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Fractals and music. A reconnaissance

So, Nat'ralists observe, a Flea
Hath smaller Fleas that on him prey,
And these have smaller fleas to bit 'em,
And so proceed ad infinitum.
(Jonathan Swift, Poems ii.651, 1773)[1]

ABSTRACT: Among the many definitions of the fractal employed by mathematicians, one of the most suggestive holds that 'the fractal is a self-similar figure displaying an invariability in respect to the transformations of scaling'. This article is an effort to present the overview of fractals in mathematics and nature and then to describe the current state of research on fractal nature of music. It is shown that self-similarity and scaling are properties of many canonic works of Western music (e.g. Johann Sebastian Bach, Ludwig van Beethoven), appearing in various forms in all historical periods. It is found in binary and ternary divisions of form and in melodic structures. It is also noted that a frequent point of reference in fractal studies of the properties of music is twentieth-century repertoire (e.g. Per Nørgård, Conlon Nancarrow, György Ligeti, Charles Wuorinen). The case of 1/f noise in which frequency (pitch) scaling naturally occurs is also discussed. Such 'scaling noise' is typical of many natural phenomena; it is observed, for example, in the variable tension of nerve cells and in heartbeats. It was also discovered in music. The article summarizes the results of the research made by Voss and Clarke (1975, 1978), Hsü and Hsü (1990, 1991), Henze and Cooper (1997) who analyzed stylistically diverse works – classical, jazz, blues, rock and non-European music – and found in them 1/f relationships referring to Fourier spectra, notes or intervals. The article reports also the psychological experiments raising the statements about a close relationship between fractal structure and the human sense of beauty. It is stressed that the fractal orientation of modern mathematics provides interesting cognitive tools allowing us to discover hitherto unexplored links between nature and art, both in the area of listeners' aesthetic preferences and also in the fascinating realm of artistic creation.

KEYWORDS: fractals, fractal nature of music, music analysis, 1/f noise, scaling, self-similarity

[1] Jonathan Swift, 'On Poetry: A Rapsody' (1733), in *The Poems of Jonathan Swift*, ed. William E. Browning, vol. 1 (London, 1910). See online version: <http://www.online-literature.com/swift/poems-of-swift/99>, accessed 9 December 2009.

Fractals in nature and in mathematics

'I coined *fractal* from the Latin adjective *fractus*. The corresponding Latin verb *frangere* means "to break"; to create irregular fragments. It is therefore sensible – and how appropriate for our needs! – that, in addition to "fragmented" (as in *fraction* or *refraction*), *fractus* should also mean "irregular", both meanings being preserved in *fragment*.' Thus wrote Benoît Mandelbrot.[2] The notion he created became an ideal method for the mathematical description of objects which resisted the traditional procedures of algebraic geometry, simplifying their properties by means of a model. Mandelbrot proposed identifying irregular geometric objects not so much with a model as with the prescription for their creation, and in the process of visualising such an algorithm he used a computer. The computer-generated set named after him contains an infinite number of small sets, very similar to one another, differing only in size (see Figure 1).

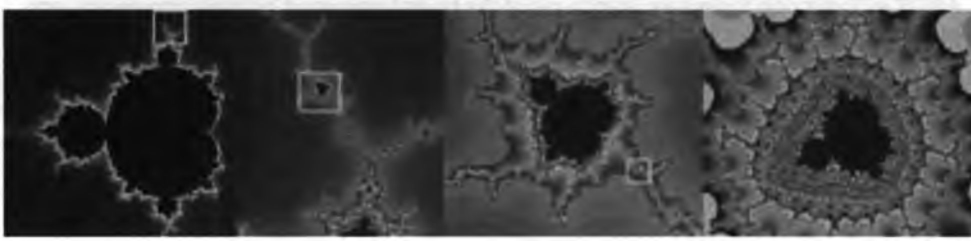

Figure 1. Quasi-self-similarity in the Mandelbrot set[3]

Among the many definitions of the fractal employed by mathematicians, one of the most suggestive holds that 'the fractal is a self-similar figure displaying an invariability in respect to the transformations of scaling'. Besides this, 'a geometric figure worthy of the name fractal must be sufficiently irregular, but not simply and purely random or 'holey', and also in some way beautiful, and at first glance it should attract attention and intrigue the observer with its sometimes baroque structure'.[4]

It is easiest to find such 'baroque' forms in nature, which does not usually take on forms that are purely triangular, square, round or spherical. The

[2] Benoît B. Mandelbrot, *The Fractal Geometry of Nature* (New York, 1982), 4.

[3] Drawing by António Miguel de Campos, taken from <http://pl.wikipedia.org/wiki/Plik:Mandelzoom.jpg>, accessed 9 December 2009.

[4] Marek Wolf, 'Prawo Zipfa, samopodobieństwo i muzyka' [Zipf's law, self-similarity and music] in *Przestrzeń w nauce współczesnej*, ed. Stefan Symiotuk and Grzegorz Nowak (Lublin, 2000). An on-line version available at (<http://www.ift.uni.wroc.pl/~mwolf/lublin.ps>), 2, accessed 9 December 2009.

coastline of India, compared to a triangle, is actually much more irregular and jagged, just like our planet, described as a rotational ellipsoid. The common occurrence of fractal structures in nature has led many scholars to state that fractal geometry is *de facto* the geometry of nature. As Mandelbrot noted: 'many facets of Nature can only be described with the help of fractals [...] Nature's patterns are irregular and fragmented [...] self-similarity is [...] the fabric of Nature.'[5] And so fractal-like objects include snowflakes and clouds, mountain ranges and river water systems, and also some plants. For example, the cauliflower has a head consisting of florets which, after separation from the whole, resemble the head scaled down. Another plant of self-similar construction is the fern (its smaller fronds resemble a large frond scaled down), which is the favourite object of computer simulations of fractals (see Figure 2).

Cauliflower
Romanesco Brassica oleracea[6]

Fern
Thelipteris noveboracensis[7]

Figure 2.

The chief defining feature of a fractal from the mathematical point of view are power-law relations expressing the property called 'scaling'. This involves

[5] Mandelbrot, *The Fractal Geometry of Nature*, 193–194, 1, 423.

[6] Example by Richard Bartz, taken from <http://commons.wikimedia.org/wiki/File:Romanesco_Brassica_oleracea_Richard_Bartz.jpg>, accessed 9 December 2009.

[7] Example taken from <http://en.wikipedia.org/wiki/File:Thelipteris_noveboracensis_ECU.jpg>, accessed 9 December 2009.

the power-law dependence of one quantity on another, as for example when calculating the surface area of a circle (πr^2), which depends in a power-law way on the figure's linear dimensions. For the fractal, however, the corresponding exponents are non-integers, which points to their 'holey', 'jagged' structure. Another property of the fractal is self-similarity, which means that a shape does not change in character when seen magnified and scaled, and that one part is similar to another part and to the whole. There are three types of self-similarity:

1. Quasi-self-similarity – its free form, in which the fractal 'appears approximately (but not exactly) identical at different scales';
2. Exact self-similarity – the strongest type of self-similarity, expressed through the creation of faithful copies of the objects as scale models;
3. Statistical self-similarity – its weakest form, in which the fractal 'has numerical or statistical measures which are preserved across scales', as in natural objects.[8]

Finally, the infinitely rich structure of fractals is due to the fact that their construction has an iterative character, repeating the same procedure an infinite number of times. Summing up, then, a fractal object must meet three conditions:

1. It must be built of a set of elements of different size, 'whose size distribution satisfies a power-law relationship spanning at least three scales';
2. It must comprise at least 'two similar regions in which the arrangement of elements either mirrors or imitates the structure of the object as a whole' and
3. Its features 'must possess sufficient detail that the overall structure cannot be more easily explained in Euclidean terms'.[9]

The most exact realisation of all these properties can be found in mathematical objects which were observed long before the notion of the fractal was introduced into the scientific language. Perhaps the most famous fractal is the Cantor set,[10] based on the Cantor function. This is formed from a line of a specific length being divided into three parts. The middle third is then removed. The remaining outer thirds are replicated below the original line and the operation involving the removal of the middle third applied to each. This iterative process is repeated ad infinitum. The 'fractal dimension' of the Can-

[8] See Saitis Charalampos, 'Fractal Art: Closer to Heaven? Modern Mathematics, the Art of Nature, and the Nature of Art' in *Proceedings of Bridges Conference* (San Sebastian, 2007), 371.

[9] See Harlan Brothers, 'What makes something Fractal?' (<http://www.brotherstechnology.com/yale/FractalMusic/FracMusicBground/Frac.html>, 2002–2009), accessed 9 December 2009.

[10] See Georg Cantor, 'Über unendliche, lineare Punktmannigfaltigkeiten', *Mathematische Annalen* 21 (1883), 545–591.

tor set,[11] which shows to what extent the fractal fills the space in which it is set, is 0.631, which means that it is situated between a point and a line (it is 'less than a line') and consists of an infinite number of segments of almost zero length (see Figure 3).

Figure 3. Cantor set[12]

'More than a line' (fractal dimension 1.261), meanwhile, is the Helge von Koch curve,[13] which is the product of the division of a line segment into three equal parts and the replacement of the middle part with a 'saw-tooth' (an equilateral triangle with no base). This gives a segment comprising four equal segments. By further iterations, we can obtain a curve made only of saw-teeth, of infinite length, but contained within a small area. When we put the Koch curves together, we obtain a Koch snowflake.

Another famous fractal is the Wacław Sierpinski triangle,[14] also known as the Sierpinski gasket or Sierpinski sieve. This is produced from an initial equilateral triangle which is divided up internally into four equal triangles, the middle one of which has its points in the centre of the sides of the higher-order triangle. This middle triangle is removed, and the operation is then repeated on the remaining three triangles. The points that remain after an infinite number of repetitions of this operation form the Sierpinski triangle, with fractal dimension 1.585 (see Figures 4–5).

[11] A fractal dimension of 0 denotes a space of zero dimension, i.e. a point. Value 1 denotes a one-dimensional space, i.e. a line; value 2, a two-dimensional space, i.e. a plane; value 3, a three-dimensional space, i.e. a solid. A fractional dimensional value means that the space is not completely filled, i.e. it is something intermediate between one dimension and another.

[12] Example taken from <http://en.wikipedia.org/wiki/File:Cantor_set_in_seven_iterations.svg>, accessed 9 December 2009.

[13] Helge von Koch, 'Une méthode géométrique élémentaire pour l'étude de certaines questions de la théorie des courbes plane', *Acta Mathematica* 30 (1906), 145–174.

[14] Wacław Sierpinski, 'Sur une courbe cantorienne qui contient une image biunivoquet et continue detoute courbe donee', *Compus Reudu de l'Académie Paris*, 162 (1916), 629–632.

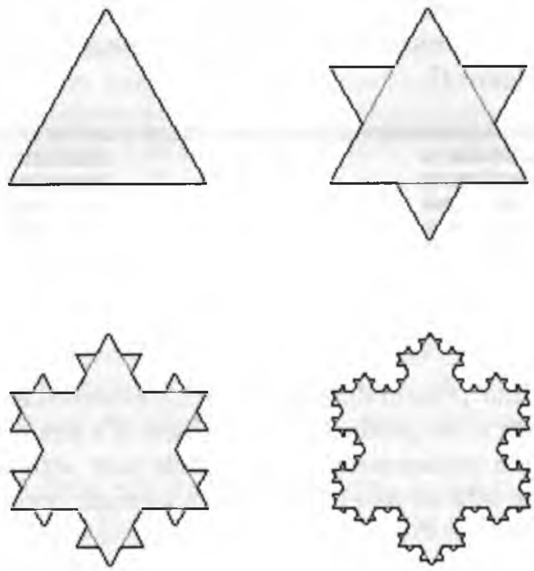

Figure 4. Koch snowflake[15]

Figure 5. Sierpinski triangle[16]

The above fractals belong to the group of iterated function systems (IFS), created iteratively through self-replication. The two other types of fractal are escape-time fractals, defined by the recurrent relationship of spatial points and forming highly impressive visualisations (e.g. the Mandelbrot set), and random fractals, generated stochastically (e.g. natural phenomena).[17] It is these that we find in music.

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15 Example taken from <http://commons.wikimedia.org/wiki/File:KochFlake.png>, accessed 9 December 2009.

16 Example taken from <http://en.wikipedia.org/wiki/File:Sierpinski_triangle_evolution.svg>, accessed 9 December 2009.

17 See Charalampos, 'Fractal Art: Closer to Heaven?', 3.

Fractals in music

Self-similarity and scaling are properties of many canonic works of Western music, appearing in various forms in all historical periods, since – as Mandelbrot himself noted – 'music displays fractal characteristics because of its inherently hierarchical nature'.[18] Consequently, scholars frequently point to the fractal lineage of all those musical processes which have involved generating a construction from a small seed-motif through the operations of symmetry. The best examples are canons and fugues. In a canon, the comes can be either an exact replica of the *dux* or its transformation; this is connected with the existence of different types of canon. One of these is the proportional canon, also called 'canon by augmentation or diminution', in which the rhythm of the *dux* is imitated in some other ratio than 1:1. The comes may therefore proceed more quickly or more slowly than the *dux*; this is connected with the phenomenon of motivic scaling. For example, in the second section of the Agnus Dei from Josquin des Prez's *Missa l'homme armé super vocem musicales*, different voices repeat the same melody and rhythmic motifs at different tempos: the middle voice one-third more quickly than the upper voice and the lower voice twice as quickly as the middle voice or two-thirds as quickly as the upper voice.[19]

We find similar procedures in many contrapuntal works by Johann Sebastian Bach, which seem to be scholars' favourite examples for demonstrating the action of the phenomenon of self-similarity and scaling in music.

Analysing the first Bourrée from the Cello Suite No. 3, Harlan J. Brothers[20] deemed it an example of structural scaling in respect to phrasing, which may be visualised as a construct analogous to the Cantor set (see Example 7). The phrasing in the first sixteen bars of this composition contains sequences of notes which are connected with each other in a particular way. They realise an AAB model in different scales, where section B lasts twice as long as section A. The basic model, m1 – two quavers and a crotchet – is first repeated (m2) and then expanded into a phrase twice as long (m3): 2 bars, 2 bars and 4 bars. This same pattern, short, short, long, in a different scale, denoted as s1, is treated analogically: 2 bars, 2 bars, 4 bars. This procedure is illustrated in a graphic notation in which 'blue groups are short elements, red groups are

[18] Gavin O'Brien, 'A Study of Algorithmic Composition and its potential for aiding laptop-based interactive performance', (M.Phil. thesis, University of Dublin, Trinity College, 2004), 26.

[19] See Harlan J. Brothers, 'Josquin des Prez' (<http://www.brotherstechnology.com/fractal-music/josquin.html>, 2002–2009), accessed 9 December 2009.

[20] 'Structural Scaling in Bach's Cello Suite No. 3', *Fractals* 15/1 (2007), 89–95.

long elements, and gray groups are of no concern. The x-axis is time and the y-axis is pitch' (see Figure 6).[21]

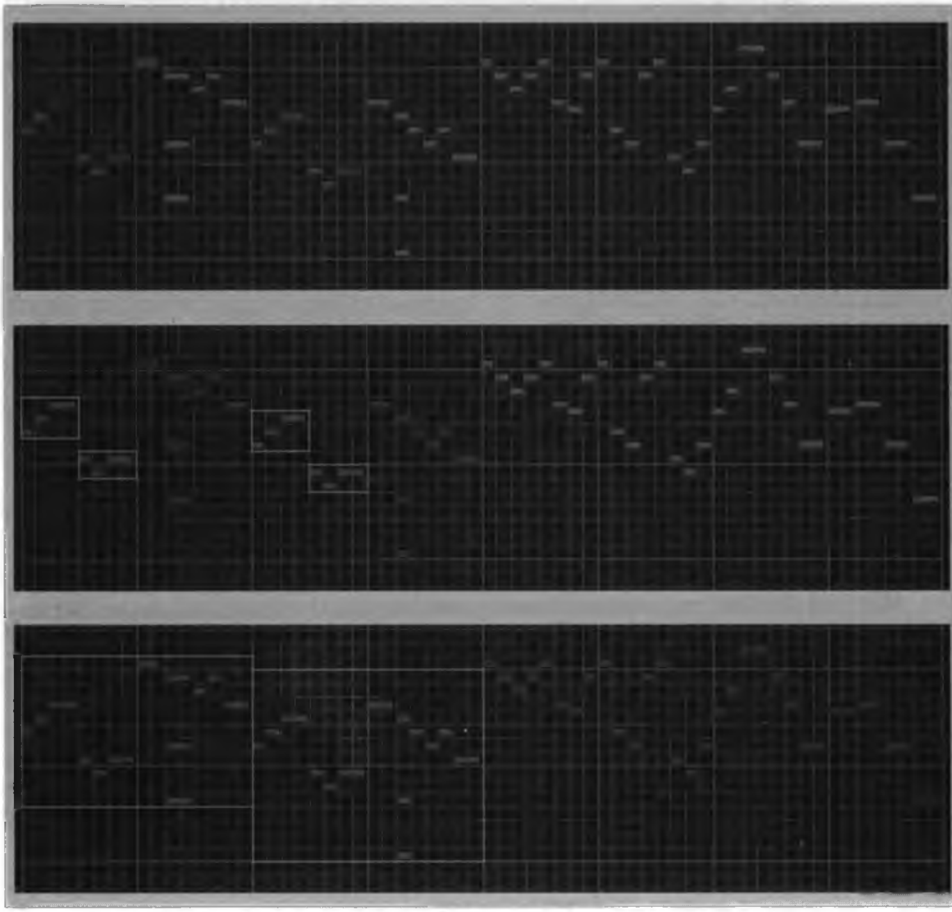


Figure 6. Graphic notation of structural scaling in Cello Suite No. 3 by J. S. Bach[22]

[21] See Brothers, 'Fractal Music. Structural Scaling' (<http://www.brotherstechnology.com/yale/FractalMusic/StrScaling/Bourree1.html>, 2004), accessed 9 December 2009.

[22] See Brothers, 'Fractals. Structural Scaling' (<http://www.brotherstechnology.com/yale/FractalMusic/StrScaling/Bourree1.html>, <http://www.brotherstechnology.com/yale/FractalMusic/StrScaling/Bourree2.html>, <http://www.brotherstechnology.com/yale/FractalMusic/StrScaling/Bourree3.html>, 2004), accessed 9 December 2009.

Example 7. J. S. Bach, Cello Suite No. 3, bars 1–16[23]

All the types of (exact) mathematic fractal mentioned above have been transferred to musical structures. For instance, the formal properties of a Cantor set have a decisive influence – in the opinion of Larry Solomon[24] – on the binary structure of the first of the cycle of six *Ecossaises* (WoO 83) by Ludwig van Beethoven and the self-similarity of the motifs employed there. In his analysis of the thirty-two bars of the score of this work, Solomon distinguishes two sections of equal length, marked A and B. Each of these sections divides into two eight-bar periods, and these, in turn, into binary four-bar phrases. The phrases are divided into two-bar sub-phrases, which are themselves divided into one-bar motifs (marked 'm') and their transformations. These motifs, meanwhile, are built from a binary group of two quavers and a crotchet. Binary one-bar units (marked 'n') are also contained in the bass part. In other words, each successive division of the thirty-two bars is a binary unit constituting a smaller replica of a larger unit (see Example 8).

[23] Analysis of the first 16 bars of the suite taken from Brothers, 'Structural Scaling in Bach's Cello Suite No. 3', 92.

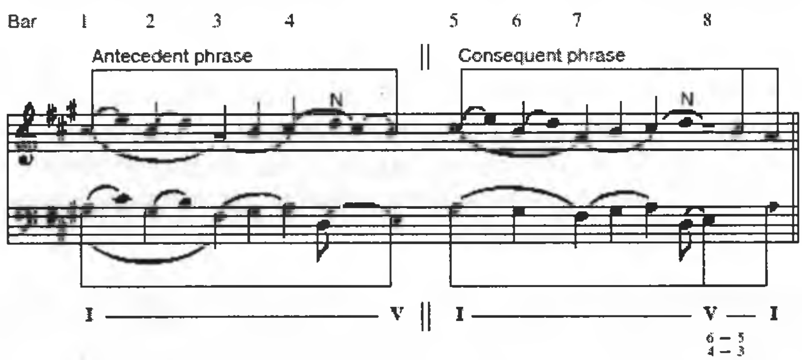
[24] Larry Solomon, 'The Fractal Nature of Music' (<http://solomonsmusic.net/fracmus.htm>, 2002), accessed 9 December 2009.

Example 8. Analysis of Ecossaise WoO 83 No. 1 by Ludwig van Beethoven[25]

Solomon stresses that the structure discovered in Beethoven is nothing exceptional, as it occurs commonly in tonal compositions based on a periodic structure. Thus another example of a fractal model of the Sierpinski triangle is the formal construct of the third movement (Scherzo) of Beethoven's Sonata, Op. 28, which displays an ABA construction with repeated binary and

[25] Example taken from Solomon, 'The Fractal Nature of Music' (see footnote 24).

ternary subdivisions, as well as many other compositions from the classical-romantic canon. An intuitive example of the seeking of fractal structures in tonal music would appear to be Schenkerian analysis.[26] After all, its basic aim is to identify self-repeating melodic and harmonic patterns that are present both on the surface of a work and in its deeper layers. The method developed by Schenker highlights self-similarity, iteration and scaling as inherent properties of tonal music. The similarity, noticeable in Schenkerian analysis, of local structures from shallower layers to the fundamental structures of the deep layer – the Ursatz (fundamental structure) – and their repetition in different scales also displays one of the characteristics of fractals given in the introduction – a sort of visual, graphic beauty (see Example 9).



Example 9. Schenkerian analysis of Wolfgang A. Mozart, Sonata in A, K331, first movement, bars 1–8[27]

A frequent point of reference in fractal studies of the properties of music is twentieth-century repertoire. Analysing etudes (canons) for pianola by Conlon Nancarrow, Julie Scrivener found they contained distinct fractal structures relating to the time and tempo of the works.[28] Treated as a model example of a musical fractal is the 'infinity series' (Dan. 'Uendelighedsrækken') of Per Nørgård, discovered in 1959. This is used to construct a melodic line and has no effect on the rhythmic or dynamic aspects of a composition. It was elaborated on the basis of the chromatic scale, but the diatonic or any other scale may also be used. The core of the whole (model) structure comprises the notes *g* and *a flat* and the interval of a rising semitone, which is projected twice, first inverted (*g-f sharp* with top beam), then uninverted (*a*

[26] See Julie Scrivener, 'Applications of Fractal Geometry to the Player Piano Music of Conlon Nancarrow', in *Proceedings of Bridges 2000: Mathematical Connections in Art, Music, and Science* [July 28–30, 2000] ed. Reza Sarhangi (Winfield, Kan., 2000), 187.

[27] Example taken from <http://openlearn.open.ac.uk/rss/file.php/stdfeed/3376/formats/AA314_2_rss.xml>, accessed 9 December 2009.

[28] Scrivener, 'Applications of Fractal Geometry', 185–192.

flat-a with bottom beam). This procedure gives rise to two new notes: *f sharp* and *a*. The next interval generated by this process is *a flat-f sharp*, two semitones downwards. This is developed again by the projection of the notes – inversion in the 'upper' system (*f sharp-a flat*) – and the retention of the original direction to the motion in the 'lower' system (*a-g*), which gives the notes *a flat* and *g*. Further starting notes produce new ones, and this gives rise to new intervals and their 'infinite' succession (see Example 10).[29]

Example 10. Generating of an infinite series through the projection of intervals[30]

The first 64 elements of the infinity series, written out in a graphic schema by Erika Christensen,[31] show – as the author states – 'a repeated succession of figures, which can be described as M- and W- shapes separated by rising axes: M/M/W/M. The interval ranges of the M- and W- shapes correspond two by two, and numerous other intervallic relationships between the figures can be observed' (see Figure 11).[32]

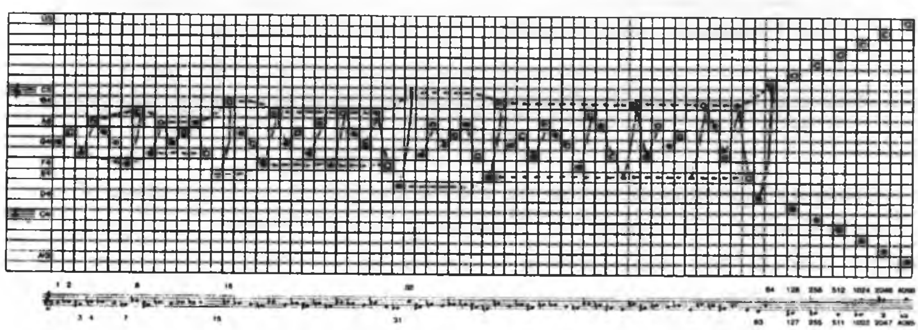


Figure 11. Unfolding and expansion of the infinity series[33]

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[29] See Jørgen Mortensen, 'Uendelighedsrækken' [infinityseries]. (<http://www.pernoergaard.dk/eng/strukturer/uendelig/uindhold.html>, 1998–99), acessed 9 December 2009.

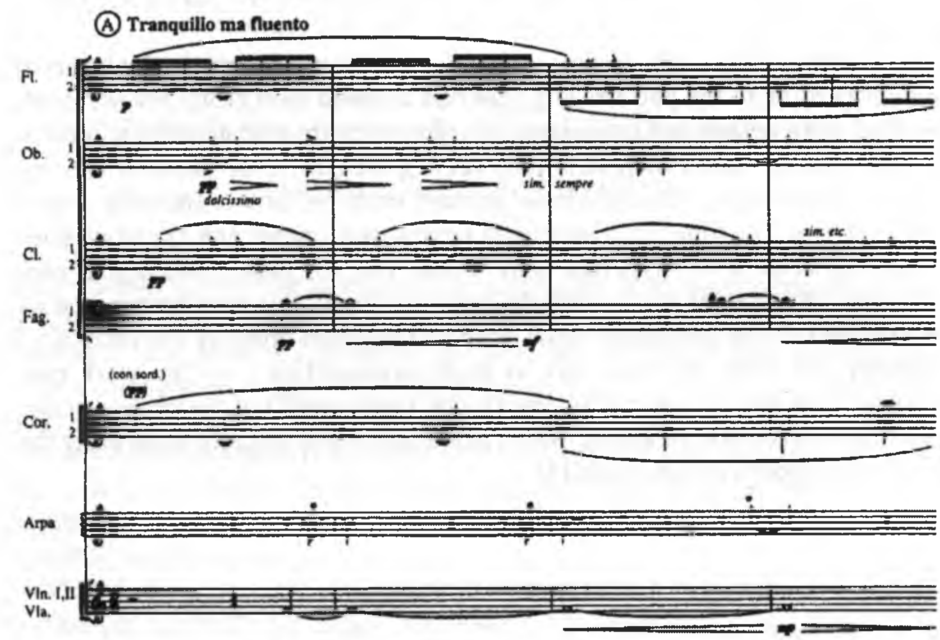
[30] Example taken from J. Mortensen, 'Construction by the projection of intervals' (<http://www.pernoergaard.dk/eng/strukturer/uendelig/ukonstruktion01.html>, 1998–99).

[31] Erik Christensen, 'Overt and hidden processes in 20th century music', *Axiomathes*, 14 (2004), 97–117.

[32] Ibid., 107.

[33] Ibid., 109. The example is a reproduction from another work by Christensen: *The Musical Timespace*. A Theory of Music Listening, vol. 2. Notation Examples and Graphs. (Aalborg University Press, 1996), 58.

The links between Nørgård's infinity series and the ideas of fractal geometry are most fully explained by the term 'open hierarchy' (Dan. 'åbent hierarki'), used by the composer. This concerns the structural relations between different levels, none of which is superior to the others. For instance, in the opening bars of *Voyage into the Golden Screen* (1967), the original series is projected in the part of the flutes. When the odd-numbered notes are removed, there remains the sequence 2, 4, 6, 8..., as in the part of the clarinet. Keeping only every fourth element of the series (4, 8, 12, 16...) brings about the projection of another transposition, as in the part of the oboe. In the French horn part, meanwhile, we find a sequence produced by extracting every fourth element (1, 5, 9, 13...), which is a replica of the basic series four times slower. In this way, the infinity sequence is present in various transpositions and modifications in different layers of the work, which reproduces a generative fractal process during which reproductions and reflections of its own shapes arise (see Example 12).[34]



Example 12. Nørgård, *Voyage into the Golden Screen*, second movement, bars 1–4

While in Nørgård's case, the compositional idea precedes the appearance in scholarship of the term 'fractal', several other twentieth-century composers quite deliberately undertook to incorporate self-similar fractal forms in their

[34] Christensen, 'Overt and hidden processes', 115.

musical structures, e.g. György Ligeti and Charles Wuorinen. The latter worked directly with Mandelbrot on performances of some of his works (e.g. *Bamboula Squared* for tape and orchestra (1984), *Natural Phantasy* for organ (1985)), and Ligeti admitted to being inspired by abstract forms generated by a computer from the Mandelbrot set. In an interview for *The New York Times*, he said: 'Yes, fractals are what I want to find in my music. They are the most complex of ornaments in the arts, like small sea horses, like the Alhambra where the walls are decorated with geometric ornaments of great minuteness and intricacy, or like the Irish Book of Kells, those marvellously decorated borders and capitals. The most complicated ornaments – perhaps not art, perhaps geometry. It is a very complex music, difficult to describe. I only want to give a metaphysic for my music. After all, music is not a science.'[35] He also pointed to places in his music where the fractal idea had a direct influence. He regarded the first work from the cycle of Etudes for piano – 'Désordre' (1985) – as an example of self-similarity with an iterative structure derived from the Koch snowflake, while he even called the fourth movement of his Piano Concerto (1985–88) 'a fractal piece'.[36]

'Désordre' presents a continuum of quavers grouped asymmetrically, most often according to the pattern 3+5. The first notes of each group are accented, doubled at an octave and prolonged, in order to create spatial melodic lines of a higher order. These melodies have varying length in the parts of the two hands. The melody in the right hand consists of three phrases (initially 4+4+6 bars) repeated 14 times in a gradually compressed metre and transpositions a step upwards in the hyperphrygian mode. The left hand, meanwhile, projects four phrases per cycle (initially 4+4+6+4 bars), this time transposed by two degrees of the pentatonic scale downwards (most often by the interval of a fourth and a third). There are 11 such transpositions. As the work progresses, there occurs a sort of scaling (increasingly small) of the phrasal structure in its successive iterations, which resembles the procedure generating the Koch snowflake[37] (see Example 13).

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35 John Rockwell, 'Laurels at an auspicious time for György Ligeti', *The New York Times*, 11 November (1986).

36 György Ligeti in conversation with Heinz-Otto Petigen and Richard Steinitz during the Huddersfield Festival, in November 1993. Cit. after Richard Steinitz, 'The dynamics of disorder', *The Musical Times* 137/1838 (1996), 8.

37 Ibid.

Example 13. Ligeti, *Désordre*, beginning

The use of procedures inspired by modern mathematical ideas in the creation of music allows one to express the conviction that Ligeti opened 'an important new chapter in the historic interaction between art and science'.[38] As a result of the growing interest in the phenomenon of fractals, there exists today a substantial repertoire of algorithmic compositions based on fractal-generating equations. Fractal algorithms are applied to pitch, dynamics, duration and other parameters in order to determine the compositional process. The types of the music thus produced depend on the kind of software employed and the methods adopted, which most often combine the procedures of scaling and iteration.

The wholly automatic composing of such structures has made it possible to discover in music 1/f noise, described by the beta superscript denoting fractal dimension ($1/f^{\beta}$). 1/f noise, also known as pink noise, is a signal with a frequency spectrum such that the power spectrum density is proportional to the reciprocal of the frequency, which means that it possesses the property of scaling, inherently contained in the soundwave. Such 'scaling noise', as Mandelbrot dubbed it, is typical of many natural phenomena; it is observed, for

[38] Richard Steinitz, 'Weeping and Wailing', *The Musical Times* 137/1842 (1996), 22.

example, in the variable tension of nerve cells and in heartbeats. Richard F. Voss and John Clarke,[39] meanwhile, discovered it in recordings of music and the human voice broadcast on radio. Stylistically diverse works – classical, jazz, blues, rock and also selected examples of non-European music – subjected to analysis displayed a power spectrum density proportional to 1/f, i.e. beta=1 or very close to 1. Only some contemporary atonal compositions (e.g. Karlheinz Stockhausen, Elliott Carter), which resembled white noise with a constant spectral density ($1/f^0$), did not possess this property (see Figure 14).[40]

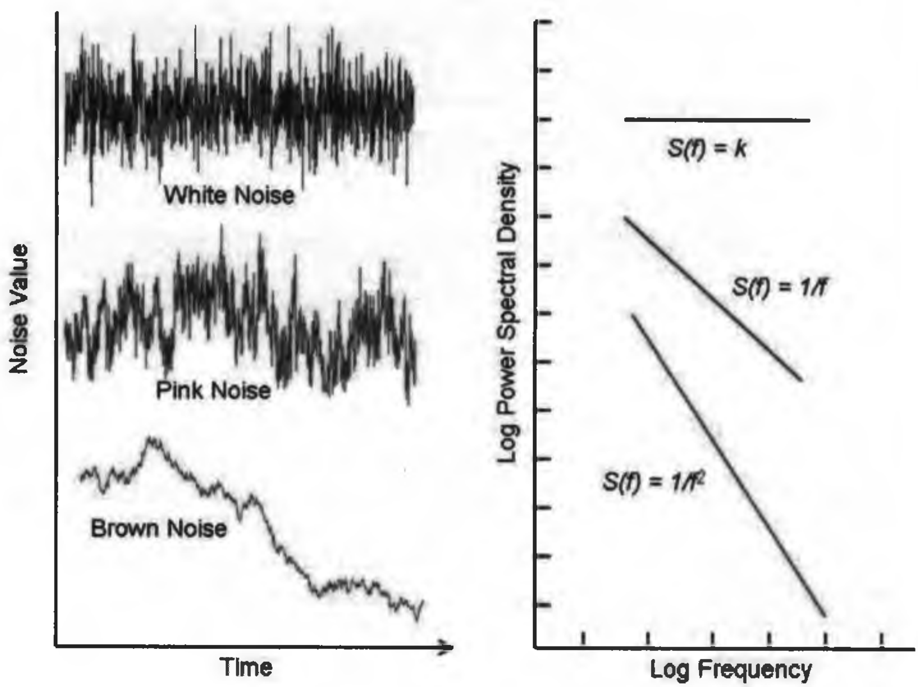


Figure 14. Realisations of the time series of various noises and their respective power spectra[41]

[39] See e.g. Richard F. Voss and John Clarke, '1/f noise in music and speech', *Nature* 258 (1975), 317–318, and Richard F. Voss and John Clarke, '1/f noise in music: music from 1/f noise', *Journal of Acoustic Society of America* 63/1 (1978), 258–263.

[40] There also exists red noise (also known as brown noise; Brownian noise), with a spectral power density of $1/f^2$.

[41] Figure taken from <http://www.scholarpedia.org/article/1/f_noise>, accessed 9 December 2009.

Research into 1/f noise in music has also been carried out by Kenneth and Andreas Hsü,[42] who focussed not on the correlation of musical notes on a global scale, but on the power-law relations between two successive intervals in a musical composition. Marking as *F* the relative frequency of the occurrence of an interval *i* semitones in length (*i* = 1 for a minor second, etc.), they demonstrated the existence of the relation $F \approx i^D$, a fractal relation characteristic of scaling. The dimension *D* for the analysed works fluctuated between 1.34 for J.S. Bach's Toccata in F sharp minor (BWV 910), through 1.73 for Wolfgang A. Mozart's Sonata in F major (KV 533) to 2.42 for J.S. Bach's Two-part Invention No. 1 in C major (BWV 772); it was similar also in the case of Swiss children's songs. Only in the case of Stockhausen's *Capricorn* could this value (identical to the fractal dimension) not be established.

Kenneth and Andreas Hsü were also interested in the problem of reducing music in order to find the smallest self-similar fragments possible.[43] They compared 'reduced landscapes' of J.S. Bach's Two-part Invention in C major, No. 1, generated from CD by means of special apparatus, reducing these 'landscapes' to 1/2, 1/4, 1/8 and more. It turned out that the visual similarity in the distinguished plots was very marked. As they noted, 'a half or quarter reduction of notes seems to give an outline of the music as it was written by J.S. Bach. The reader can find the audio self-similarity by playing the reduced notes on a piano, as we did. To a novice, the half- or quarter-Bach sounds like J.S.Bach, although he gains the impression that the composition has an economy of [trills] and ornamentations. Further reductions to 1/8, 1/16, and 1/32 tend to eliminate more of the "irregularity of the silhouette", yet the distinguishing overall line of the music landscape is preserved.'[44]

Only the final reduction, to 1/64, gives three notes which the authors see as the fundamental notes on which the entire composition was built. Thus the investigative procedure led the authors to the conclusion that 'reduced landscapes' open up the prospect of creating music for novices, whom this sort of skeleton of a work may help to understand its properties better and more quickly (see Figure 15).[45]

[42] Kenneth J. Hsü and Andreas J. Hsü, 'Fractal Geometry of Music', *Proceedings of National Academy of Sciences USA* 87 (1990), 938–941.

[43] 'Self-similarity of the "1/f noise" called music', *Proceedings of National Academy of Sciences USA* 88 (1991), 3507–3509.

[44] Ibid., 3508.

[45] Ibid., 3509.

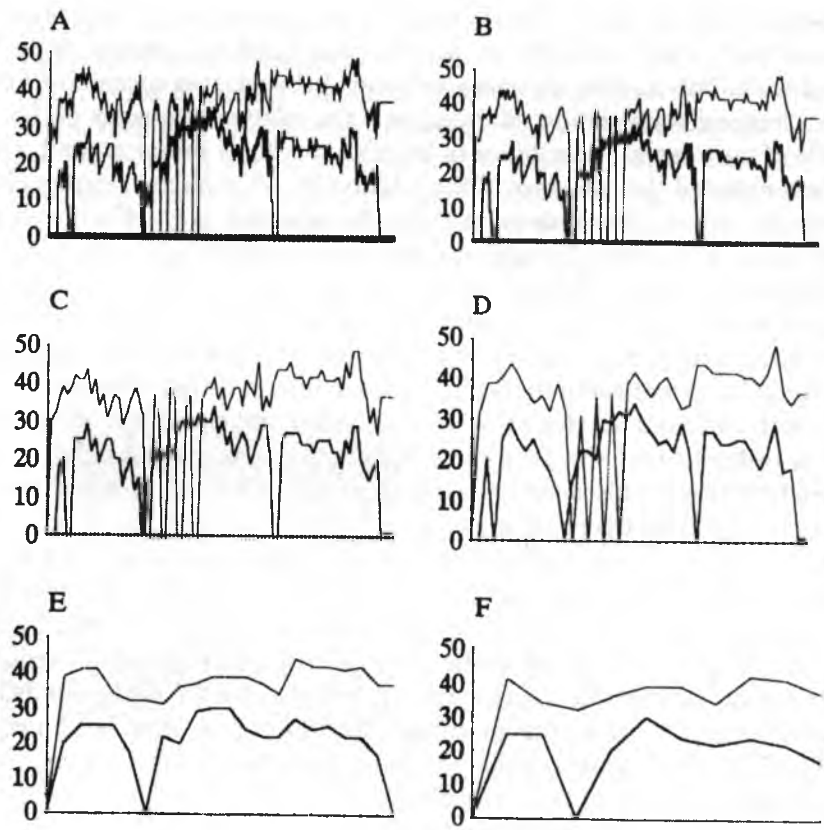


Figure 15. Reduction of J. S. Bach's Two-part Invention in C major, BWV 772. A – whole, B – 1/2, C – 1/2, D – 1/8, E – 1/16, F – 1/32[46]

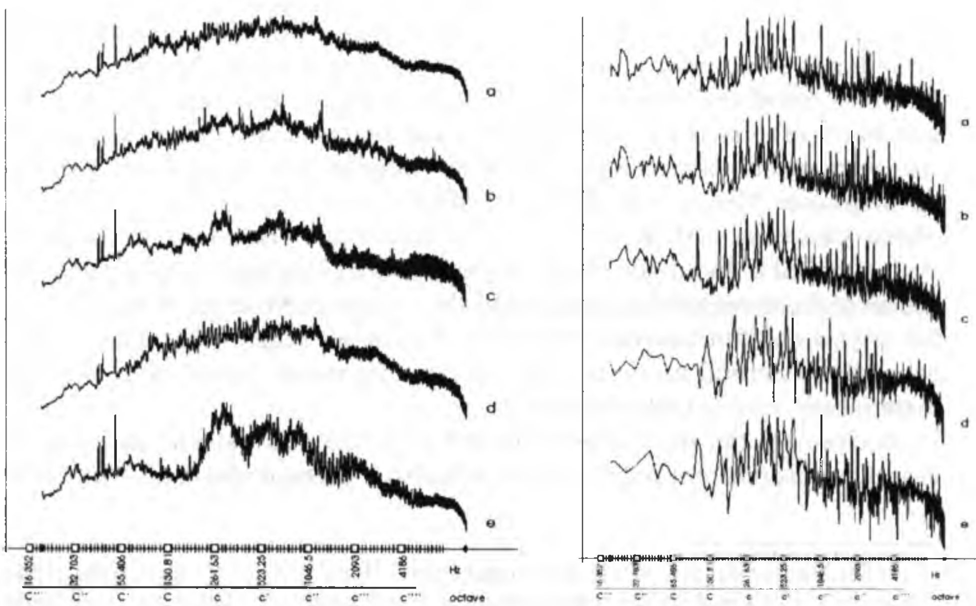
Similar reductions were made by Sabine Henze and David Cooper,[47] who divided up Fourier spectra of musical works from different historical periods and cultural circles in order to find the point at which self-similarity disappears. They presented the results on diagrams generated by means of the Cool Edit program, on which amplitudes are measured in decibels and the spectra are presented as log-log plots.

All the analyses showed considerable similarities, but their lower limits proved to be considerably differentiated. They most often fell between 2 and 6

[46] Ibid., 3508.

[47] See Sabine Henze and David Cooper, 'Fractal Characteristics of the Fourier Spectra of Recordings of Musical Compositions', Electronic Studio of the Department of Music at the University of Leeds, 1997. An on-line version of the paper available at <http://www.leeds.ac.uk./music/Studio/lcmg/es-rrs.html>, 1-10, accessed February 2009.

seconds, as in works by J.S. Bach (Fugue in C minor), Beethoven (Third Symphony, first movement) and Pierluigi da Palestrina (*Osculetur me osculo oris sui*), and even in examples of music from West Java (Indonesia). Only twentieth-century repertoire fell outside these parameters. In the case of Ligeti's *Aventures*, the average limit was too small and – as the authors of the experiment noted – 'differences in the spectrum are already obvious by a quarter of the piece (around 2:55 min) and are strongly amplified by an eighth of the piece'.[48] The most self-similar construction was shown by Steve Reich's *Music for Mallet Instruments*, in which the self-similar fragments lasted barely 0.03 sec (see Figure 16).



Ligeti, *Aventures*
a) first half of the piece (5:52 min); b) second half of the piece; c) quarter (2:56 min), not centred, long gaps; d) quarter, not centred, big highlights; e) eighth (1:28 min), not centred, long gaps

Reich, *Music for Mallet Instruments*
a-c) one sec.; d-e) 0.03 sec.

Figure 16. Reduction of spectra of works by Ligeti and Reich[49]

[48] Ibid., 3.
[49] Ibid., 6–7.

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The above-mentioned applications of fractal geometry in music will unquestionably lead to a strengthening of music's ties, observed since ancient times, with nature and with mathematics. Although the fractal aspect of music is just one of the many elements defining a musical composition, its investigation may reveal an answer to the fundamental question of the essence of musical beauty, of the existence of patterns that transcend the boundaries of cultures and tastes. These would appear to be patterns that are neither too monotonous and regular, on one hand, nor tormenting our ears with complete disorder, on the other.

This assumption is confirmed by psychological experimentation. Richard Voss has examined, for instance, how three types of melody generated from three kinds of acoustic noise (white, pink (1/f) and brown) are interpreted. Listeners found melodies based on pink noise (1/f) 'most appealing' in relation to the others, as they represented a satisfying balance between excessive randomness and excessive predictability.[50] Similar conclusions were reached by Stephanie Mason and Michael Saffle,[51] generating melodies and polyphonic structures with the help of L-system curves[52]. These proved 'similar or even identical to hundreds of existing melodies by classical and popular composers'.[53] In summarising, they stated that 'aspects of certain theories about the origins and fundamental structures of melodies suggest that much – perhaps all – beautiful music is, in some essential sense, fractal in its melodic material and internal self-similarity'.[54]

A close relationship between fractal structure and the human sense of beauty is observed not only in music, but also – perhaps above all – in works

[50] See Martin Gardner, 'White, Brown and Fractal Music' in *Fractal Music, Hypercards and more..., Mathematical Recreations from SCIENTIFIC AMERICAN Magazine* (New York, 1992), 15.

[51] Stephanie Mason and Michael Saffle, 'L-Systems, Melodies and Musical Structure', *Leonardo Music Journal* 4 (1994), 31–38.

[52] The curves discovered by Aristid Lindenmayer (a Hungarian biologist), known as 'L-systems curves', are a sort of formal grammar describing the growth of plants. In Mason and Saffle's experiment, melodies were generated from them by interpreting horizontal segments as durations and vertical segments as pitches.

[53] Mason and Saffle, 'L-Systems, Melodies and Musical Structure', 35.

[54] Ibid. Complementary research made by different scholars proved that fractal dimension could be used effectively in distinguishing different kinds of music and in aesthetic evaluation. See for example Maxime Bigerelle and Alain Iost, 'Fractal dimension and classification of music', *Chaos, Solitons and Fractals* 11 (2000) 2179–2192; Bill Manaris, Juan Romero, Penousal Machado, Dwight Krehbiel, Timothy Hirzel, Walter Pharr, Robert B. Davis, 'Zipf's Law, Music Classification, and Aesthetics', *Computer Music Journal* 29(1) (2005), 55–69.

of visual art. This was noted by Mandelbrot himself, who emphasised that in abstract art there exists 'a sharp distinction between such art that has a fractal base and such art that does not, and that the former type is widely considered the more beautiful'.[55] His intuitions have supported, for example, the experiments of Richard P. Taylor, who tested people's reactions to a series of mechanically-generated false 'Pollocks'. Taylor noted that viewers were particularly fond of pictures with a fractal dimension of 1.3–1.5.[56]

Regardless of the ultimate answer to the question as to the essence of beauty in art and whether it is connected with a fractal dimension, the fractal orientation of modern mathematics provides revolutionary cognitive tools allowing us to discover hitherto unexplored links between nature and art, both in the area of receivers' aesthetic preferences and also in the most fascinating and at the same time most mysterious realm of artistic creation.

Translated by John Comber

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[55] See Gardner, 'White, Brown and Fractal Music', 11.

[56] Richard P. Taylor, 'Order in Pollock's Chaos', *Scientific American* (December 2002), 116–121.