DOI: 10.5604/01.3001.0014.1195

PRZEGLĄD STATYSTYCZNY
R. LXIII – ZESZYT 3 – 2016

ZBIGNIEW ŚWITALSKI[1]

STABILITY AND GENERALIZED COMPETITIVE EQUILIBRIA IN A MANY-TO-MANY GALE-SHAPLEY MARKET MODEL[2]

1. INTRODUCTION

In their well-known paper, Gale, Shapley (1962) modelled the process of assigning applicants to colleges, where the problem was to match applicants with colleges in some “optimal” way. Such kind of matching process can be treated as a market process, in which applicants are interpreted as “buyers”, colleges – as “sellers”, and the traded “goods” as seats in particular colleges.

During the last 50 years modelling the so-called markets with two-sided preferences using the idea of Gale and Shapley became very popular. Different kinds of such markets (for example labor markets or auction markets) are described, e.g., by Roth, Sotomayor (1992).

In the simplest version of a market with two-sided preferences we have two disjoint finite sets of buyers and sellers. The buyers have preferences over the sellers, and the sellers have preferences over the buyers (both represented by linear orders). We assume also that each seller owns a certain number of identical objects which he wants to sell, and each buyer wants to buy at most one object (this resembles the “college admissions” market – traditionally called many-to-one market).

In the last 10 years very general market models based on Gale-Shapley theory have been built, for example contract theory of Hatfield, Milgrom (2005). Models with contracts are very intensively used in the modern theory of markets with indivisible goods. Preferences in these models are often represented by the so-called choice functions (see, e.g., Hatfield et al., 2013).

The main theoretical tool used in the Gale-Shapley theory (and hence in the theory of markets with two-sided preferences and in the contract theory) is the notion of stable matching. A matching u assigning buyers to sellers is stable if there is no pair (b, s) such that buyer b and seller s would have simultaneously any incentive to change the matching u.

[1] University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 4a Prof. Z. Szafrana St., 65-516 Zielona Góra, Poland, e-mail: z.switalski@wmie.uz.zgora.pl.

[2] The research was financially supported by the Polish National Science Centre (NCN) – grant DEC-2011/01/B/HS4/00812.

It can be proved (see Roth, Sotomayor, 1992) that stable matchings in the GS model form the set of core allocations for the respective cooperative game, i.e. allocations such that no coalition of agents can improve the situation of all members of the coalition. Hence, a stable matching is sometimes interpreted as a kind of cooperative equilibrium for the respective market model (see, e.g., Sotomayor, 2007).

But considering any market model, we can also ask about the notion of competitive market equilibrium (in the sense of Walras). The notion of competitive equilibrium is one of the fundamental notions in economic theory, so for a market model with two-sided preferences it is quite natural to ask two questions:

1. Is there any reasonable way to define competitive equilibrium for such a model?
2. If a competitive equilibrium is defined, what are the relationships between the notion of such equilibrium and the notion of stable matching (cooperative equilibrium) for such a model?

The solution of the problem of relationships between competitive equilibria and stable matchings for such kind of markets can help to solve the problems of existence of such equilibria and to find methods of looking for them. For example, in the simplest version of GS theory the problem of existence of stable matchings and the problem how to find them is fully solved (see Gale, Shapley, 1962). Hence, proving the equivalence between stability and competitive equilibria for the simplest version of GS model (see, e.g., theorem 1 below) means automatically proving the existence of such equilibria and gives the method of finding them.

For traditional, continuous models of market equilibrium it can be often proved that the competitive equilibrium is in the core (see, e.g., Moore, 2007). There are also many discrete matching models, different from the GS model, for which exact relationships between competitive and cooperative equilibria are established. These are models related mainly to the so-called "assignment games" in the sense of Shapley, Shubik (1971/72).

The main difference between the GS models and the SS (= Shapley-Shubik) models is that in the GS models buyers' preferences do not depend on prices (they are exogenously given, as in the neoclassical consumer theory), contrary to the SS models, in which buyers' preferences are determined by quasi-linear utilities depending on prices.

In the simplest case of SS model, it can be proved that the core allocations (which are optimal assignments in this case) are exactly competitive equilibria allocations (see, e.g., Shapley, Shubik, 1971/72; Shoham, Leyton-Brown, 2009, theorem 2.3.5, p. 31). There are many other similar results for different variants of the SS model (see, e.g., Camina, 2006; Sotomayor, 2007).

In the contract theory of Hatfield et al. (2013) the result about strict relationship between stable matchings and competitive equilibria can also be proved (the utility function used by Hatfield et al. (2013) is quasi-linear similarly as in the SS models).

As to the GS models, there was (to the best of our knowledge) no research on this topic until the papers of Świtalski (2008, 2010) and Azevedo, Leshno (2011) appeared. Perhaps the reason for that was the skeptical view of Shapley and Scarf (1974, p. 35) on the problem of introducing the concept of market equilibrium for the GS model:

"*It does not appear to be possible to set up a conventional market for this model* [= GS *model*] *in such a way that a competitive price equilibrium will exist and lead to an allocation in the core*".

In the papers of Świtalski (2008, 2010) different kinds of generalized equilibria (the so-called stable equilibria, order equilibria and boundary equilibria) for the GS models were defined. In Azevedo, Leshno (2011, p. 18) the so-called "supply and demand lemma" was proved. In this lemma it was shown that stable matchings in the college admissions problem can be characterized with the help of families of "cutoffs" (cutoff for a given college is the score of marginal accepted student for this college). If we interpret cutoffs as some kind of prices in the respective market model, then their result can be reformulated in the following way: "a matching u is stable if and only if it is a competitive price equilibrium allocation" (see theorem 1 below).

In the paper of Świtalski (2015) a generalization of Azevedo and Leshno's result was proved (the model described there is one-to-one with preferences of the agents represented by weak orders).

In the presented paper we prove some far-reaching generalization of the result of Azevedo and Leshno. We consider a certain variant of a many-to-many market model, based on GS model, with choice functions representing preferences of buyers and weak orders representing preferences of sellers (and with quotas for all agents). Equilibria in our model (we call them order equilibria, see Świtalski, 2010) are defined by general conditions represented by families of subsets $\{W(s)\}$ (indexed by sellers s) which can be treated as a generalization of "cutoff" or price conditions in the Azevedo, Leshno's (2011) model. For such models we study relationships between equilibria and stability under different assumptions about choice functions. Using the results of Alkan, Gale (2003) we also prove the result on existence of order equilibria for our model (and show that in some cases the so-called strongly order equilibria may not exist).

The simplest one-to-one version of our model (with equilibria defined as usual price equilibria) can be identified with the model of matching markets with budget constraints described by Chen et al. (2014) – with the assumption that the utility functions of the buyers are constant (do not depend on prices). Yet, the stability concept used by Chen et al. differs from the standard one and hence their result (2014, theorem 3.1) cannot be treated as a special case of our results.

Our results cannot also be treated as a special case of the results for trading neworks obtained by Hatfield et al. (2013). Preferences of buyers in their theory depend on prices (they use quasi-linear utility functions) and in our paper we treat buyers' preferences as exogenously given as in the standard GS model.

The paper is organized as follows. In section 2 we describe the simplest one-to-one version of our model and translate Azevedo and Leshno's result in terms of equilibrium theory (theorem 1). In section 3 we construct general model and state the main results of the paper (lemmas 1 and 2, theorem 2). In section 4 we consider the problem of existence of equilibria for our model (theorem 3 and example 1).

2. THE SIMPLEST ONE-TO-ONE MODEL

The model of Gale, Shapley (1962) is concerned with the problem of "optimization" of the process of admitting applicants to colleges. Applicants have preferences over colleges and colleges have preferences over applicants. Gale and Shapley find a method of assigning applicants to colleges which gives stable matching u, i.e. a matching for which there is no pair (b, s) such that the applicant b prefers college s to $u(b)$ ($u(b)$ is the college to which b is admitted) and s prefers b to an applicant admitted to s.

A stable matching may be treated as some kind of equilibrium state in the college admissions market. Yet, it is defined in a completely different way than the classical notion of competitive, Walrasian equilibrium.

We can treat applicants as "buyers" and colleges as "sellers" in the market, but to define the competitive equilibrium we need prices on the sellers' side of the market and budget constraints on the buyers' side.

We can observe that the role of prices can be played by scores with the help of which applicants are classified in many admission systems (see, e.g., Biro, Kiselgof, 2013). In such a system each applicant has a number of scores achieved in different disciplines (maths, physics, biology and so on) and each college ranks students according to the sum of scores achieved in the disciplines which are taken into account by this college.

Let $p(s)$ be the sum of scores of the worst (marginal) applicant admitted to college s (under some assignment u), and let $r(b, s)$ be the sum of scores applicant b achieves in the disciplines required by the college s. Then the inequality $r(b, s) \geq p(s)$ is a necessary and sufficient condition guaranteeing b to be admitted to college s under u ($p(s)$ can be treated as a score-limit in the sense of Biro, Kiselgof (2013) or as a "cutoff" in the sense of Azevedo, Leshno (2011). Hence we can think of $p(s)$ as a kind of price of a seat in college s, and of $r(b, s) \geq p(s)$ as a kind of budget constraint for applicant b when he is interested in being admitted to s (applicant b can be admitted to any college s for which $r(b, s) \geq p(s)$ is satisfied).

Observe that "budget constraints" in the college admissions market depend on both the "buyers"-side and the "sellers"-side of the market.

Given prices and budget constraints, we can now easily define competitive equilibrium in the Walras sense.

We start from the simplest one-to-one model (resembling the marriage model of Gale, Shapley, 1962). Using this model**,** we explain the main idea of the paper. Let B be a finite n-element set of buyers and S – a finite n-element set of sellers. Each seller owns exactly one indivisible object which he wants to sell (objects can be houses, cars, horses, paintings and so on). Each buyer wants to buy exactly one object.

We identify sellers with objects which they own, hence the phrase "object s" should be understood as a shortened version of "object owned by seller s". We assume that buyers have preferences over sellers (equivalently – over the objects) and sellers have preferences over buyers. Preferences are represented by strict linear orders, i.e. to each agent (buyer b or seller s) an ordered list of agents from the opposite set, indicating preferences of b or s, is assigned (there are no indifferences). We use notation $b >_s c$ meaning that buyer b is better than buyer c for seller s, and $s >_b t$ meaning that seller s is better than seller t for buyer b.

For each buyer b and each seller s we define a reservation price $r(b, s)$ interpreted as maximal price that b is willing to pay for object s (in the college market the role of reservation price is played by the sum of scores applicant b achieves in the disciplines required by college s).

We assume that preferences of the sellers are determined by the reservation prices of the buyers, i.e. for any buyers b and c and seller s we have

$$b >_s c \iff r(b, s) > r(c, s) \tag{1}$$

(to avoid indifferences, we assume here that, for a given s, the numbers $r(b, s)$ are different).

Formula (1) is obvious for the college market (in the score system college s ranks applicants according to the sums of scores). For general markets formula (1) says that seller s prefers a buyer who can pay more over a buyer who can pay less for the object owned by s (hence s can sell this object to b at a higher price than to c).

A *matching* of buyers with sellers (or an allocation of objects among buyers) is a set of pairs (b, s) such that each agent (b or s) occurs in exactly one pair. If u is a matching and $(b, s) \in u$, then we write also $u(b) = s$ or $u(s) = b$.

A matching u is *stable* if there is no pair (b, s) satisfying the condition:

$$s >_b u(b) \quad \text{and} \quad b >_s u(s). \tag{2}$$

A pair satisfying (2) is called a *blocking pair* for u.

Assume now that each seller announces a price $p(s)$ for the object he owns. A sequence of prices $p(s)$ ($s \in S$) is called the *price vector p*. Prices p are called equilibrium prices if there is a matching u such that each buyer b gets object s (i.e. $u(b) = s$) which is the best object for him among all objects satisfying the inequality $r(b, s) \geq p(s)$ (i.e. all feasible objects for b). Such a matching is called equilibrium allocation associated with p. The following result is a kind of reformulation of Azevedo and Leshno's result (2011, see also Świtalski, 2015).

□ **Theorem 1.** A matching u is stable if and only if it is an equilibrium allocation associated with some price vector p.

In the next section we study a market model which generalizes the presented above one-to-one model. Namely, we consider a many-to-many model in which preferences of buyers are represented by choice functions and preferences of sellers are weak orders. The notion of equilibrium which we use is very general. The budget constraints $r(b, s) \geq p(s)$ are replaced by certain conditions defined by families $\{W(s)\}$ of subsets of the set B (the set of buyers). For such defined models we study relationships between stable matchings and equilibria allocations (lemmas 1 and 2). Theorem 2 formulated at the end of section 3 includes theorem 1 as a special case.

3. THE GENERAL MODEL

Gale, Shapley (1962) defined a college admission model which is traditionally called many-to-one model (there can be many applicants that a fixed college wants to admit, but each applicant wants to be admitted to only one college). Many authors (e.g., Echenique, Oviedo, 2006; Klaus, Walzl, 2009; Kominers, 2012), starting from Gale-Shapley model, described many-to-many market models, especially for different kinds of labor markets. For example, Echenique, Oviedo (2006) consider a market consisting of firms and consultants, where each firm wants to hire a set of consultants and each consultant wants to work for a set of firms. Other examples (mentioned by Echenique, Oviedo, 2006) are the markets for medical interns in the U.K. or teacher (university professor) markets in some countries (where teachers (professors) can work in more than one school (university)).

In our paper we also consider a many-to-many model. To start with the formal description of this model, we define two finite and non-empty sets: a set of buyers (e.g. firms) B and a set of sellers (e.g. consultants) S.

The symbol $B \times S$ denotes Cartesian product of B and S, i.e. the set of ordered pairs (b, s) such that $b \in B$ and $s \in S$.

For any relation $u \subset B \times S$ and for any $b \in B$, $s \in S$, we define the sets of "neighbouring" elements:

$$u(b) = \{s \in S\colon (b, s) \in u\}, \tag{3}$$

$$u(s) = \{b \in B\colon (b, s) \in u\}. \tag{4}$$

We assume that a non-empty set of acceptable pairs $F \subset B \times S$ is defined. A pair (b, s) belongs to F if buyer b is acceptable for s and seller s is acceptable for b. Hence, according to (3) and (4), the sets $F(b)$ and $F(s)$ can be defined. The set $F(b)$ can be interpreted as the set of acceptable sellers for buyer b, and $F(s)$ – as the set of acceptable buyers for seller s.

From the point of view of contract theory (Hatfield et al., 2013) acceptable pairs can be interpreted as possible transactions (trades) which can be realized in the market. In other words, $(b, s) \in F$ means that buyer b can sign a contract with seller s. A contract is signed if b and s agree to the conditions of the contract (e.g. price). We assume that b can sign many contracts with different sellers, but only one contract with a given seller s (and s can sign many contracts with other buyers, but only one contract with a given buyer b).

In our model we introduce quotas for buyers and sellers. Let $q(b) \geq 1$ be the quota for b, which is a maximum number of contracts which b can sign with different sellers and $q(s) \geq 1$ – the quota for s, which is a maximum number of contracts which s can sign with different buyers. We assume that $\# F(b) \geq q(b)$ and $\# F(s) \geq q(s)$ ($\# A$ denotes the cardinality of a set A).

Preferences of the sellers are represented by weak orders. Namely, we assume that in every set $F(s)$, a weak order (transitive and complete relation) $\geq_s$ is defined (i.e. the seller s may be indifferent between some two buyers). The symbols $>_s$ and $\approx_s$ will denote the respective strict order and indifference relation. Hence the notation $b >_s c$ means that buyer b is better than buyer c for seller s, and $b \approx_s c$ means that s is indifferent between b and c.

Preferences of buyers are represented by choice functions. Choice functions are a standard tool in economic and decision theory (see, e.g., Aizerman, Aleskerov, 1995; Aleskerov, Monjardet, 2002) and they are very often used for the models of markets with two-sided preferences, especially for the labor markets (see, e.g., Echenique, 2007; Klaus, Walzl, 2009; Hatfield et al., 2013). Defining a choice function means that we know what choice will be made by a decision maker when she is confronted with a given set of decision alternatives. Formally, a choice function is a mapping C from the family T of all subsets of a given set (set of all possible alternatives) to the same family, assigning a set $C(X) \subset X$ to every $X \in T$. The set $C(X)$ is interpreted as the set of elements chosen from X by a decision maker.

Usually, in the papers on many-to-many markets or contract theory (see, e.g., Echenique, Oviedo, 2006; Klaus, Walzl, 2009; Kominers, 2012), choice functions are generated by preferences over the subsets from the family T. In our paper we do not assume a priori that there is some order relation in T and that $C(X)$ is some "best" set (with respect to this order) in the family of all subsets of X.

In our model we assume that a choice function is defined for every feasible set $F(b)$ (for a given buyer b). Hence, for every buyer b and every set of feasible sellers $X \subset F(b)$, a set $C(b, X) \subset X$ is defined. The set $C(b, X)$ is interpreted in the following way. Assume that b considers a certain set of feasible sellers X. Then her decision will be to choose the set $C(b, X)$ as the set of sellers, with whom she will sign a contract. Of course the number of such sellers should not exceed $q(b)$, hence we assume that:

$$\text{(i)} \quad C(b, X) = X, \quad \text{if} \quad \# X < q(b),$$

$$\text{(ii)} \quad \# C(b, X) = q(b), \quad \text{if} \quad \# X \geq q(b).$$

(we consider here the so-called quota-filling choice functions in the sense of Alkan, Gale, 2003).

In what follows we consider the following properties of the function C:

The outcast property (see, e.g., Aizerman, Aleskerov, 1995, p. 20; Aleskerov, Monjardet, 2002, p. 39; Echenique (2007) defines an equivalent property called independence of irrelevant alternatives):

For every $b \in B$ and $X, Y \subset F(b)$ we have

$$Y \subset X \setminus C(b, X) \;\Rightarrow\; C(b, X \setminus Y) = C(b, X). \tag{5}$$

The outcast property means that if we delete a set, consisting of not chosen elements, from the set X, then the resulting choice will remain the same.

We note that the outcast property implies the following properties:

$$C(b, C(b, X)) = C(b, X), \tag{6}$$

$$C(b, C(b, X) \cup \{s\}) = C(b, X) \tag{7}$$

for any $s \in X$ (to prove (6) we take $Y = X \setminus C(b, X)$ in (5), and to prove (7) we take $Y = X \setminus (C(b, X) \cup \{s\})$ in (5)).

So, we can add an element s to the set of chosen elements and this operation does not change the set of chosen elements.

The heritage property (see, e.g., Aizerman, Aleskerov, 1995, p. 18; Aleskerov, Monjardet, 2002, p. 36; in the matching literature this kind of property is sometimes called substitutability, see, e.g., Echenique, 2007):

For every $b \in B$ and $X, Y \subset F(b)$ we have

$$Y \subset X \Rightarrow Y \cap C(b, X) \subset C(b, Y). \tag{8}$$

The heritage property means that any element (seller) chosen from X, which belongs to a smaller set Y, should also be chosen from Y.

Choice functions satisfying both the outcast and heritage properties are called path independent (or Plott) choice functions (see Danilov, Koshevoy, 2005). Example of Plott choice function is the choice determined by a linear order (then $C(b, X)$ is the set of $q(b)$ best sellers in X (if $\# X \geq q(b)$)).

We define a generalized GS-model as a 6-tuple (B, S, F, C, P, q), where F is the set of acceptable pairs, C is the family of choice functions (defined for all $b \in B$), P is the family of weak orders (defined for all $s \in S$), and q is the vector of quotas (defined for all $b \in B$ and all $s \in S$).

For a given generalized GS-model (B, S, F, C, P, q) we define now the notion of matching and (strongly) stable matching (definitions 1–5). We define matching as a set of pairs and it is easy to see that our definition is equivalent to the standard definition of Echenique, Oviedo (2006) (they define matching as some mapping from $B \cup S$ into the set of all subsets of $B \cup S$). Our definition of blocking pair is a combination of a standard definition for a many-to-one model (see Roth, Sotomayor, 1992,

p. 129) with the definition of Echenique, Oviedo (2006, p. 240) for a many-to-many model (for the buyers' side of the market). To unify the definitions we introduce the non-standard notion of "improving the situation of an agent" (definitions 2 and 3). Strongly stable matchings are defined similarly as in Manlove (2002).

Definition 1. A relation $u \subset B \times S$ is a matching if

(i) $u \subset F$,

(ii) $\# u(b) \leq q(b), \quad \forall b \in B$,

(iii) $\# u(s) \leq q(s), \quad \forall s \in S$.

A matching u can be interpreted as a set of actual contracts signed by agents from the sets B and S (transactions realized in the market). Obviously, according to (i), such contracts should be taken from F – the set of all possible (potential) contracts.

Definition 2. Let $u \subset B \times S$ be a matching. We say that a seller $s \in F(b)$ improves the situation of a buyer $b \in F(s)$ (we write $s >_{\mathrm{b}} u(b)$) if $s \in C\,(b, u(b) \cup \{s\})$.

Definition 3. Let $u \subset B \times S$ be a matching. We say that a buyer $b \in F(s)$ improves the situation (weakly improves the situation) of a seller $s \in F(b)$ (we write $b >_{\mathrm{s}} u(s)$ or $b \geq_{\mathrm{s}} u(s)$ respectively) if at least one of the following conditions holds:

(i) $\# u(s) < q(s)$,

(ii) $\exists\, c \in u(s), \quad b >_{\mathrm{s}} c \quad (b \geq_{\mathrm{s}} c)$.

Definition 4. A pair $(b, s) \in B \times S$ is a blocking pair (weakly blocking pair) for a matching $u \subset B \times S$ if

(i) $(b, s) \in F \setminus u$,

(ii) $s >_{\mathrm{b}} u(b)$,

(iii) $b >_s u(s) \quad (b \geq_{\mathrm{s}} u(s))$.

Definition 5. A matching $u \subset B \times S$ is stable (strongly stable) if there are no blocking pairs (weakly blocking pairs) for u.

Now we can define a generalized equilibrium in a generalized GS-model (B, S, F, C, P, q). In the simplest one-to-one model the set of feasible sellers (objects) for a given buyer b was defined with the help of inequality

$$r(b, s) \geq p(s). \tag{9}$$

Inequality (9) is a necessary and sufficient condition for buyer b to obtain object s (or to sign a contract with seller s). There can be markets in which sellers can state some other conditions for buyers needed to sign contracts (for example there can be some law requirements needed to buy some products). In general, we can assume that the conditions required by seller s determine a set of buyers $W(s) \subset F(s)$ such that being in the set $W(s)$ is for buyer b a necessary and sufficient condition to sign a contract with s (for example, the price condition (9) determines the set $W(s) = \{b \in F(s): r(b, s) \geq p(s)\}$). The sets $F(s)$ are fixed, but the sets $W(s)$ can vary and we can define equilibrium in the generalized models with respect to the families $W = \{W(s)\}$ $(s \in S)$. A family $W = \{W(s)\}$ $(s \in S)$ is called a system of conditions. In definitions 6 and 7 (below) we define generalized equilibria without imposing any special assumptions on the sets $W(s)$.

First, we define the set of feasible sellers under the system $W = \{W(s)\}$ for buyer b as:

$$F(W, b) = \{s \in S: b \in W(s)\}.$$

Obviously, $F(W, b) \subset F(b)$ and hence we can define the set of the "best" sellers (contracts) for b under the system W as

$$M(W, b) = C(b, F(W, b)).$$

Buyer b demands objects from the set $M(W, b)$ (wants to sign contracts with the sellers from $M(W, b)$) and hence the demand set for seller s (under the conditions W) can be defined as:

$$D(W, s) = \{b \in F(s): s \in M(W, b)\}.$$

Demand set $D(W, s)$ is the set of buyers for whom s is among the "best" sellers. It is easy to see (from the above definitions) that $D(W, s) \subset W(s)$.

Definition 6. A system of conditions $W = \{W(s)\}$ is an *equilibrium system* if

(i) $\# D(W, s) \leq q(s)$, for all $s \in S$.

(ii) $W(s) = F(s)$, for all $s \in S$ such that $\# D(W, s) < q(s)$.

Inequality (i) guarantees that, under the conditions W, each buyer can sign all the contracts which are best for her without exceeding the supply limits $q(s)$. Condition (ii) states that it is not possible to weaken the conditions $\{W(s)\}$ in order to increase the demand of the buyers in the situation when supply limits are not reached (it is a generalization of the standard condition of zeroing the prices of unassigned goods for one-to-one matching models – such condition guarantees that we cannot decrease prices of these goods to increase the demand for them, see e.g., Mishra, Talman, 2010; Chen at al., 2014; Świtalski, 2015).

Let $W = \{W(s)\}$ be an equilibrium system. We define a matching associated with W as:

$$u(W) = \{(b, s) \in F: b \in D(W, s)\}.$$

Definition 7. An *equilibrium* (in a generalized GS-model (B, S, F, C, P, q)) is a pair (u, W) such that W is an equilibrium system and $u = u(W)$.

If (u, W) is an equilibrium we say that $u = u(W)$ is an equilibrium allocation associated with W.

To state our results about relationships between stability and equilibria we need to restrict the notion of equilibrium to the so-called order equilibrium which is an equilibrium for which the sets $W(s)$ are "compatible" with the order relations $>_s$.

Definition 8. A system of conditions $W = \{W(s)$ is compatible (strongly compatible) with the sellers' preferences if

$$b \in W(s) \wedge c >_s b \;\Rightarrow\; c \in W(s), \quad \text{for all} \quad s \in S,$$
$$(b \in W(s) \wedge c \geq_s b \;\Rightarrow\; c \in W(s), \quad \text{for all} \quad s \in S).$$

Hence, the system W is compatible with the sellers' preferences if all the buyers who are preferred over a certain buyer satisfying $W(s)$, also satisfy $W(s)$ (in other words the conditions which determine the set of "possible" buyers are ordinal). Observe that strong compatibility implies compatibility. It is also easy to see that the price conditions (9) in a one-to-one model with preferences of the sellers defined by (1) are compatible with the sellers' preferences.

Definition 9. An equilibrium (u, W) in a generalized GS-model (B, S, F, C, P, q) is an *order equilibrium* (strongly order equilibrium) if W is compatible (strongly compatible) with the sellers' preferences.

We note here that the notion of order equilibrium needs representation of the sellers' preferences by weak orders. This is the reason why in our model we have an asymmetry in the representation of preferences (choice functions for the preferences of the buyers and weak orders for the preferences of the sellers).

Let (B, S, F, C, P, q) be a generalized GS-model. We say that C satisfies the outcast (heritage) property if all the choice functions in the family C satisfy this property.

In the next two lemmas we show that if C satisfies both the outcast and heritage properties, then a matching $u \subset B \times S$ is stable (strongly stable) iff it is an order (strongly order) equilibrium allocation (i.e. iff there exists a system of conditions W such that (u, W) is an order (strongly order) equilibrium allocation). The final result is formulated as theorem 2 below.

Lemma 1. Let $M = (B, S, F, C, P, q)$ be a generalized GS-model such that C satisfies the outcast property. If (u, W) is an order (strongly order) equilibrium for the model M, then the matching u is stable (strongly stable).

Proof. Assume that u is not stable (not strongly stable), so there is a pair $(b, s) \in F \setminus u$ such that $s >_b u(b)$ and $b >_s u(s)$ ($b \geq_s u(s)$). First we prove that $s \in F(W, b)$. Consider two cases:

1. $\# u(s) = q(s)$. Hence $b >_s c$ ($b \geq_s c$) for some $c \in u(s) = D(W, s) \subset W(s)$, and so $c \in W(s)$.
   Thus, by definition 8, $b \in W(s)$, and so $s \in F(W, b)$.
2. $\# u(s) < q(s)$. By definition 6 we have $W(s) = F(s)$, and hence $b \in W(s)$ (because $(b, s) \in F \setminus u$ implies $b \in F(s)$). Thus $s \in F(W, b)$.

We also have (by $s >_b u(b)$):

$$s \in C\ (b, u(b) \cup \{s\}) \tag{10}$$

and, by the definition of $u(W)$:

$$u(b) = \{s: (b, s) \in u\} = \{s: b \in D(W, s)\} = \{s: b \in F(s) \wedge s \in M(W, b)\}$$
$$= M(W, b) = C(b, F(W, b)).$$

Hence, by (10) and (7) (we take $X = F(W, b)$), we have:

$$s \in C(b, C(b, F(W, b)) \cup \{s\}) = C(b, F(W, b)) = u(b).$$

This fact contradicts the assumption $(b, s) \in F \setminus u$ and implies that u is stable (strongly stable). ■

Lemma 2. Let u be a stable (strongly stable) matching in a generalized GS-model $M = (B, S, F, C, P, q)$ such that C satisfies the heritage property. Then there exists a system of conditions W compatible (strongly compatible) with P such that (u, W) is an order (strongly order) equilibrium.

Proof. For any matching $u \subset B \times S$ we can define a system of conditions:

$$W(s) = \begin{cases} u(s) \cup \{b \in F(s) \colon \exists\, c \in u(s),\, b >_s c \ (b \geq_s c)\}, & \text{if } \#\, u(s) = q(s), \\ F(s), & \text{if } \#\, u(s) < q(s). \end{cases} \tag{11}$$

In the case when $\#\, u(s) = q(s)$, $W(s)$ consists of all the buyers matched with s and all the buyers which are better (not worse) than at least one buyer matched with s. It is easy to see that $W(s)$ are compatible (strongly compatible) with the sellers' preferences.

To prove that (u, W) is an order (strongly order) equilibrium, we should show (by definitions 6 and 7) that $\#\, D(W, s) \leq q(s)$ for all $s \in S$, $\#\, D(W, s) < q(s)$ implies $W(s) = F(s)$ and that $u(W) = u$. Observe that $u(W) = u$ is equivalent to $u(W)(s) = u(s)$ for all $s \in S$ and $u(W)(s) = D(W, s)$. Hence, to show that (u, W) is an order (strongly order) equilibrium, it is sufficient to show that $u(s) = D(W, s)$ for all $s \in S$ (by (11) and by condition (iii) in the definition 1).

1. First we prove that $u(s) \subset D(W, s)$. Let $b \in u(s)$. We want to prove that $b \in D(W, s)$. By the definition of $W(s)$ (11), $b \in W(s)$. Hence, by the definition of $F(W, b)$, $s \in F(W, b)$. Obviously, $b \in F(s)$ ($u \subset F$). Thus, to state that $b \in D(W, s)$, it suffices to show that $s \in M(W, b) = C(b, F(W, b))$.

Assume that $s \notin C(b, F(W, b))$. Hence $\#\, F(W, b) > q(b)$ (if $\#\, F(W, b) \leq q(b)$, then $\#\, C(b, F(W, b)) = \#\, F(W, b)$, and so $s \in F(W, b)$ would imply $s \in C(b, F(W, b))$).

Summing up, we have the following facts:

(i) $u(b), M(W, b) \subset F(W, b)$,
(ii) $\#\, M(W, b) = q(b)$,
(iii) $\#\, F(W, b) > q(b)$,
(iv) $s \in u(b)$, $s \notin M(W, b)$,
(v) $\#\, u(b) \leq q(b)$ (see (ii), definition 1).

The facts (i)−(v) imply the existence of $t \in M(W, b)$ such that $t \notin u(b)$. Obviously, $u(b) \cup \{t\} \subset F(W, b)$ and hence, by heritage property (8), we have:

$$(u(b) \cup \{t\}) \cap C(b, F(W, b)) \subset C(b, u(b) \cup \{t\}).$$

Hence, because $t \in M(W, b) = C(b, F(W, b))$, we have $t \in C(b, u(b) \cup \{t\})$. Thus $t >_b u(b)$.

Now we will prove that $b >_t u(t)$ ($b \geq_t u(t)$). Consider two cases:

1. $\#\, u(t) = q(t)$. We have $t \in M(W, b) \subset F(W, b)$, hence $b \in W(t)$. We also have $b \notin u(t)$ (because $t \notin u(b)$), hence (by the definition of $W(t)$) there exists $c \in u(t)$ such that $b >_t c$ ($b \geq_t c$). Thus $b >_t u(t)$ ($b \geq_t u(t)$).
2. $\#\, u(t) < q(t)$. Then $b >_t u(t)$ ($b \geq_t u(t)$) by definition 3.

Hence we have obtained $t >_b u(b)$ and $b >_t u(t)$ ($b \geq_t u(t)$). It is easy to see that $(b, t) \in F \setminus u$ (because $t \in F(W, b) \subset F(b)$ and $t \notin u(b)$). Thus, by definition 4, (b, t) is a (weakly) blocking pair for u, a contradiction to the stability of u.

2. Now we prove that $D(W, s) \subset u(s)$ for all $s \in S$. Let $b \in D(W, s)$. Then $b \in F(s)$ (hence $s \in F(b)$) and $s \in M(W, b)$. Assume that $b \notin u(s)$ (hence $s \notin u(b)$). Thus $(b, s) \in F \setminus u$. We will prove that $b >_s u(s)$ ($b \geq_s u(s)$) and $s >_b u(b)$, thus showing that (b, s) is a (weakly) blocking pair for u, a contradiction to the stability of u.

(i) To show that $b >_s u(s)$ ($b \geq_s u(s)$), we can use the same reasoning as in the second part of the proof in p. 1 (t should be changed by s).

(ii) To show that $s >_b u(b)$ observe that $u(b) \subset M(W, b)$ (if $t \in u(b)$, then $b \in u(t)$, and, by proof in p. 1, $u(t) \subset D(W, t)$, hence $b \in D(W, t)$, and so $t \in M(W, b)$). By the definition of $M(W, b)$, $\# M(W, b) \leq q(b)$. Hence, by $s \notin u(b)$ and $u(b) \subset M(W, b)$, we have $\# u(b) < q(b)$ and so, by definition 2 and the definition of choice function, $s >_b u(b)$. ■

From lemmas 1 and 2 we obtain the following result.

□ **Theorem 2.** If $M = (B, S, F, C, P, q)$ is a generalized GS-model such that C is a Plott choice function, then $u \subset B \times S$ is stable (strongly stable) if and only if it is an order (strongly order) equilibrium allocation.

4. THE EXISTENCE OF EQUILIBRIA

Let $M = (B, S, F, C, P, q)$ be a generalized GS-model. We can ask about the existence of an order (strongly order) equilibrium for such a model. Using theorem 2 and some existence results from GS theory we can prove the following

□ **Theorem 3.** If $M = (B, S, F, C, P, q)$ is a generalized GS-model such that C is a Plott choice function, then there exists an order equilibrium (u, W) for M.

Proof. For any preference relation $P(s)$ (weak preference order for a seller s in the model M) take a linear extension $L(s)$ (i.e. a linear order $L(s)$ such that $P(s) \subset L(s)$). Linear orders $L(s)$ determine, in an obvious way, Plott choice functions for the sellers (we take, for any $X \subset F(s)$, the set of $q(s)$ best buyers in the set X, or the set X, if $\# X \leq q(s)$). Having Plott choice functions on both sides of the market, we can use Alkan and Gale theory (2003) to deduce that there exists a stable matching u in the model $N = (B, S, F, C, L, q)$ (Alkan, Gale, 2003, theorem 1, p. 298 – the properties of consistency and persistency used by Alkan and Gale are equivalent to the outcast and heritage properties, respectively). It is easy to see that u is also stable for the model M. Hence, by lemma 2, we can find a system of conditions W, compatible with the seller' preferences P, such that (u, W) is an order equilibrium, and this completes the proof. ■

Unfortunately, similar result cannot be proved for strongly order equilibria. Namely, there can be generalized GS models, for which there are no strongly stable matchings, and hence, by theorem 2, no strongly order equilibria. The following example shows such a situation.

Example 1. Let $B = \{b, c\}$, $S = \{s\}$, $F = \{(b, s), (c, s)\}$, $C(b, \{s\}) = \{s\}$, $C(c, \{s\}) = \{s\}$. Let s be indifferent between b and c and let all quotas be equal to 1. The only possible matchings for M (according to definition 1) are $u = \{(b, s)\}$, $v = \{(c, s)\}$ and empty matching $\varnothing$. It is easy to see that (c, s) is a weakly blocking pair for u ($s >_c u(c)$, because $s \in C(c, u(c) \cup \{s\}) = C(c, \varnothing \cup \{s\}) = C(c, \{s\}) = \{s\}$, and $c \geq_s u(s)$, because $c \geq_s b$ and $b \in u(s) = \{b\}$), (b, s) is a weakly blocking pair for v, and both (c, s) and (b, s) are weakly blocking pairs for $\varnothing$. Hence we have no strongly stable matching for the model $M = (B, S, F, C, P, q)$ (although, as it is also easy to see, both u and v are stable for M).

5. CONCLUDING REMARKS

In our paper we have investigated relationships between the concept of stability and the concept of generalized competitive equilibrium (called here order equilibrium) for some variant of a many-to-many Gale-Shapley market model. The results we have obtained can help to prove existence results for equilibria for such kind of models. In the existing literature there are many similar results for market models of the Shapley-Shubik type, but little has been proved till now for models of the GS type (with preferences not depending on prices). Hence our paper fills a gap in this area.

Preferences in our model are represented by choice functions (for the buyers) and weak orders (for the sellers), so there is an asymmetry here. The reason is that to define an order equilibrium (which is a generalization of price equilibrium) it is necessary to have an ordering relation on the sellers' side of the market. An interesting question could be: can we avoid such an asymmetry by introducing a concept of equilibrium which would not depend on special representation of preferences by weak orders? Another question is the possibility of using in our model choice functions without quota restrictions (as in the model of Echenique, Oviedo, 2006). It could be also interesting to study in detail relationships between stability and price equilibria (a special case of order equilibria) for many-to-many GS models. We leave these problems for further research.

REFERENCES

Aizerman M., Aleskerov F., (1995), *Theory of Choice*, North-Holland, Amsterdam.

Aleskerov F., Monjardet B., (2002), *Utility Maximization, Choice and Preference*, Springer, Berlin.

Alkan A., Gale D., (2003), Stable Schedule Matching under Revealed Preference, *Journal of Economic Theory*, 112, 289–306.

Azevedo E. M., Leshno J. D., (2011), A Supply and Demand Framework for Two-Sided Matching Markets, *working paper*, *www.aeaweb.org* (last access: 25.11.2015).

Biro P., Kiselgof S., (2013), College Admissions with Stable Score-Limits, *discussion paper*, Institute of Economics, Research Centre for Economic and Regional Studies, Hungarian Academy of Sciences, Budapest.

Camina E., (2006), A Generalized Assignment Game, *Mathematical Social Sciences*, 52, 152–161.

Chen N., Deng X., Ghosh A., (2014), Competitive Equilibria in Matching Markets with Budgets, *arxiv.org*.

Crawford V. P., Knoer E. M., (1981), Job Matching with Heterogeneous Firms and Workers, *Econometrica*, 49, 437–450.

Danilov V., Koshevoy G., (2005), Mathematics of Plott Choice Functions, *Mathematical Social Sciences*, 49, 245–272.

Echenique F., (2007), Counting Combinatorial Choice Rules, *Games and Economic Behavior*, 58, 231–245.

Echenique F., Oviedo J., (2006), A Theory of Stability in Many-to-Many Matching Markets, *Theoretical Economics*, 1, 233–273.

Gale D., Shapley L. S., (1962), College Admissions and the Stability of Marriage, *American Mathematical Monthly*, 69, 9–15.

Hatfield J. W., Milgrom P. L., (2005), Matching with Contracts, *American Economic Review*, 95, 913–935.

Hatfield J. W., Kominers S. D., Nichifor A., Ostrovsky M., Westkamp A., (2013), Stability and Competitive Equilibrium in Trading Networks, *Journal of Political Economy*, 121 (5), 966–1005.

Klaus B., Walzl M., (2009), Stable Many-to-Many Matchings with Contracts, *Journal of Mathematical Economics*, 45, 422–434.

Kominers S. D., (2012), On the Correspondence of Contracts to Salaries in (Many-to-Many) Matching, *Games and Economic Behavior*, 75, 984–989.

Manlove D. F., (2002), The Structure of Stable Marriage with Indifference, *Discrete Applied Mathematics*, 122, 167–181.

Mishra D., Talman D., (2010), Characterization of the Walrasian Equilibria of the Assignment Model, *Journal of Mathematical Economics*, 46, 6–20.

Moore J. C., (2007), *General Equilibrium and Welfare Economics*, Springer, Berlin.

Roth A. E., Sotomayor M. A., (1992), *Two-Sided Matching. A Study in Game-Theoretic Modeling and Analysis*, Cambridge University Press.

Sotomayor M., (2007), Connecting the Cooperative and Competitive Structures of the Multiple Partners Assignment Game, *Journal of Economic Theory*, 134, 155–174.

Shapley L. S., Scarf H., (1974), On Cores and Indivisibility, *Journal of Mathematical Economics*, 1, 23–37.

Shapley L. S., Shubik M., (1971/72), The Assignment Game I: The Core, *International Journal of Game Theory*, 1, 111–130.

Shoham Y., Leyton-Brown K., (2009), *Multiagent Systems – Algorithmic, Game-Theoretic and Logical Foundations*, Cambridge University Press.

Świtalski Z., (2008), Stability and Equilibria in the Matching Models, *Scientific Research of the Institute of Mathematics and Computer Science, Częstochowa University of Technology*, 2 (7), 77–85.

Świtalski Z., (2010), Equilibria in Markets with Two-Sided Preferences (in Polish), *Przegląd Statystyczny*, 4 (57), 54–69.

Świtalski Z., (2015), Some Properties of Competitive Equilibria and Stable Matchings in a Gale-Shapley Market Model, *Studia Ekonomiczne*, *Zeszyty Naukowe Uniwersytetu Ekonomicznego w Katowicach*, 248, 222–232.

STABILNOŚĆ I UOGÓLNIONE RÓWNOWAGI KONKURENCYJNE
W MODELU RYNKU GALE'A-SHAPLEYA TYPU „MANY-TO-MANY"

Streszczenie

W artykule zdefiniowano, dla pewnego wariantu modelu rynku Gale'a-Shapleya (typu „many-to--many"), pojęcie uogólnionej równowagi konkurencyjnej i pokazano że, przy odpowiednich założeniach, skojarzenia stabilne w tym modelu mogą być reprezentowane jako alokacje równowag konkurencyjnych (i vice versa). Przedstawione wyniki są daleko idącymi uogólnieniami „lematu o podaży i popycie" z pracy Azevedo, Leshno (2011) dotyczącego modelu rekrutacji kandydatów do szkół.

Wykorzystując wyniki Alkana, Gale'a (2003) udowodniono również twierdzenie o istnieniu uogólnionych równowag dla podanego modelu.

Słowa kluczowe: skojarzenie stabilne, teoria Gale'a-Shapleya, model „many-to-many", równowaga konkurencyjna, dyskretny model rynku, teoria kontraktów

STABILITY AND GENERALIZED COMPETITIVE EQUILIBRIA
IN A MANY-TO-MANY GALE-SHAPLEY MARKET MODEL

Abstract

We define, for some variant of a many-to-many market model of Gale-Shapley type, a concept of generalized competitive equilibrium and show that, under suitable conditions, stable matchings in such a model can be represented as competitive equilibria allocations (and vice versa). Our results are far-reaching generalizations of the "discrete supply and demand lemma" of Azevedo, Leshno (2011) for the college admissions market.

Using the results of Alkan, Gale (2003), we also prove a theorem on existence of generalized equilibria in our model.

Keywords: stable matching, Gale-Shapley theory, many-to-many model, competitive equilibrium, discrete market model, contract theory