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Relevant paraconsistent logic

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Roman Tuziak in his *Logika sprzeczności. Uwagi o logice parakonsystentnej* (hereafter: Tuziak, 2019) describes the first system of propositional paraconsistent calculus constructed by Stanisław Jaśkowski in 1948. In order to examine inconsistent situations he intended to find a system of calculus which:

1) when applied to the contradictory systems would not always entail their overcompleteness (*i.e.* triviality),
2) would be rich enough to enable practical inference,
3) would have an intuitive justification.

Discussive logic, which has his solution to the problem, was concerned with cases of discussions in which appear assertions that might seem or be contradictory to each other. As notes Jean-Yves Beziau (Beziau, 2000) the central problem of paraconsistent logic is to find a negation which is a paraconsistent negation in the sense that from two contradictory premises *a* and ¬*a*, we should not be able to deduce every formula *b*, and which at the same time is a paraconsistent negation in the sense that it has enough strong properties to be called a negation. Jaśkowski's system does not seem to be a solution to his problem.

Many logicians have developed Jaśkowski's ideas. For instance, since 1958 the Brazilian logician Newton da Costa, together with his collaborators, has formulated many systems of inconsistent theories. In the system C_n one can find the following conditions:

1) The law of non-contradiction should not be valid in general,
2) From two contradictory premises *a* and ¬*a*, we should not be able to deduce every formula *b*,
3) The systems should contain as many schemas as possible of classical propositional calculus (CL).

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Da Costa extends a positive (negation-free) part of classical logic by adding to it a negation weaker than that of classical logic.

An interesting approach is represented by relevance logic. As is known in the late 1950s Anderson and Belnap developed the system in order to avoid the so-called paradoxes of implication. Since formulas such as $A \to (B \to A)$ and $A \to (\neg A \to B)$, are not valid in relevance systems, so the logics are paraconsistent (the failure of the first formula prevents us from getting the other one from contraposition).

Finally, a different approach was developed by Diderick Batens and his co-laborators. The main feature of adaptive logic is that they are externaly dynamic, which means that the inference relation is non-monotonic and internally dynamic, which means that earlier conclusions may be withdrawn in case of the inconsistent behavior of some formulas in later lines of the proof.

As notes Tomasz Skura and Roman Tuziak (Skura & Tuziak, 2005) a paraconsistent logic should be as rich as possible. In other words a system should contain as many classical theorems as possible. In formal terms this means that paraconsistent logic should be maximal. To be precise, by a logic we shall mean a (Tarskian) consequence relation $\vdash$ between finite sets of formulas and formulas that satisfies the following conditions.

(refl) $A \vdash A$

(mon) If $X \vdash A$, then $X,Y \vdash A$

(trans) If $X \vdash A$ and $X, A \vdash B$, then $X \vdash B$

(struct) If $X \vdash A$ and s is a substitution, then $sX \vdash sA$.

A logic $\vdash$ is said to be subclassical, if $\vdash \subseteq \vdash_{CL}$, where $\vdash_{CL}$ is defined thus:

$X \vdash_{CL} A$, if for every valuation v in the 2-element Boolean algebra, if $v(X) \subseteq \{1\}$ then $v(A) = 1$.

We say that a subclassical logic $\vdash$ is paraconsistent, if the sequent $\sigma = p, \neg p / q$ is not in $\vdash$. And we say that a paraconsistent logic $\vdash$ is maximal, if there is no paraconsistent logic $\vdash$` such that $\vdash$ is a proper subset of $\vdash$`.

If a paraconsistent logic has the connective of implication a binary connective $\to$ satisfies the condition:

(modus ponens) $A, A \to B \vdash B$.

Then $\vdash B$ whenever both $\vdash A$ and $\vdash A \to B$.

A logic $T \subseteq CL$ is said to be paraconsistent, if the formula $Z = p \to (\neg p \to q)$ is not in T. (Here CL is the set of theorems of (classical logic) $\vdash_{CL}$). And we say that a paraconsistent logic T is maximal, if there is no paraconsistent logic T` such that T is a proper subset of T`. From these definitions we obtain the following useful facts.

PROPOSITION 1: A paraconsistent logic T is maximal if for every formula $A \in CL - T$, there is a substitution s such that $sA \to Z \in T$.

PROPOSITION 2: A paraconsistent logic $\vdash$ is maximal if for every sequent $X/A \in \vdash_{CL} - \vdash$ there is a substitution s such that $p, \neg p, sA \vdash q$ and for every $F \in X$ we have $p, \neg p \vdash sF$.

Both Skura/Tuziak and Arieli/Avron/Zamansky (Arieli, Avron, & Zamansky, 2010) come to a conclusion that the simplest and the most natural a paraconstistent logic is three-valued. The first authors are interested in algebras M = (M, ¬, ∧, ∨, →) with M = {f, ∗, t} such that ({f, t}, ¬, ∧, ∨) is a Boolean algebra and (M, ∧, ∨) is a lattice, that is, f ≤ ∗ ≤ t and x ∧ y (x ∨ y) is the meet (join) of x, y. Moreover we assume that the operation → should not be quite arbitrary. What conditions should it satisfy? They consider three kinds of implication: Lukasiewiczian, intuitionistic, and relevant ones. The corresponding algebras will be denoted by L, H, R respectively. For each of them we have two kinds of negation: one will be denoted by — and the other by ~. They are defined thus: −∗ = ∗ and ~∗ = t (the case ¬∗ = f excludes paraconsistency). Thus we have 6 algebras: L_{-}, $L_{\sim}$, H_{-}, $H_{\sim}$, R_{-}, $R_{\sim}$. For each of them we can choose either d = {t} or D = {∗, t} as a set of designated values. If M is one these algebras, the symbols $\mathbf{M}_d$, and $\mathbf{M}_D$ will denote the matrices (M, d) and (M, D), respectively. Let's focus on relevance logics. The relevance principle was first introduced by Alan R. Anderson and Nuel D. Belnap. As is known these systems developed as attempts to avoid the paradoxes of material and strict implication. The material implication (A→B) is true whenever A is false or B is true — *i.e.*, (¬A∨B). So if A is true, then the material implication is true when B is true. Among the paradoxes of material implication are the following:

A→(B→A) (Positive Paradox),

¬A→(A→B),

(A→B) ∨ (B→C).

In turn, the strict implication (A→B) is true whenever it is not possible that A is true and B is false — *i.e.*, ¬◊(A ∧ ¬B). Among the paradoxes of strict implication are the following:

(A ∧ ¬A)→B,

A→(B→B),

A→(B ∨ ¬B).

It is obvious for example that rejection of the Positive Paradox prevents from receiving implicative version of the explosion's principle: A → (¬A → B). To derive the last one enable the Positive Paradox with the axiom of contraposition (A11 below). R may be axiomatized as follows:

A1. A → A (self-implication, SI)

A2. (A ∧ B) → A, (A ∧ B) → B (∧-elimination, ∧-E)

A3. ((A → B) ∧ (B → C)) → (A → (B ∧ C)) (∧-introduction, ∧-I)

A4. A → (A ∨ B), B → (A ∨ B) (∨-introduction, ∨-I)

A5. ((A → C) ∧ (B → C)) → ((A ∨ B) → C) (∨-elimination, ∨-E)

A6. (A ∧ (B ∨ C)) → ((A ∧ B) ∨ (A ∧ C)) (distributivity, D)

A7. (A → B) → ((B → C) → (A → C)) (suffixing, SF)

A8. (A → (A → B)) → (A → B) (contraction, CTR)

A9. $(A \to (B \to C)) \leftrightarrow (B \to (A \to C))$ (permutation, PM)
A10. $(A \to \neg A) \to \neg A$ (reductio, RD)
A11. $(A \to B) \to (\neg B \to \neg A)$ (contraposition, CP)
A12. $\neg\neg A \to A$ (double negation, DN).
RM is obtained by adding to R the axiom scheme:
A13. $A \to (A \to A)$ (mingle).
Mingle can be replaced equivalently with the converse of Contraction (A8):
$(A \to B) \to (A \to (A \to B))$ (expansion).

As noted by Priest/Routley (Priest & Routley, 1984) we have many examples of paraconsistent theories: the Newton-Leibniz version of the calculus, Cantor's set theory, early quantum mechanics, Hegel's dialectic, etc. A particularly interesting example of is naive set theory. Naive set theory is the theory in a first order (intensional) language whose only predicate is "$\in$", and whose postulates are

1) $\exists\, y\, \forall x\, (x \in y \leftrightarrow \varphi)$ where φ is arbitrary,
2) $\{\, \forall x\, (x \in z \leftrightarrow x \in y)\} \Vdash z = y$.

This theory captures the naive notion of set, viz. a set is the extension of an arbitrary property. The assumption that any property may be used to form a set, without restriction, leads to paradoxes. One common example is Russell's paradox: there is no set consisting of "all sets that do not contain themselves".

One of the attempts to invalidate explosion is to treat negation as an intensional operator. A Routley (1972) (Priest, 2002) proposed a structure, (W, *, v), where W is a set (of worlds), * is a map from W to W, and v maps sets of pairs comprising a world and propositional parameter to {1, O}. The truth conditions for conjunction and disjunction are the standard:

$v_w(A \wedge B) = 1$ iff $v_w(A) = 1$ and $v_w(B) = 1$
$v_w(A \vee B) = 1$ iff $v_w(A) = 1$ or $v_w(B) = 1$.
The truth conditions for negation are:
$v_w(\neg A) = 1$ iff $v_{w^*}(A) = 0$
if $w^* = w$, these conditions just reduce to the classical ones.

Using Dunn's semantics *FDE*, it means semantics for the logic of First Degree Entailment, there is possibility to modify it. In natural deduction terms, *FDE* can be characterized by the rules $\wedge$-I, $\wedge$-E, $\vee$-I and $\vee$-E, together with the rules:

$\dfrac{\neg (A \wedge B)}{\neg A \vee \neg B}$ De Morgan

$\dfrac{\neg A \wedge \neg B}{\neg (A \vee B)}$

$\dfrac{A}{\neg\neg A}$ Double Negation

Modification of the logic relies on dropping the rule for double negation, and replacing it with:

A

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.

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$$\frac{B \quad \neg B}{\neg A}$$

One can add further conditions on * without ruining its paraconsistency. The most notable is: $w^{**} = w$. In this situation, we restore the rule for double negation.

Given an *FDE* interpretation, v, define a Routley evaluation on the worlds w and w*, as follows:

$v_w(A) = 1$ iff $1 \in v(A)$

$v_w{}^*(A) = 1$ iff $0 \notin v(A)$.

Following a Routley interpretation one can treat also → intensionally. The simplest way is to give it the *S5* truth conditions:

$v_w(A \to B) = 1$ iff for all $w' \in W$ ($v_{w'}(A) = 1 \Rightarrow v_{w'}(B) = 1$)

In such an interpretation either $A \to B$ is true at all worlds, or at none. With the Routley * and the semantics for negation, it follows that the same is true of negated conditionals. It also follows that:

$v_w (A \to B) = 1$ iff $v_w{}^* (A \to B) = 1$ iff $v_w \neg (A \to B) \neq 1$.

For the achievement maximal paraconsistent logics Skura/Tuziak (cf. Skura & Tuziak, 2005) define the operation → as follows:

$* \to f = f$, $* \to * = *$, $* \to t = t = f \to *$, $t \to * = f$.

Note that $\vdash_{-d}$ and $\vdash_{\sim d}$ are not paraconsistent. Also $T_{-d} \subseteq T_{-D}$ and $T_{\sim d} \subseteq T_{\sim D}$. Therefore it suffices to consider the logics $\vdash_{-D}$, $\vdash_{\sim D}$ and their corresponding sets of theorems. Both $\vdash_{\sim D}$ and $\vdash_{\sim D}$ are maximal paraconsistent logics, and so are the sets of their theorems. To see this take $G_* = p$, $G_t = q \to q$, $G_f = \neg(q \to q)$ for $\vdash_{-D}$, and $G_* = p$, $G_t = \neg p$, $G_f = \neg\neg p$ for $\vdash_{\sim D}$. We note that T_{-D} is not a subset of $T_{\sim D}$ (because $p \to (\neg p \to p)$ is in T_{-D} but not in $T_{\sim D}$).

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