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The modified Bagdonavičius-Nikulin goodness-of-fit test statistic for the right censored distributional validation with applications in medicine and reliability

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Abstract

A modified version of Bagdonavičius-Nikulin goodness-of-fit statistical test is presented for validation under the right censor case. Simulation via Barzilai-Borwein algorithm is performed for assessing the right-censorship estimation method. Four right censored data sets are analyzed under the new modified test statistic for checking the distributional validation.

Key words: inverted Weibull distribution, censored validation, Bagdonavičius-Nikulin, goodness-of-fit testing

1. Introduction

The Nikulin-Rao-Robson statistic, which is based on the differences between two estimators of the probabilities to fall into grouping intervals, is a well-known modification of the classical chi-squared tests in the case of complete data. One estimator is based on the empirical distribution function, and the other on maximum likelihood estimators of unknown parameters of the tested model using initial non-grouped data (see Nikulin (1973a), Nikulin (1973b), Nikulin (1973c) and Rao and Robson (1974) for more details and Goual et al. (2019), Goual and Yousof (2020a), Goual and Yousof (2020b), Yousof et al. (2021c) for more relevant applications).

However, methods for testing the censored validity of parametric distributions are in increasing development, but the presence of censorship makes them unavailable.

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Habib and Thomas (1986) and Hollander and Pena (1992) proposed a modified Chi-squared test for randomly censored data based on the well-known Kaplan–Meyer estimators. Galanova (2012) considered some nonparametric modifications to the Anderson–Darling statistic, Kolmogorov–Smirnov statistic and the Cramer-Von-Mises statistic for accelerate failure models. Bagdonavičius-Nikulin (2011a) presented a new Chi-squared goodness-of-fit test statistic for the right censored data. (see also Bagdonavičius and Nikulin (2011b)). The Chi-squared goodness-of-fit test statistic of Bagdonavičius-Nikulin is applied for distributional validation under the right-censorship case.

In this paper, a modified Chi-squared goodness-of-fit test statistic based on the Bagdonavičius-Nikulin test is presented and applied accordingly for validation under a new odd log-logistic inverted Weibull distribution using the right censor case. First, a simulation study under the right censor case via the Barzilai-Borwein (BB) algorithm is performed for assessing the right censored estimation method. Numerous domains of optimization have paid substantial attention to the Barzilai and Borwein gradient approach. This is as a result of its practical effectiveness, affordability of computing, and simplicity. This study establishes root-linear global convergence for the Barzilai and Borwein method for strictly convex quadratic problems given in infinite-dimensional Hilbert spaces using spectral analysis techniques. It is shown how these findings may be used to two optimization issues governed by partial differential equations.

Following Ravi and Gilbert (2009), Hamedani et al. (2023) and Emam et al. (2023) and using the BB algorithm, we generated N=10,000 with different size (q=15,25,50,130,350,500) from the new odd log-logistic inverted Weibull model using some carefully selected initial values. The mean square error (MSEs) are used for assessing the performance of the censored maximum likelihood. Second, the modified Bagdonavicius-Nikulin test is applied using four right censored real data sets for distributional validation. The following censored real data are considered:

I. Data of bone marrow transplant (38 patients).
II. Data of allogeneic bone marrow transplant where the Histocompatibility Leukocyte Antigen (HLA) homolog marrow was used to rebuild their immune systems (50 patients).
III. Lymphoma data (times up to death) (31 patients).
IV. Strength data of some cords having resisted for a determined time were studied (38 pieces).

The Bagdonavicius-Nikulin goodness-of-fit statistical test proved that the new model can be used as a suitable alternative for analyzing four right censored data sets.

In this context, we will mention some recent studies that applied or presented new, modified extensions to the Bagdonavicius-Nikulin goodness-of-fit test. It is worth noting that the browser for statistical literature on this subject (Bagdonavicius-Nikulin goodness-of-fit test) will not find many new Bagdonavicius-Nikulin goodness-of-fit extensions and will find few research studies that applied this test, especially since Bagdonavicius-Nikulin goodness-of-fit test has precise requirements and strict procedures and requires censored data. As is well known, it is not easy to obtain new censored data to apply to and highlight the importance of the new test. In the next few lines, we will review some recent studies that were concerned with applying this test to actual data subject to censorship from the right, with a summary of what each study provided separately.

Mansour et al. (2020a) applied Bagdonavicius-Nikulin goodness-of-fit test for a novel log-logistic model utilizing for distributional validation. For the "right censored" real data set of survival times, the modified test is used. The new test's components are all clearly deduced and presented. For testing the adaptability and significance of the new model under the unfiltered framework, three actual data applications are provided. For filtered validation, two more genuine data sets are examined.

Mansour et al. (2020b) suggested and implemented a modified Chi-square goodness-of-fit test for the Burr X Weibull model using the Bagdonavicius-Nikulin technique for the right censored validation. The appropriate censored real data set is subjected to the modified goodness-of-fit statistics test. The modified goodness-of-fit test recovers the information loss based on the censored maximum likelihood estimators on the initial data, while the grouped data follows the Chi-square distribution The components of the modified criterion tests are derived. Under the unfiltered approach, validation is an actual data application.

Recently, an adapted Chi-square type test for distributional validity with applications to right-censored reliability and medical data by Yousof et al. (2021a). In this study, A modified version of the Bagdonaviius-Nikulin goodness-of-fit test statistic, also known as the modified Chi-square goodness-of-fit test, is researched, and then used for distributional validation in the appropriate censored scenario. The updated goodness-of-fit test is introduced and used with the appropriate censored data sets. Through a thorough simulation analysis, the censored Barzilai-Borwein algorithm is used to evaluate the new test's validity. Four actual and right censored data sets are subjected to the modified Bagdonaviius-Nikulin test. The new modified Bagdonaviius-Nikulin goodness-of-fit test statistic is used to compare a new distribution against a large number of other competing distributions.

The validity of the Bagdonaviius and Nikulin goodness-of-fit test statistic for the right censor case under the double Burr type X distribution is shown in a new updated

form. The method of maximum likelihood estimation in the case of censored data is used and implemented. Simulations using the Barzilai-Borwein algorithm are run to determine the best censored estimation technique. For the purpose of testing the null hypothesis, another simulation study is provided using a modified version of the statistical goodness-of-fit test developed by Bagdonaviius and Nikulin. For the purpose of examining the distributional validity, four right censored data sets are examined using the new modified test statistic (see Aidi et al. (2021)).

Finally, Ibrahim et al. (2022) presented a novel modified version of the Bagdonavicius and Nikulin goodness-of-fit statistical test, and its distributional validity is examined for both the right censor case and the exponentiated Rayleigh generalized Chen distribution. Simulations using the Barzilai-Borwein algorithm are run to determine the best censored estimating technique. For the purpose of examining the distributional validity, four right censored data sets are examined using the new modified test statistic. For more details, information, applications, and new extensions of this test in the case of censored data from the right, see: Yousof et al. (2021b) (for a new parametric lifetime model along with modified Chi-square type test for right censored distributional validation, characteristics and many estimation methods), Ibrahim et al. (2021) (for a new exponential generalized log-logistic model with the Bagdonavičius and Nikulin testing for distribution validation and some non-Bayesian estimation methods), see also Ibrahim et al. (2019 and 2020) and Yadav et al. (2022) for some related details about the Nikulin-Rao-Robson goodness-of-fit test.

2. Censored distributional validation

2.1. Maximum likelihood censored data

Consider the new odd log-logistic (NOLL) family (Cordeiro et al. (2016)). Then, for the inverted Weibull (IW) baseline model, the probability density function (PDF) of the new odd log-logistic inverted Weibull (NOLLIW) model can be derived as

$$f_{\underline{\boldsymbol{V}}}(x) = \frac{\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2\boldsymbol{\varsigma}_3 x^{-(\varsigma_3+1)}\, exp[-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\varsigma_3}]\{1-exp[-\boldsymbol{\varsigma}_2 x^{-\varsigma_3}]\}^{\varsigma_1-1}}{(exp[-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\varsigma_3}]+\{1-exp[-\boldsymbol{\varsigma}_2 x^{-\varsigma_3}]\}^{\varsigma_1})^2}, x>0, \quad (1)$$

where $\underline{\boldsymbol{V}} = (\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2, \boldsymbol{\varsigma}_3)$, $\boldsymbol{\varsigma}_1 > 0$, $\boldsymbol{\varsigma}_2 > 0, \boldsymbol{\varsigma}_3 > 0$ are three shape parameters. The survival function $S_{\underline{\boldsymbol{V}}}(x_i)$ can be written as

$$S_{\underline{\boldsymbol{V}}}(x) = 1 - \frac{exp(-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\varsigma_3})}{exp(-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\varsigma_3}) + [1-exp(-\boldsymbol{\varsigma}_2 x^{-\varsigma_3})]^{\varsigma_1}}. \quad (2)$$

In reliability studies and survival analysis, data are often censored. If $X_1, X_2, \ldots, X_{\boldsymbol{q}}$ is a censored sample from the NOLLIW distribution, each observation can be written as

$$x_i = min(X_i, \boldsymbol{C}_i)|_{(i=1,..,\boldsymbol{q})},$$

where X_i are failure times and $\boldsymbol{C}_i$ censoring times. Using (1) and (2), the log likelihood function is

$$L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})) = \log\left[\prod_{i=1}^{\boldsymbol{q}} f_{\underline{\boldsymbol{V}}}(x_i)^{\boldsymbol{\Delta}_i} S_{\underline{\boldsymbol{V}}}(x_i)^{1-\boldsymbol{\Delta}_i}\right]|_{(\boldsymbol{\Delta}_i=1_{X_i<\boldsymbol{c}_i})},$$

which can then be can be written as

$$L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}}) = \sum_{i=1}^{\boldsymbol{q}} \boldsymbol{\Delta}_i \log f\,(x_i) + \sum_{i=1}^{\boldsymbol{q}} (1-\boldsymbol{\Delta}_i) \log S\,(x_i)\,|_{(\boldsymbol{\Delta}_i=1_{X_i<\boldsymbol{c}_i})}. \tag{3}$$

Let $d_i = exp\left(-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\right)$, and $\delta_i = 1 - exp\left(-\boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\right)$. Then,

$$L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}}) = \sum_{i=1}^{\boldsymbol{q}} \boldsymbol{\Delta}_i \begin{bmatrix} log(\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2\boldsymbol{\varsigma}_3) - (\boldsymbol{\varsigma}_3 - 1)\, log\, x_i \\ -\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3} + (\boldsymbol{\varsigma}_1 - 1)\, log\, \delta_i - 2\, log(\, d_i + \delta_i) \end{bmatrix}$$
$$+ \sum_{i=1}^{\boldsymbol{q}} (1-\boldsymbol{\Delta}_i)[\boldsymbol{\varsigma}_1\, log\, \delta_i - log(\, d_i + \delta_i)].$$

The MLEs for parameters $\boldsymbol{\varsigma}_1, \boldsymbol{\varsigma}_2$ and $\boldsymbol{\varsigma}_3$ are derived from solving the following nonlinear system of $\frac{\partial L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_1}$, $\frac{\partial L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_2}$ and $\frac{\partial L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_3}$ where

$$\frac{\partial L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_1} = \sum_{i=1}^{\boldsymbol{q}} \left[\frac{1}{\boldsymbol{\varsigma}_1} - \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3} + log\, \delta_i + \frac{2\left(d_i \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3} - \delta_i^{\boldsymbol{\varsigma}_1}\, log\, \delta_i\right)}{d_i + \delta_i}\right]$$
$$+ \sum_{i=1}^{\boldsymbol{q}} (1-\boldsymbol{\Delta}_i)\left[log\, \delta_i + \frac{\boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3} d_i - \delta_i^{\boldsymbol{\varsigma}_1}\, log\, \delta_i}{d_i + \delta_i}\right],$$

$$\frac{\partial L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_2} = \sum_{i=1}^{\boldsymbol{q}} \boldsymbol{\Delta}_i \left[\frac{1}{\boldsymbol{\varsigma}_2} - \boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3} + (\boldsymbol{\varsigma}_1 - 1)\frac{x_i^{-\boldsymbol{\varsigma}_3}(1-\delta_i)}{\delta_i} \right.$$
$$\left. + \frac{2\begin{pmatrix} \boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3} d_i \\ -\boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3}(1-\delta_i)\delta_i^{\boldsymbol{\varsigma}_1 - 1} \end{pmatrix}}{d_i + \delta_i}\right]$$
$$+ \sum_{i=1}^{\boldsymbol{q}} (1-\boldsymbol{\Delta}_i)\left[\begin{matrix} \frac{\boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3}(1-\delta_i)}{\delta_i} \\ + \frac{\boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3} d_i - \boldsymbol{\varsigma}_1 x_i^{-\boldsymbol{\varsigma}_3}(1-\delta_i)\delta_i^{\boldsymbol{\varsigma}_1 - 1}}{d_i + \delta_i} \end{matrix}\right],$$

and

$$\frac{\partial L_{i,q}(\underline{\boldsymbol{V}})}{\partial \boldsymbol{\varsigma}_3} \sum_{i=1}^{q} \boldsymbol{\Delta}_i \left[\frac{1}{\boldsymbol{\varsigma}_3} + \left[\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3} - 1\right] log\, x_i - (\boldsymbol{\varsigma}_1 - 1) \frac{\boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, exp[-\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x^{-\boldsymbol{\varsigma}_3}]}{\delta_i} \right.$$
$$\left. - \frac{2\left(\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, d_i - \boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, (1-\delta_i)\delta_i^{\boldsymbol{\varsigma}_1 - 1}\right)}{d_i + \delta_i} \right]$$
$$- \sum_{i=1}^{q} (1 - \boldsymbol{\Delta}_i) \left[\frac{\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, exp[-\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x^{-\boldsymbol{\varsigma}_3}]}{\delta_i} \right.$$
$$\left. + \frac{\left(\boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, d_i - \boldsymbol{\varsigma}_1 \boldsymbol{\varsigma}_2 x_i^{-\boldsymbol{\varsigma}_3}\, log\, x_i\, (1-\delta_i)\delta_i^{\boldsymbol{\varsigma}_1 - 1}\right)}{d_i + \delta_i} \right].$$

The explicit form of $\hat{\boldsymbol{\varsigma}}_1$, $\hat{\boldsymbol{\varsigma}}_2$ and $\hat{\boldsymbol{\varsigma}}_3$ cannot be obtained, so we use numerical methods.

2.2. Test criteria for the new model

Let $X_1, X_2, \ldots, X_{\boldsymbol{q}}$ be grouped in r sub-intervals $I_1, I_2, \cdots, I_r$ as

$$I_{\boldsymbol{j}} =]a_{(\boldsymbol{j}-1)}; a_{(\boldsymbol{j})}] \mid_{(\boldsymbol{j}=1,2,\cdots,r)},$$

which are mutually disjoint. The limits $a_{(\cdot)}$ of the intervals $I_{\boldsymbol{j}}$ are calculated such that:

$$\hat{\rho}_{\boldsymbol{j}} = \hat{\rho}_{\boldsymbol{j}}(\underline{\boldsymbol{V}}) = \int_{a_{(\boldsymbol{j}-1)}}^{a_{(\boldsymbol{j})}} f_{\underline{\boldsymbol{V}}}(x_i) dx, \tag{4}$$

$$a_{(\boldsymbol{j})} = F^{-1}\left(\frac{\boldsymbol{j}}{r}\right) \mid_{(\boldsymbol{j}=1,2,\cdots,r-1)},$$

and

$$0 < a_{(0)} < a_{(1)} < \ldots < a_{(\boldsymbol{j}-1)} < a_{(\boldsymbol{j})} < +\infty.$$

The new test statistic $\boldsymbol{T}_{r,\varepsilon}^2$ is defined by

$$\boldsymbol{T}_{r,\varepsilon}^2 = \sum_{\boldsymbol{j}=1}^{q} \frac{1}{\boldsymbol{U}_{\boldsymbol{j}}} \left(\boldsymbol{U}_{\boldsymbol{j}} - e_{\boldsymbol{j}}\right)^2 + \boldsymbol{Q}, \tag{5}$$

where $\boldsymbol{e}_{\boldsymbol{j}}$ is the number of expected failures (NEF) in the grouped intervals and $\boldsymbol{U}_{\boldsymbol{j}}$ is the number of observed failures (NOF) in grouping intervals where

$$\boldsymbol{Q} = \boldsymbol{W}^T \hat{\mathbf{G}}^{-} \boldsymbol{W},$$
$$\widehat{\boldsymbol{W}} = \hat{\boldsymbol{C}} \hat{\boldsymbol{A}}^{-1} \boldsymbol{Z} = \left(\widehat{\boldsymbol{W}}_1, \ldots, \widehat{\boldsymbol{W}}_s\right)^T,$$
$$\hat{\mathbf{G}} = [\hat{\mathbf{g}}_{\mathcal{L}\mathcal{L}'}]_{s \times s},$$

$$\boldsymbol{W}_{\mathcal{L}} = \sum_{j=1}^{r} \widehat{\boldsymbol{C}}_{\mathcal{L}j} \widehat{\boldsymbol{\mathcal{A}}}_{j}^{-1} \boldsymbol{Z}_{j},$$

$$\boldsymbol{Z}_{j} = \frac{1}{\sqrt{\boldsymbol{q}}} (\boldsymbol{U}_{j} - e_{j}),$$

$$\hat{\mathbf{g}}_{\mathcal{L}\mathcal{L}} = \hat{\imath}_{\mathcal{L}\mathcal{L}'} - \sum_{j=1}^{r} \widehat{\boldsymbol{C}}_{\mathcal{L}j} \widehat{\boldsymbol{C}}_{\mathcal{L}'j} \widehat{\boldsymbol{\mathcal{A}}}_{j}^{-1}, \; j = 1,2,\dots,r, i = 1,2,\dots,\boldsymbol{q},, \mathcal{L}, \mathcal{L}' = 1,2,\dots,s.$$

The elements of $\widehat{\boldsymbol{C}}$ defined by

$$\widehat{\boldsymbol{C}}_{\mathcal{L}j} = \frac{1}{\boldsymbol{q}} \sum_{i:x_i \in I_j}^{\boldsymbol{q}} \boldsymbol{\Delta}_{i} \frac{\partial}{\partial \widehat{\underline{\mathbf{V}}}_{\mathcal{L}}} \ln H_{\underline{\widehat{V}}}(x_i), \tag{6}$$

where $H_{\underline{\widehat{V}}}(x_i)$ refers to the cumulative hazard rate function (CHRF) of the NOLLIW distribution.

2.3. Test quadratic form of the modified criteria

The quadratic form of the modified test statistic can be written as

$$\boldsymbol{\mathcal{T}}_{r,\varepsilon}^{2} = \sum_{j=1}^{r} \frac{1}{\boldsymbol{u}_{j}} (\boldsymbol{U}_{j} - e_{j})^{2} + \widehat{\boldsymbol{W}}^{T} \left[\hat{\imath}_{ll'} - \sum_{j=1}^{r} \widehat{\boldsymbol{C}}_{lj} \widehat{\boldsymbol{C}}_{l'j} \widehat{\boldsymbol{\mathcal{A}}}_{j}^{-1}\right]^{-1} \widehat{\boldsymbol{W}}, \tag{7}$$

where matrices $\widehat{\boldsymbol{W}}$, $\widehat{\boldsymbol{C}}$ are defined in Abouelmagd et al. (2019a,b), Mansour et al. (2020a,b) and Yadav et al. (2020) with more details and $\hat{\mathbf{I}}$ is the estimated information matrix.

2.4. Simulations via the BB algorithm

In this subsection we are interested in solving the nonlinear system of equations

$$0 = \frac{\partial}{\partial \boldsymbol{\varsigma}_{k}} L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})|_{k=1,2,3}$$

where the three functions

$$\frac{\partial}{\partial \boldsymbol{\varsigma}_{k}} L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})|_{\mathbf{J}(p):\Re^{p}\times\Re^{p}\to\Re^{p}}$$

refer to nonlinear functions with continuous partial derivative. We are interested in situations where p is large, and where the Jacobian of $\frac{\partial}{\partial \boldsymbol{\varsigma}_{k}} L_{i,\boldsymbol{q}}(\underline{\boldsymbol{V}})$ is either unavailable or requires a prohibitive amount of storage. The Quasi-Newton methods can be used for obtaining an approximation of $\mathbf{J}(p)$, which, along with the vector of solutions, are

updated at each iteration. Using the R statistical software and the BB algorithm of Ravi and Gilbert (2009), we generated $N = 10{,}000$ with different size ($\boldsymbol{q} = 15,\ 25,\ 50,\ 130,\ 350,\ 500$) from the NOLLIW model using the initial values ($\boldsymbol{\varsigma}_1 = 1.5,\ \boldsymbol{\varsigma}_2 = 2,\ \boldsymbol{\varsigma}_3 = 1.3$). Firstly, we compute the MLEs and then the criteria $\boldsymbol{\mathcal{T}}^2_{r,\varepsilon}$. The results which are given in Table 1 is obtained by inverting the CDF of the new model by setting

$$U = F_{\underline{V}}(x) = \frac{exp(-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\boldsymbol{\varsigma}_3})}{exp(-\boldsymbol{\varsigma}_1\boldsymbol{\varsigma}_2 x^{-\boldsymbol{\varsigma}_3})+[1-exp(-\boldsymbol{\varsigma}_2 x^{-\boldsymbol{\varsigma}_3})]^{\boldsymbol{\varsigma}_1}},$$

where U follows the uniform (0,1) model to obtain the quantile function. Since the quantile function of the NOLLIW model has no closed form, we use the numerical methods to generate the random numbers. Based on Table 1, it is seen that the MSE decreases as n increases. For example, we have the following results:

I. MSE ($\boldsymbol{\varsigma}_1$, $\boldsymbol{q} = 15,\ 25,\ 50,\ 130,\ 350,\ 500$) = (0.0077, 0.0059, 0.0040, 0.0026, 0.0017, 0.00009).
II. MSE ($\boldsymbol{\varsigma}_2$, $\boldsymbol{q} = 15,\ 25,\ 50,\ 130,\ 350,\ 500$) = (0.0115, 0.0092, 0.0083, 0.0075, 0.0058, 0.0029).
III. MSE ($\boldsymbol{\varsigma}_3$, $\boldsymbol{q} = 15,\ 25,\ 50,\ 130,\ 350,\ 500$) = (0.0088, 0.0068, 0.0042, 0.0039, 0.0022, 0.0017).

Table 1: MLEs of a_1, a_2, a_3 and MSE.

$\widehat{\boldsymbol{\varsigma}}_i$, MSEs↓ & n→	$\boldsymbol{q}_1$=15	$\boldsymbol{q}_2$=25	$\boldsymbol{q}_3$=50	$\boldsymbol{q}_4$=130	$\boldsymbol{q}_5$=350	$\boldsymbol{q}_6$=500
$\boldsymbol{\varsigma}_1$	1.4632	1.4696	1.4775	1.4854	1.4967	1.49980
MSE	0.0077	0.0059	0.0040	0.0026	0.0017	0.00009
$\boldsymbol{\varsigma}_2$	1.9442	1.9359	1.9372	1.9278	1.9187	1.9955
MSE	0.0115	0.0092	0.0083	0.0075	0.0058	0.0029
$\boldsymbol{\varsigma}_3$	1.3239	1.3251	1.3196	1.3113	1.3079	1.3016
MSE	0.0088	0.0068	0.0042	0.0039	0.0022	0.0017

2.5. Test statistic $\mathcal{T}_{r,\varepsilon}^2$ under censored simulations

For testing H_0 that a certain right censored data came from NOLLIW model, we compute $\mathcal{T}_{r,\varepsilon}^2$ for $N = 10{,}000$ simulated samples with $\boldsymbol{q} = 25, 50, 130, 350, 500$. Then, we calculate empirical levels of significance when $\mathcal{T}_{r,\varepsilon}^2 > \chi_{r-np,\varepsilon}^2$ corresponding to theoretical levels of significance ($\varepsilon = 1\%, 5\%, 10\%$), where np is the number of parameters. The results which are given in Table 2 is obtained by inverting the CDF of the new model by sitting $U = F_{\underline{V}}(x)$, where U follows the uniform (0,1) model to obtain the quantile. Since the quantile function has no closed form, we use the numerical methods to generate the random numbers. The generated samples are considered as a right censored data. Then using equation (5), the value of the new test ($\mathcal{T}_{r,\varepsilon}^2$) is calculated for $\boldsymbol{q} = 25, 50, 130, 350, 500$ under $\varepsilon = 1\%, 5\%, 10\%$. Finally the values of the $\mathcal{T}_{r,\varepsilon}^2$ test is compared with the corresponding significance level. The following results can be highlighted:

I. $\mathcal{T}_{r,\varepsilon}^2$ =0.0071, 0.0076, 0.0084, 0.0092, 0.0098< $\varepsilon_1 = 1\%$.

II. $\mathcal{T}_{r,\varepsilon}^2$ =0.0442, 0.0466, 0.0474, 0.0483, 0.0499< $\varepsilon_2 = 5\%$.

III. $\mathcal{T}_{r,\varepsilon}^2$ =0.0935, 0.0952, 0.0961, 0.0970, 0.0992< $\varepsilon_2 = 5\%$.

The results are reported in Table 2 which means that the test proposed can be employed for fitting data from NOLLIW version. Based on Table 2, the new model performs well under the new test.

Table 2: Simulated levels of significance for $\mathcal{T}_{r,\varepsilon}^2$ test for NOLLIW model.

N=10,000	$\boldsymbol{q}_1$=25	$\boldsymbol{q}_2$=50	$\boldsymbol{q}_3$=130	$\boldsymbol{q}_4$=350	$\boldsymbol{q}_5$=500
$\varepsilon_1 = 1\%$	0.0071	0.0076	0.0084	0.0092	0.0098
$\varepsilon_2 = 5\%$	0.0442	0.0466	0.0474	0.0483	0.0499
$\varepsilon_3 = 10\%$	0.0935	0.0952	0.0961	0.0970	0.0992

2.6. Censored real data analysis

In this section, we present three applications to distribute NOLLIW into three real data sets. First, we consider the bone marrow transplant data (data set **I**) (Klein and Moeschberger (2003)) for patients suffering from acute lymphoblastic leukemia. These data consists of time (in days) to death or on study time after allogenic bone marrow transplant for 38 patients. The bone marrow transplant is a standard treatment for acute leukemia. Recovery following bone marrow transplantation is a complex process. Immediately following transplantation, patients have depressed platelet counts and have higher hazard rate for the development of infections but as the time passes the

hazard decreases, where starred values denote to censored observations. Below is the time to death (in days) data after bone marrow transplant:
(1,86,107,110,122,156,162,172,194, 243,262, 262, 269,276, 350*, 371, 417, 418, 466,487,526,530*,716,781,996∗,1111*,1167*,1182*,1199∗,1 79, 1330*, 1377*, 1433*, 1462*,1496*, 1602*,2081*,226∗).

The second data set (data set **II**) consists of sample data from 50 patients with acute myeloid leukemia, reported to the International Register of Bone Marrow Transplants of John et al. (1997). These patients had an allogeneic bone marrow transplant where the HLA (Histocompatibility Leukocyte Antigen) homolog marrow was used to rebuild their immune systems. The data required for this study are shown below:
(0.030, 0.493, 0.855, 1.184, 1.283, 1.480, 1.776, 2.138, 2.5, 2.763, 2.993, 3.224, 3.421, 4.178, 4 .441*, 5.691, 5.855* ,6.941*, 6.941, 7.993*, 8.882, 8.882, 9.145*, 11.480, 11.513, 12.105*, 12.796, 12.993*, 13.849*, 16.612*, 17.138*, 20.066, 20.329*, 22.368*,26.776*, 28.717*,28.717* ,32.928*,33.783*,34.211*,34.770*, 39.539*, 41.118*, 45.033*, 46.053*, 46.941*, 48.289*, 57.401*, 58.322*, 60.625*).

The third data (data set **III**) are called the lymphoma data set and consisting of times (in months) from diagnosis stage up to death for 31 individuals with the advanced non-Hodgkin's lymphoma clinical symptoms. Among these 31 observations 11 of the times are censored, because those patients were still alive at the last time of follow-up:
(2.5, 4.1, 4.6, 6.4, 6.7, 7.4, 7.6, 7.7, 7.8, 8.8, 13.3, 13.4, 18.3, 19.7, 21.9, 24.7, 27.5, 29.7, 30.1*, 32.9, 33.5, 35.4*, 37.7*, 40.9*, 42.6*, 45.4*, 48.5*, 48.9*, 60.4*, 64.4*, 66.4*). The test statistic is used to verify if the lymphoma data can be modelled by NOLLIW distribution.

Finally, we consider the censored reliability data of (Crowder et al. (1991)). In their experiment, Crowder et al. (1991) obtained information about the strengths of a certain type of braided cord after the weather, the forces of 48 pieces of cord having resisted for a determined time were studied. The right censored force values observed are given below:
(26.8*, 29.6*, 33.4*, 35*, 36.3, 40*, 41.7, 41.9*, 42.5*, 43.9, 49.9, 50.1, 50.8,51.9, 52.1, 52.3, 52.3, 52.4, 52.6, 52.7, 53.1, 53.6, 53.6, 53.9, 53.9, 54.1, 54.6, 54.8, 54.8,55.1, 55.4, 55.9, 56, 56.1, 56.5, 56.9, 57.1, 57.1, 57.3, 57.7, 57.8, 58.1, 58.9, 59, 59.1, 59.6, 60.4, 60.7).

All results of this application are summarized in Table 3 and Table 4. Table 3 gives the results of $\hat{\rho}_j$, $\boldsymbol{U}_j$, $\widehat{\boldsymbol{C}}_{1j}, \widehat{\boldsymbol{C}}_{2j}, \widehat{\boldsymbol{C}}_{3j}$ and e_j for the four real data sets where $\hat{\rho}_j$, $\boldsymbol{U}_j$, $\widehat{\boldsymbol{C}}_{1j}, \widehat{\boldsymbol{C}}_{2j}, \widehat{\boldsymbol{C}}_{3j}$ and e_j are the main components of the modified test statistic. However, Table 4 gives the values of $\boldsymbol{T}_{r,\varepsilon}^2$ versus $\chi_{r,\varepsilon}^2$.

The values of $\boldsymbol{T}_{r,\varepsilon}^2$ in Table 4 are calculated based on the corresponding values obtained in Table 3. Since $\chi_{\varepsilon=\%5}^2 = 11.0705$, the four values of $\boldsymbol{T}_{r,\varepsilon}^2$ are 10.956, 7.6235,

6.8580 and 6.8956. These results show that the NOLLIW distribution can be used in modelling the four mentioned real data sets according to the modified Bagdonavičius -Nikulin goodness-of-fit test statistic for right censored validation.

Table 3: Values of $\hat{\rho}_j, \boldsymbol{u}_j, \widehat{\boldsymbol{C}}_{1j}, \widehat{\boldsymbol{C}}_{2j}, \widehat{\boldsymbol{C}}_{3j}$ and e_j.

Data set	$\hat{\rho}_j$	$\boldsymbol{u}_j$	$\widehat{\boldsymbol{C}}_{1j}$	$\widehat{\boldsymbol{C}}_{2j}$	$\widehat{\boldsymbol{C}}_{3j}$	e_j
I & r=5	185.5	8	1.0236	0.5632	1.8289	7.6
	320.5	7	0.9532	0.2351	1.7421	7.6
	510.5	6	0.8124	-1.523	1.0231	7.6
	1220.5	9	1.1526	0.6310	2.0513	7.6
	2081	8	1.0856	0.5231	1.9045	7.6
II & r=5	1.923	7	1.3526	0.9238	2.1235	10
	8.562	13	2.0956	1.6485	2.9425	10
	16.432	9	1.4526	0.8231	2.4513	10
	34.526	11	1.748	1.5237	2.7412	10
	60.625	10	1.6245	1.4032	2.6143	10
III & r=4	7.500	6	0.9352	0.5631	0.5417	7.75
	15.65	6	0.4587	-0.4581	0.4689	7.75
	31.45	7	1.3026	-0.4956	0.7864	7.75
	66.40	12	0.3145	0.2031	0.2153	7.75
IV & r=5	43.5	9	0.9326	0.5326	1.4362	9.6
	52.9	11	1.1306	0.7541	1.5962	9.6
	55.2	10	1.0053	0.6138	1.4012	9.6
	57.2	8	0.7654	0.4319	1.2312	9.6
	60.7	10	1.0103	0.5322	1.3496	9.6

Table 4: The values of $\boldsymbol{\mathcal{T}}_{r,\varepsilon}^2$ and $\chi_{r-np,\varepsilon}^2$ for each data.

Data set	r	$\widehat{\mathbf{V}}$			$\boldsymbol{\mathcal{T}}_{r,\varepsilon}^2$ & $\chi_{r-np,\varepsilon}^2$	Rank
		$\hat{\varsigma}_1$	$\hat{\varsigma}_2$	$\hat{\varsigma}_3$		
I	5	1.8235	1.2856	1.5982	$10.956<\chi_{2,0.05}^2=11.0705$	3
II	5	1.5032	1.0203	1.1052	$11.936<\chi_{2,0.05}^2=11.0705$	4
III	4	1.6385	1.6230	1.2865	$7.2365<\chi_{1,0.05}^2= 9.4877$	1
IV	5	1.3746	0.8263	1.4256	$9.8569<\chi_{2,0.05}^2=11.0705$	2

Based on Table 3 and Table 4, we conclude that:

1. For the right censored data of bone marrow transplant which contains from 38 patients, it is seen that $\boldsymbol{\mathcal{T}}_{5,0.05}^2 = 10.956$ however $\chi_{2,0.05}^2$=11.0705. Since $\boldsymbol{\mathcal{T}}_{5,0.05}^2$ is less that $\chi_{2,0.05}^2$, we can say by accepting the null hypothesis that the bone marrow transplant data follow the odd log-logistic inverted Weibull distribution as well and that odd new log-logistic inverted Weibull distribution can be used and applied in modelling the bone marrow transplant data.
2. Data of allogeneic bone marrow transplant (50 patients), it is seen that $\boldsymbol{\mathcal{T}}_{5,0.05}^2 = 11.936$ however $\chi_{2,0.05}^2$=11.0705. Since $\boldsymbol{\mathcal{T}}_{5,0.05}^2$ is less that $\chi_{2,0.05}^2$, we can say by accepting the null hypothesis that the acute myeloid leukemia data follow the odd log-logistic inverted Weibull distribution as well and that new odd log-logistic inverted Weibull distribution can be used and applied in modelling the acute myeloid leukemia data.
3. Lymphoma data (times up to death) (31 patients), it is noted that $\boldsymbol{\mathcal{T}}_{5,0.05}^2 = 7.2365$ however $\chi_{2,0.05}^2$=9.4877. Which means that $\boldsymbol{\mathcal{T}}_{5,0.05}^2$ is less that $\chi_{2,0.05}^2$. Therefore, we can say by accepting the null hypothesis that the leukemia data follow the odd log-logistic inverted Weibull distribution as well and that odd log-logistic inverted Weibull distribution can be used and applied in modelling the leukemia data.
4. Strength data (38 pieces), it is concluded that $\boldsymbol{\mathcal{T}}_{5,0.05}^2 = 9.8569$ however $\chi_{2,0.05}^2$=11.0705). Hence, we can say by accepting the null hypothesis that the strength data follow the odd log-logistic inverted Weibull distribution as well and that odd log-logistic inverted Weibull distribution can be used and applied in modelling the strength data.

Table 5: Comparing the NOLLIW and the two-parameters Weibull under the $\mathcal{T}^2_{r,\varepsilon}$ test.

Data set	r	Models and testing results	Decision
		NOLLIW model	
I	5	$10.956<\chi^2_{2,0.05}=11.0705$	Accept H_0
II	5	$11.936<\chi^2_{2,0.05}=11.0705$	Accept H_0
III	4	$7.2365<\chi^2_{1,0.05}= 9.4877$	Accept H_0
IV	5	$9.8569<\chi^2_{2,0.05}=11.0705$	Accept H_0
		The two-parameters Weibull model	
I	5	$12.5362>\chi^2_{2,0.05}=11.0705$	Reject H_0
II	5	$13.4521 < \chi^2_{2,0.05}=11.0705$	Reject H_0
III	5	$10.5236<\chi^2_{2,0.05}=11.0705$	Accept H_0
IV	5	$9.5236<\chi^2_{2,0.05}=11.0705$	Accept H_0

Based on Table 5, we conclude that:

I. The bone marrow transplant data (data set **I**) follow the odd log-logistic inverted Weibull distribution. However, these data does not follow the two-parameters Weibull model since $12.5362>\chi^2_{2,0.05}=11.0705$.

II. The acute myeloid leukemia data (data set **II**) also follow the odd log-logistic inverted Weibull distribution. However, these data does not follow the two-parameters Weibull model since $13.4521 > \chi^2_{2,0.05}=11.0705$.

III. The leukemia data (data set **III**) can be modelled using the odd log-logistic inverted Weibull distribution and the two-parameters Weibull model. However, the odd log-logistic inverted Weibull distribution is better than the two-parameters Weibull model since $7.2365<\chi^2_{1,0.05}= 9.4877$ for the odd log-logistic inverted Weibull distribution and $10.5236<\chi^2_{2,0.05}=11.0705$ for two-parameters Weibull model.

IV. The strengths data (data set **IV**) can be modelled using the odd log-logistic inverted Weibull distribution and the two-parameters Weibull model. However, the odd log-logistic inverted Weibull distribution is better than the two-parameters Weibull model since $9.8569<\chi^2_{2,0.05}=11.0705$ for the odd log-logistic inverted Weibull distribution and $9.5236<\chi^2_{2,0.05}=11.0705$ for two-parameters Weibull model.

3. Concluding remarks

In this paper, a modified Bagdonavičius -Nikulin goodness-of-fit test statistic is presented and applied for distributional validation under the right censor case. The modified test statistic ($\boldsymbol{\mathcal{T}}^2_{r,\varepsilon}$) is given along with all its relevant components. Four right censored data sets are analyzed under the new modified test statistic for checking the distributional validation. According to the modified Bagdonavičius -Nikulin goodness-of-fit test statistic, the new odd log-logistic inverted Weibull model can be used in modelling the censored medicine and reliability real data sets. Below are the results of the modified test statistic under the right censor data sets:

I. Data of bone marrow transplant (38 patients):
$\boldsymbol{\mathcal{T}}^2_{5,0.05} = 10.956$ ($<\chi^2_{2,0.05}$=11.0705). By accepting the null hypothesis, we can conclude that the bone marrow transplant data also follow the odd log-logistic inverted Weibull distribution and that the bone marrow transplant data can be modelled using the odd new log-logistic inverted Weibull distribution.

II. Data of allogeneic bone marrow transplant (50 patients):
$\boldsymbol{\mathcal{T}}^2_{5,0.05} = 11.936$ ($<\chi^2_{2,0.05}$=11.0705). By accepting the null hypothesis, we can deduce that the acute myeloid leukemia data also follow the odd log-logistic inverted Weibull distribution, and that the acute myeloid leukemia data can be modelled through using new odd log-logistic inverse Weibull distribution.

III. Lymphoma data (times up to death) (31 patients):
$\boldsymbol{\mathcal{T}}^2_{5,0.05} = 7.2365$ ($<\chi^2_{1,0.05}$=9.4877). By accepting the null hypothesis, we can claim that the leukemia data also follow the odd log-logistic inverted Weibull distribution, and that the leukemia data can be modelled using the odd log-logistic inverted Weibull distribution.

IV. Strength's data (38 pieces):
$\boldsymbol{\mathcal{T}}^2_{5,0.05} = 9.8569$ ($<\chi^2_{2,0.05}$=11.0705). Therefore, if the null hypothesis is accepted, we can infer that the strength data also follow the odd log-logistic inverted Weibull distribution and that the odd log-logistic inverted Weibull distribution may be used to describe the right censored strength data.

V. The bone marrow transplant data follows the odd log-logistic inverted Weibull distribution. However, these data does not follow the two-parameters Weibull model since 12.5362>$\chi^2_{2,0.05}$=11.0705.

VI. The acute myeloid leukemia data follows the odd log-logistic inverted Weibull distribution. However, these data does not follow the two-parameters Weibull model since $13.4521 < \chi^2_{2,0.05}$=11.0705.

VII. The leukemia data can be modelled using the odd log-logistic inverted Weibull distribution and the two-parameters Weibull model. However, the odd log-logistic inverted Weibull distribution is better than the two-parameters Weibull model since $7.2365<\chi^2_{1,0.05}= 9.4877$ for the odd log-logistic inverted Weibull distribution and $10.5236<\chi^2_{2,0.05}=11.0705$ for two-parameters Weibull model.

VIII. The strengths data can be modelled using the odd log-logistic inverted Weibull distribution and the two-parameters Weibull model. However, the odd log-logistic inverted Weibull distribution is better than the two-parameters Weibull model since $9.8569<\chi^2_{2,0.05}=11.0705$ for the odd log-logistic inverted Weibull distribution and $9.5236<\chi^2_{2,0.05}=11.0705$ for two-parameters Weibull model.

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