat{\theta})$ approaches zero, the $MSE(\hat{\theta})$ decreases. Furthermore, if $\hat{\theta}$ is considered as the median of the interval, $\theta_m = (\theta_0 + \theta_1)/2$, then $(\theta_1 - \hat{\theta}) = \frac{\theta_1 - \theta_0}{2}$ and $(\theta_0 - \hat{\theta}) = \frac{\theta_0 - \theta_1}{2}$. In this case, the equation(16) can be written as:

$$\begin{aligned}\widetilde{\theta}_{sh} &= \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln\Big[\frac{V(\hat{\theta}) + \frac{(\theta_1 - \theta_0)^2}{4}}{V(\hat{\theta}) + \frac{(\theta_0 - \theta_1)^2}{4}}\Big] \\ &= \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln(1) = \hat{\theta},\end{aligned}$$

$\widetilde{\theta}_{sh}$ approaches $\hat{\theta}$.

Note that the expectation and variance of $\widetilde{\theta}_{sh}$ is not easy since $\widetilde{\theta}_{sh}$ is not linear $\hat{\theta}$. Golosnoy and Liesenfeld (2011) suggest to find $\widetilde{\theta}_{sh}$ by using the first order Taylor expansion around the median point θ_m. We also define $\theta_d = (\theta_1 - \theta_0)/2$, so the equation would be as follows:

$$\widetilde{\theta}_{sh} = \theta_m + (\hat{\theta} - \theta_m)\frac{\partial \hat{\theta}(\theta_m)}{\partial \hat{\theta}} + \frac{(\hat{\theta} - \theta_m)^2}{2} \frac{\partial^2 \hat{\theta}(\theta_m)}{\partial \hat{\theta}^2} = 0$$

where

$$\frac{\partial \hat{\theta}(\theta_m)}{\partial \hat{\theta}} = 1 + \frac{V(\hat{\theta})}{\theta_1 - \theta_0}\left(\frac{\theta_0 - \hat{\theta}}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2} + \frac{\theta_1 - \hat{\theta}}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\right),$$

and

$$\frac{\partial^2 \hat{\theta}(\theta_m)}{\partial \hat{\theta}^2} = 0.$$

The resulting estimator is

$$\widetilde{\theta}_{sh} = \hat{\theta}\left(1 - \frac{V(\hat{\theta})}{V(\hat{\theta}) + \left(\left(\frac{\theta_1 - \theta_0}{2}\right)^2\right)}\right) + \theta_m \frac{V(\hat{\theta})}{V(\hat{\theta}) + \left(\left(\frac{\theta_1 - \theta_0}{2}\right)^2\right)}$$

We also define $\theta_d = \frac{\theta_1 - \theta_0}{2}$, so the equation would be as follows:

$$\widetilde{\widetilde{\theta}}_{sh} = \hat{\theta}\left[1 - \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}\right] + \theta_m \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}. \quad (18)$$

For $\frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}$ is constant, its variance is equal zero. So, we can easily show that

$$E(\widetilde{\widetilde{\theta}}_{sh}) = \theta - (\theta - \theta_m)\frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2},$$

and

$$V(\widetilde{\widetilde{\theta}}_{sh}) = V(\hat{\theta})\Big(1 - \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}\Big)^2.$$

Let $1 - \frac{V(A\bar{X})}{V(A\bar{X}) + \theta_d^2} = B_2$ then

$$\widetilde{\widetilde{\theta}}_{sh} = AB_2\bar{X} + (1 - B_2)\theta_m,$$

so

$$\begin{aligned}
MSE(\widetilde{\widetilde{\theta}}_{sh}) &= E\Big(\widetilde{\widetilde{\theta}}_{sh} - \theta\Big)^2 \\
&= E\Big(B_2A\bar{X} + (1 - B_2)\theta_m - \theta\Big)^2 \\
&= E\Big(B_2(A\bar{X} - \theta) + B_2\theta + (1 - B_2)\theta_m - \theta\Big)^2 \\
&= E\Big(B_2(A\bar{X} - \theta) + (1 - B_2)\theta + (1 - B_2)\theta_m\Big)^2 \\
&= E\Big(B_2(A\bar{X} - \theta) + (1 - B_2)(\theta - \theta_m)\Big)^2 \\
&= B_2^2 MSE(\hat{\theta}_{Sh-outlier}) + (1 - B_2)^2(\theta - \theta_m)^2,
\end{aligned}$$

resulting

$$MSE(\widetilde{\widetilde{\theta}}_{sh}) \leq MSE(\hat{\theta}_{Sh-outlier}).$$

4. Interval shrinkage estimation under LINEX loss function

In decision theory and quality assurance filed, loss functions are used to reflect the monetary loss or economic loss caused by deterioration of the product characteristics from the target quality. However, Berger (1985) even emphasized that the loss function should be bounded and concave, because the loss function also mimics the negative of the utility, whereas the squared-error loss, Taguchi quadratic loss in quality control, or absolute error loss is unbounded and even disturb the convexity. In some decision problems, some types of asymmetric losses are proposed. One of the most eminent examples is LINEX, which was proposed by Varian (1975) and populated by Zellner (1986).

Consider LINEX loss function for $\tilde{\tilde{\theta}}$

$$L(\Delta) = e^{a\Delta} - a\Delta - 1, \quad \Delta = \frac{\tilde{\tilde{\theta}}}{\theta}.$$

which

$$\Delta = \frac{\tilde{\tilde{\theta}}}{\theta} = \frac{AB_2}{\theta}\bar{X} + (1-B_2)\frac{\theta_m}{\theta}.$$

where $A = \frac{2n}{2n-k}$, $B_2 = 1 - \frac{V(A\bar{X})}{V(A\bar{X})+\theta_d^2}$. In this case the risk under LINEX loss function is obtained by

$$R = E(L(\Delta)) = E(e^{a\Delta} - a\Delta - 1) = E(e^{a\Delta}) - aE(\Delta) - 1$$

where

$$\begin{aligned} aE(\Delta) &= aE\left(\frac{\tilde{\tilde{\theta}}}{\theta}\right) = \frac{a}{\theta}E(\tilde{\tilde{\theta}}) = \frac{a}{\theta}E(AB_2\bar{X} + (1-B_2)\theta_m) \\ &= \frac{aAB_2}{\theta}E(\bar{X}) + (1-B_2)\frac{\theta_m}{\theta} \\ &= \frac{aAB_2}{\theta}\left(\frac{2n-k}{2n}\theta\right) + (1-B_2)\frac{\theta_m}{\theta} \\ &= \frac{aAB_2(2n-k)}{2n} + (1-B_2)\frac{\theta_m}{\theta} \end{aligned}$$

$$\begin{aligned} E(e^{a\Delta}) &= E\left(e^{\frac{a\tilde{\theta}}{\theta}}\right) = E\left(e^{\frac{a}{\theta}(AB_2\bar{X}+(1-B_2)\theta_m)}\right) \\ &= e^{\frac{a(1-B_2)\theta_m}{\theta}} E\left(e^{\frac{aAB_2\bar{X}}{\theta}}\right) = e^{\frac{a(1-B_2)\theta_m}{\theta}} E\left(e^{\frac{aAB_2}{n\theta}(X_1+X_2+\ldots+X_n)}\right) \\ &= e^{\frac{a(1-B_2)\theta_m}{\theta}} E\left(e^{\frac{aAB_2}{\theta}X_1} e^{\frac{aAB_2}{\theta}X_2} \ldots e^{\frac{aAB_2}{\theta}X_n}\right) = e^{\frac{a(1-B_2)\theta_m}{\theta}} \left[E\left(e^{\frac{aAB_2}{n\theta}X}\right)\right]^n \end{aligned}$$

such that

$$E\left(e^{\frac{aAB_2}{n\theta}X}\right)=\frac{k}{n}\int_0^{\theta}\frac{1}{\theta}e^{\frac{aAB_2}{n\theta}x}dx+\frac{n-k}{n}\int_0^{\infty}\frac{1}{\theta}e^{\frac{aAB_2}{n\theta}x}e^{\frac{-x}{\theta}}dx$$

$$=\frac{k}{n\theta}\left[\frac{n\theta}{aAB_2}e^{\frac{aAB_2}{n\theta}x}|_0^{\theta}\right]+\frac{n-k}{n\theta}\int_0^{\infty}e^{-\left(1-\frac{aAB_2}{n}\right)\frac{x}{\theta}}dx$$

$$=\frac{k}{n\theta}\left[\frac{n\theta}{aAB_2}e^{\frac{aAB_2}{n\theta}}-\frac{n\theta}{aAB_2}\right]=\frac{n-k}{n\theta}\left(\frac{n\theta}{n-aAB_2}\right)$$

$$=\frac{k}{aAB_2}e^{\frac{aAB_2}{n}}-\frac{k}{aAB_2}+\frac{n-k}{n-aAB_2}$$

Hence,

$$R=e^{\frac{a(1-B_2)\theta m}{\theta}}\left[\frac{k}{aAB_2}e^{\frac{aAB_2}{2}}-\frac{K}{aAB_2}+\frac{n-k}{n-aAB_2}\right]^n-\frac{aAB_2(2n-k)}{2n}+(1-B_2)\frac{\theta_m}{\theta}-1$$

5. Numerical Study

To compare the performance of mean square error (MSE) and LINEX risk of the interval shrinkage estimator $\widetilde{\widetilde{\theta}}_{sh}$, we carry out simulation study using R software and the results are shown in Tables 1 to 4. The shape parameter takes different values. Samples were generated with sizes $n = 10(10)(50)$ using of R software. The MSE and LINEX loss function of the interval shrinkage estimator decrease when the sample size increases. Meantime, for $k = 1, n = 9$ means that one sample is generated from the uniform distribution and 9 samples are generated from the exponential distribution. Here, the number of replicated cases is $N = 1000$. In cases of $a = -0.01$ and $a = 0.01$, they are very close to each other. It is also worth mentioning, based on the results of Tables 1 and 2, when the value of "a" tends to zero, the results are the same.

Table 1. $k = 1, \quad \theta = 4, \quad \theta_g = 3.2, \quad \theta_0 = 3.7, \quad \theta_1 = 4.2$

	a=-1		a=-0.25		a=-0.01	
n	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$
10	0.0082	0.0054	0.0082	0.0008	0.0077	0.0012
20	0.0042	0.0052	0.0055	0.0007	0.0060	0.0011
30	0.0037	0.0048	0.0043	0.0007	0.0060	0.0011
40	0.0027	0.0039	0.0030	0.0006	0.0033	0.0013
50	0.0027	0.0038	0.0025	0.0005	0.0023	0.0013

Table 2. $k=2, \quad \theta=4, \quad \theta_g=3.2, \quad \theta_0=3.7, \quad \theta_1=4.2$

	a=-1		a=-0.25		a=-0.01	
n	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$
10	0.0012	0.0024	0.0021	0.0004	0.0013	0.0019
20	0.0016	0.0021	0.0017	0.0003	0.0016	0.0014
30	0.0015	0.0029	0.0017	0.0003	0.0017	0.0013
40	0.0019	0.0029	0.0017	0.0004	0.0017	0.0010
50	0.0016	0.0049	0.0016	0.0004	0.0017	0.0017

Table 3. $k=1, \quad \theta=4, \quad \theta_g=3.2, \quad \theta_0=3.7, \quad \theta_1=4.2$

	a=0.01		a=0.25		a=1	
n	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$
10	0.0083	0.0013	0.0082	0.0024	0.0082	0.0006
20	0.0058	0.0013	0.0046	0.0020	0.0052	0.0005
30	0.0044	0.0010	0.0041	0.0024	0.0043	0.0004
40	0.0029	0.0013	0.0032	0.0024	0.0028	0.0003
50	0.0026	0.0015	0.0026	0.0016	0.0026	0.0002

Table 4. $k=2, \quad \theta=4, \quad \theta_g=3.2, \quad \theta_0=3.7, \quad \theta_1=4.2$

	a=0.01		a=0.25		a=1	
n	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$	$MES(\widetilde{\widetilde{\theta}}_{sh})$	$R(\widetilde{\widetilde{\theta}}_{sh})$
10	0.0025	0.0005	0.0017	0.0005	0.0018	0.0002
20	0.0018	0.0010	0.0017	0.0010	0.0016	0.0002
30	0.0017	0.0017	0.0017	0.0014	0.0015	0.0002
40	0.0016	0.0016	0.0016	0.0013	0.0015	0.0004
50	0.0018	0.0018	0.0015	0.0012	0.0017	0.0001

6. Practical Example

In order to illustrate the methodology proposed in this paaper, we consider, Nelson (1982) concerning the data on time to break-down of an insulating fluid between electrodes at a voltage of 34 KV (Kilo-Volts). Data are as follows:
0.19 0.78 0.96 1.31 2.78 3.16 4.15 4.67 4.85 6.50 7.35 8.01 8.27 12.06 31.75 32.52 33.91 36.71 72.89

In the initial evaluation, one-sample Kolmogorov-Smirnov test results show that the data follow an exponential distribution. Figures 1 and 2 have been reported to be checked for the presence of outlier's data. Figure 1 shows the presence of one outlier. Investigation of this

result is based on theoretical and interval shrinkage estimation. Here, to find the number of outliers or k, we consider $\theta \in (14, 15)$ and based on the sample information
$n = 19$, $\sum_{i=1}^{19} x_i = 272.82$; $\bar{x} = 14.35895$. Note that in Table 5, to determine the value of k we have

$$\hat{V}(\tilde{\tilde{\theta}}) = \hat{V}(\hat{\theta}) \left(1 - \frac{\hat{V}(\hat{\theta})}{\hat{V}(\hat{\theta}) + \theta_d^2}\right)$$

such that

$$\hat{V}(\hat{\theta}) = \left[\frac{2n}{2n-k}\right]^2 \left(1 - \frac{2k}{3n} - \frac{k^2}{4n^2}\right) \frac{\hat{\theta}^2}{n}$$

According to the estimator of $\hat{V}(\tilde{\tilde{\theta}})$, it can be said that the increase in the value of k is greater than the increase in the estimator. But by rotating the value of the maximum likelihood, the value of k is determined.

Table 5.

k	$\tilde{\tilde{\theta}}_{sh}$	$V(\tilde{\tilde{\theta}}_{sh})$	$\sum^*$	$L(\tilde{\tilde{\theta}}\|x)$
0	14.49682	0.0055031	1	5.790283×10^{-31}
1	14.50518	0.0051386	19.29345	$5.879082 \times 10^{-31*}$
2	14.51281	0.0047826	172.2583	$5.831662 \times 10^{-31*}$
3	14.51969	0.00443596	943.3984	5.635563×10^{-31}
4	14.52582	0.00409897	3540.8260	5.287464×10^{-31}

According to the results of Table 5, the likelihood function with respect to k is maximized when k is equal to 1. So, the number of outliers is 1 and $\tilde{\tilde{\theta}}_{sh} = 14.51281$.

7. Conclusion

In an experimental situation, many a time an experimenter comes across some of the observations which are far removed from the main body of the data and hence are outliers. In this paper, shrinkage and interval shrinkage estimators are discussed for the first time with the presence of outliers generated from a uniform distribution and it is shown that the interval shrinkage estimator is better than the shrinkage estimator. Using different loss functions can also improve the performance of the estimator. It may be mentioned that the proposed method can be extended for Bayesian interval shrinkage estimation and other positive data distribution as well as for the presence of outliers from other distributions.

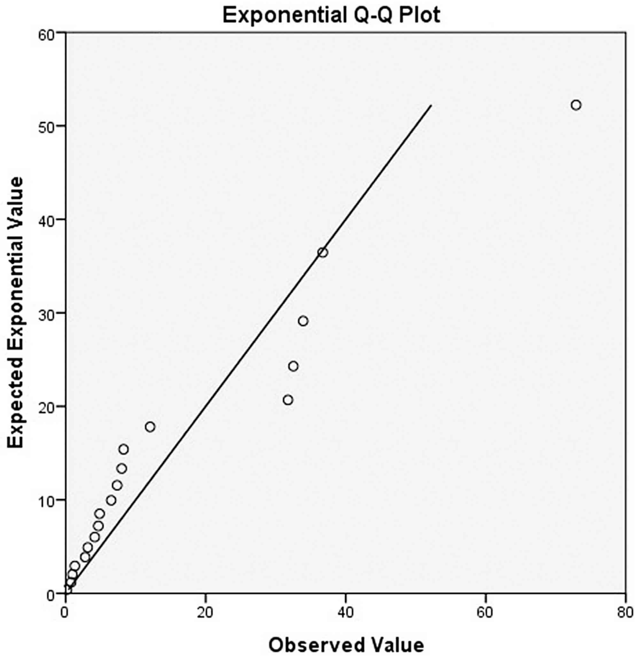

Figure 1. Exponential Q-Q Plot

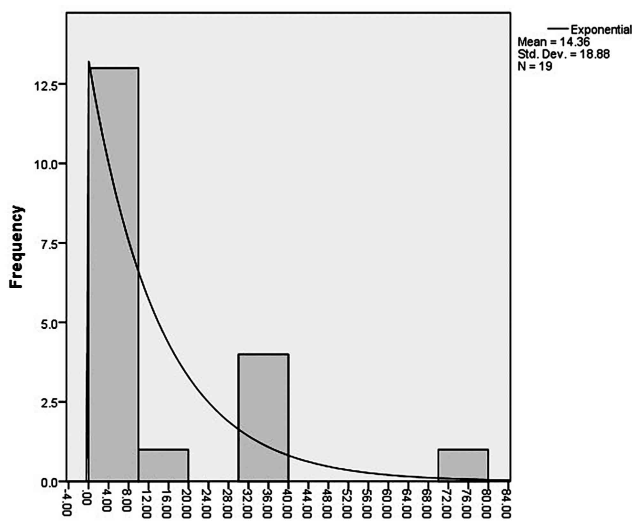

Figure 2. Frequency Distribution

8. Acknowledgment

The author would like to thank the editor and the two reviewers for their valuable comments which improved the quality of the paper.

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